ACI 318-14 CODE EDITION

TENTH EDITION

DESIGN OF REINFORCED CONCRETE

JACK C. McCORMAC • RUSSELL H. BROWN

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Design of Reinforced Concrete

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Preface

Audience

This textbook presents an introduction to reinforced concrete design. We authors hope the material is written in such a manner as to interest students in the subject and to encourage them to continue its study in the years to come. The text was prepared with an introductory three-credit course in mind, but sufficient material is included for an additional three-credit course.

New to This Edition

Updated Code

With the tenth edition of this text, the contents have been updated to conform to the *2014 Building Code of the American Concrete Institute* (ACI 318-14). Changes to this edition of the code are discussed in Section 1.7 of the text under the heading **Summary of 2014 ACI Code Changes**.

Chapter on Concrete Masonry Updated to ACI 530-13 Code

The new chapter on strength design of reinforced concrete masonry that was added to the ninth edition of the text has been updated to conform to the 2013 issue of ACI 530. Because this code revision involved a lot of reorganization, most of the code references are different. This chapter is available only online (www.wiley.com/college/mccormac). Because strength design of reinforced concrete masonry is so similar to that of reinforced concrete, the authors felt that this would be a logical extension to the application of the theories developed earlier in the text. The design of masonry lintels, walls loaded out-of-plane, and shear walls are included. An example of the design of each type of masonry element is also included to show the student some typical applications. Spreadsheets to assist in the design of these elements are also included on the Wiley website.

Organization

The text is written in the order that the authors feel would follow the normal sequence of presentation for an introductory course in reinforced concrete design. In this way, it is hoped that skipping back and forth from chapter to chapter will be minimized. The material on columns is included in three chapters (Chapters 9, 10, and 11). Some instructors do not have time to cover the material on slender columns, so it was put in a separate chapter (Chapter 11). The remaining material on columns was separated into two chapters in order to emphasize the difference between columns that are primarily axially loaded (Chapter 9) and those with significant bending moment combined with axial load (Chapter 10).

Instructor and Student Resources

The website for the book is located at www.wiley.com/college/mccormac and contains the following resources.

For Instructors

Solutions Manual A password-protected Solutions Manual, which contains complete solutions for all homework problems in the text, is available for download. Most are handwritten, but some are carried out using spreadsheets or Mathcad.

Figures in PPT Format Also available are the figures from the text in PowerPoint format, for easy creation of lecture slides.

Lecture Presentation Slides in PPT Format Presentation slides developed by Dr. Terry Weigel of the University of Louisville are available for instructors who prefer to use PowerPoint for their lectures. The PowerPoint files are posted rather than files in PDF format to permit the instructor to modify them as appropriate for his or her class.

Sample Exams Examples of sample exams are included for most topics in the text. Problems in the back of each chapter are also suitable for exam questions.

Course Syllabus A course syllabus along with a typical daily schedule is included in editable format.

Visit the Instructor Companion Site portion of the book website at www.wiley.com/college/mccormac to register for a password. These resources are available for instructors who have adopted the book for their course. The website may be updated periodically with additional material.

For Students and Instructors

Excel Spreadsheets Excel spreadsheets were created to provide the student and the instructor with tools to analyze and design reinforced concrete elements quickly to compare alternative solutions. Spreadsheets are provided for most chapters of the text, and their use is self-explanatory. Many of the cells contain comments to assist the new user. The spreadsheets can be modified by the student or instructor to suit their more specific needs. In most cases, calculations contained within the spreadsheets mirror those shown in the example problems in the text. The many uses of these spreadsheets are illustrated throughout the text. At the end of most chapters are example problems demonstrating the use of the spreadsheet for that particular chapter. Space does not permit examples for all of the spreadsheet capabilities. The examples chosen were thought by the authors to be the most relevant.

Visit the Student Companion Site portion of the book website at www.wiley.com/college/mccormac to download these spreadsheets.

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JACK C. MCCORMAC
RUSSELL H. BROWN

Introduction

CHAPTER 1

1.1 Concrete and Reinforced Concrete

Concrete is a mixture of sand, gravel, crushed rock, or other aggregates held together in a rocklike mass with a paste of cement and water. Sometimes one or more admixtures are added to change certain characteristics of the concrete such as its workability, durability, and time of hardening.

As with most rocklike substances, concrete has a high compressive strength and a very low tensile strength. *Reinforced concrete* is a combination of concrete and steel wherein the steel reinforcement provides the tensile strength lacking in the concrete. Steel reinforcing is also capable of resisting compression forces and is used in columns as well as in other situations, which are described later.

1.2 Advantages of Reinforced Concrete as a Structural Material

Reinforced concrete may be the most important material available for construction. It is used in one form or another for almost all structures, great or small—buildings, bridges, pavements, dams, retaining walls, tunnels, drainage and irrigation facilities, tanks, and so on.

The tremendous success of this universal construction material can be understood quite easily if its numerous advantages are considered. These include the following:

1. It has considerable compressive strength per unit cost compared with most other materials.
2. Reinforced concrete has great resistance to the actions of fire and water and, in fact, is the best structural material available for situations where water is present. During fires of average intensity, members with a satisfactory cover of concrete over the reinforcing bars suffer only surface damage without failure.
3. Reinforced concrete structures are very rigid.
4. It is a low-maintenance material.
5. As compared with other materials, it has a very long service life. Under proper conditions, reinforced concrete structures can be used indefinitely without reduction of their load-carrying abilities. This can be explained by the fact that the strength of concrete does not decrease with time but actually increases over a very long period, measured in years, because of the lengthy process of the solidification of the cement paste.
6. It is usually the only economical material available for footings, floor slabs, basement walls, piers, and similar applications.
7. A special feature of concrete is its ability to be cast into an extraordinary variety of shapes from simple slabs, beams, and columns to great arches and shells.
8. In most areas, concrete takes advantage of inexpensive local materials (sand, gravel, and water) and requires relatively small amounts of cement and reinforcing steel, which may have to be shipped from other parts of the country.
9. A lower grade of skilled labor is required for erection as compared with other materials such as structural steel.

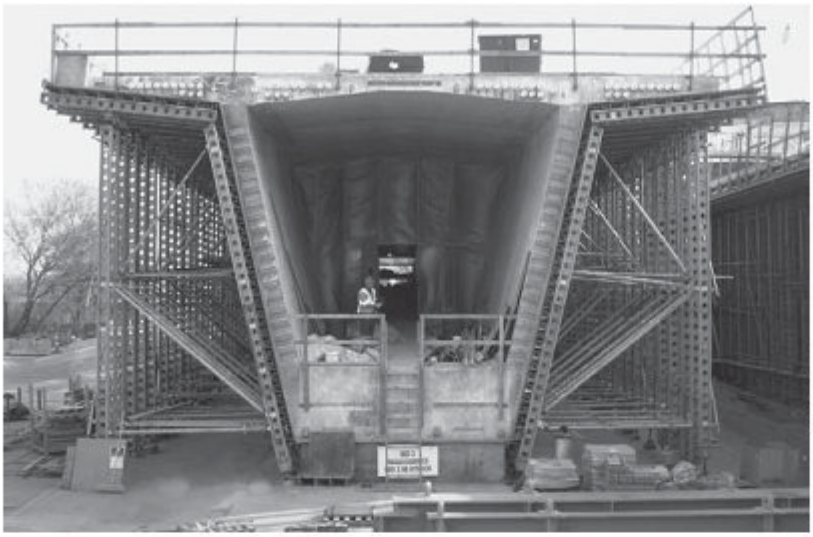
Courtesy of Portland Cement Association.

NCNB Tower in Charlotte, North Carolina, completed 1991.

1.3 Disadvantages of Reinforced Concrete as a Structural Material

To use concrete successfully, the designer must be completely familiar with its weak points as well as its strong ones. Among its disadvantages are the following:

1. Concrete has a very low tensile strength, requiring the use of tensile reinforcing.
2. Forms are required to hold the concrete in place until it hardens sufficiently. In addition, falsework or shoring may be necessary to keep the forms in place for roofs, walls, floors, and similar structures until the concrete members gain sufficient strength to support themselves. Formwork is very expensive. In the United States, its costs run from one-third to two-thirds of the total cost of a reinforced concrete structure, with average values of about 50%. *It should be obvious that when efforts are made to improve the economy of reinforced concrete structures, the major emphasis is on reducing formwork costs.*
3. The low strength per unit weight of concrete leads to heavy members. This becomes an increasingly important matter for long-span structures, where concrete's large dead weight has a great effect on bending moments. Lightweight aggregates can be used to reduce concrete weight, but the cost of the concrete is increased.
4. Similarly, the low strength per unit volume of concrete means members will be relatively large, an important consideration for tall buildings and long-span structures.



Courtesy of EFCO Corp.

Pre-cast segments for the I-35W St Anthony Falls Bridge, Minneapolis, Minnesota.

5. The properties of concrete vary widely because of variations in its proportioning and mixing. Furthermore, the placing and curing of concrete is not as carefully controlled as is the production of other materials, such as structural steel and laminated wood.

Two other characteristics that can cause problems are concrete's shrinkage and creep. These characteristics are discussed in Section 1.12 of this chapter.

1.4 Historical Background

Most people believe that concrete has been in common use for many centuries, but this is not the case. The Romans did make use of a cement called *pozzolana* before the birth of Christ. They found large deposits of a sandy volcanic ash near Mt. Vesuvius and in other places in Italy. When they mixed this material with quicklime and water as well as sand and gravel, it hardened into a rocklike substance and was used as a building material. One might expect that a relatively poor grade of concrete would result, as compared with today's standards, but some Roman concrete structures are still in existence today. One example is the Pantheon (a building dedicated to all gods), which is located in Rome and was completed in 126 CE.

The art of making pozzolanic concrete was lost during the Dark Ages and was not revived until the eighteenth and nineteenth centuries. A deposit of natural cement rock was discovered in England in 1796 and was sold as "Roman cement." Various other deposits of natural cement were discovered in both Europe and America and were used for several decades.

The real breakthrough for concrete occurred in 1824, when an English bricklayer named Joseph Aspdin, after long and laborious experiments, obtained a patent for a cement that he called portland cement because its color was quite similar to that of the stone quarried on the Isle of Portland off the English coast. He made his cement by taking certain quantities of clay and limestone, pulverizing them, burning them in his kitchen stove, and grinding the resulting clinker into a fine powder. During the early years after its development, his cement was used primarily in stuccos.[1] This wonderful product was adopted very slowly by the building industry and was not even introduced in the United States until 1868; the first portland cement was not manufactured in the United States until the 1870s.

The first uses of concrete are not very well known. Much of the early work was done by the Frenchmen François Le Brun, Joseph Lambot, and Joseph Monier. In 1832, Le Brun built a concrete

[1] Kirby, R. S. and Laurson, P. G., 1932, *The Early Years of Modern Civil Engineering* (New Haven: Yale University Press), p. 266.

Itar-Tass Photos/NewsCom.

Installation of the concrete gravity base substructure (CGBS) for the LUNA oil-and-gas platform in the Sea of Okhotsk, Sakhalin region, Russia.

house and followed it with the construction of a school and a church with the same material. In about 1850, Lambot built a concrete boat reinforced with a network of parallel wires or bars. Credit is usually given to Monier, however, for the invention of reinforced concrete. In 1867, he received a patent for the construction of concrete basins or tubs and reservoirs reinforced with a mesh of iron wire. His stated goal in working with this material was to obtain lightness without sacrificing strength.[2]

From 1867 to 1881, Monier received patents for reinforced concrete railroad ties, floor slabs, arches, footbridges, buildings, and other items in both France and Germany. Another Frenchman, François Coignet, built simple reinforced concrete structures and developed basic methods of design. In 1861, he published a book in which he presented quite a few applications. He was the first person to realize that the addition of too much water to the mix greatly reduced concrete's strength. Other Europeans who were early experimenters with reinforced concrete included the Englishmen William Fairbairn and William B. Wilkinson, the German G. A. Wayss, and another Frenchman, François Hennebique.[3,4]

William E. Ward built the first reinforced concrete building in the United States in Port Chester, New York, in 1875. In 1883, he presented a paper before the American Society of Mechanical Engineers in which he claimed that he got the idea of reinforced concrete by watching English laborers in 1867 trying to remove hardened cement from their iron tools.[5]

Thaddeus Hyatt, an American, was probably the first person to correctly analyze the stresses in a reinforced concrete beam, and in 1877, he published a 28-page book on the subject, entitled *An Account of Some Experiments with Portland Cement Concrete, Combined with Iron as a Building Material*. In this book he praised the use of reinforced concrete and said that "rolled beams (steel) have to be taken largely on faith." Hyatt put a great deal of emphasis on the high fire resistance of concrete.[6]

E. L. Ransome of San Francisco reportedly used reinforced concrete in the early 1870s and was the originator of deformed (or twisted) bars, for which he received a patent in 1884. These bars, which

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[2] Ibid., pp. 273–275.

[3] Straub, H., 1964, *A History of Civil Engineering* (Cambridge: MIT Press), pp. 205–215. Translated from the German *Die Geschichte der Bauingenieurkunst* (Basel: Verlag Birkhauser), 1949.

[4] Kirby and Laurson, *The Early Years of Modern Civil Engineering*, pp. 273–275.

[5] Ward, W. E., 1883, "Béton in Combination with Iron as a Building Material," *Transactions ASME*, 4, pp. 388–403.

[6] Kirby and Laurson, *The Early Years of Modern Civil Engineering*, p. 275.

were square in cross section, were cold-twisted with one complete turn in a length of not more than 12 times the bar diameter.[7] (The purpose of the twisting was to provide better bonding or adhesion of the concrete and the steel.) In 1890 in San Francisco, Ransome built the Leland Stanford Jr. Museum. It is a reinforced concrete building 312 ft long and 2 stories high in which discarded wire rope from a cable-car system was used as tensile reinforcing. This building experienced little damage in the 1906 earthquake and the fire that ensued. The limited damage to this building and other concrete structures that withstood the great 1906 fire led to the widespread acceptance of this form of construction on the West Coast. Since the early 1900s, the development and use of reinforced concrete in the United States has been very rapid.[8,9]

1.5 Comparison of Reinforced Concrete and Structural Steel for Buildings and Bridges

When a particular type of structure is being considered, the student may be puzzled by the question, "Should reinforced concrete or structural steel be used?" There is much joking on this point, with the proponents of reinforced concrete referring to steel as that material that rusts and those favoring structural steel referring to concrete as the material that, when overstressed, tends to return to its natural state—that is, sand and gravel.

There is no simple answer to this question, inasmuch as both of these materials have many excellent characteristics that can be utilized successfully for so many types of structures. In fact, they are often used together in the same structures with wonderful results.

The selection of the structural material to be used for a particular building depends on the height and span of the structure, the material market, foundation conditions, local building codes, and architectural considerations. For buildings of fewer than 4 stories, reinforced concrete, structural steel, and wall-bearing construction are competitive. From 4 to about 20 stories, reinforced concrete and structural steel are economically competitive, with steel having been used in most of the jobs over 20 stories in the past. Today, however, reinforced concrete is becoming increasingly competitive for buildings over 20 stories, and there are a number of reinforced concrete buildings of greater height around the world. The 74-story, 859-ft-high Water Tower Place in Chicago is the tallest reinforced concrete building in the world. The 1465-ft CN tower (not a building) in Toronto, Canada, is the tallest reinforced concrete structure in the world.

Although we would all like to be involved in the design of tall, prestigious reinforced concrete buildings, there are just not enough of them to go around. As a result, nearly all of our work involves much smaller structures. Perhaps 9 out of 10 buildings in the United States are 3 stories or fewer in height, and more than two-thirds of them contain 15,000 sq ft or less of floor space.

Foundation conditions can often affect the selection of the material to be used for the structural frame. If foundation conditions are poor, using a lighter structural steel frame may be desirable. The building code in a particular city may favor one material over the other. For instance, many cities have fire zones in which only fireproof structures can be erected—a very favorable situation for reinforced concrete. Finally, the time element favors structural steel frames, because they can be erected more quickly than reinforced concrete ones. The time advantage, however, is not as great as it might seem at first because, if the structure is to have any type of fire rating, the builder will have to cover the steel with some kind of fireproofing material after it is erected.

Making decisions about using concrete or steel for a bridge involves several factors, such as span, foundation conditions, loads, architectural considerations, and others. In general, concrete is an excellent compression material and normally will be favored for short-span bridges and for cases where rigidity is required (as, perhaps, for railway bridges).

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[7] American Society for Testing Materials, 1911, *Proceedings*, 11, pp. 66–68.
[8] Wang, C. K. and Salmon, C. G., 1998, *Reinforced Concrete Design*, 6th ed. (New York: HarperCollins), pp. 3–5.
[9] "The Story of Cement, Concrete and Reinforced Concrete," *Civil Engineering*, November 1977, pp. 63–65.

1.6 Compatibility of Concrete and Steel

Concrete and steel reinforcing work together beautifully in reinforced concrete structures. The advantages of each material seem to compensate for the disadvantages of the other. For instance, the great shortcoming of concrete is its lack of tensile strength, but tensile strength is one of the great advantages of steel. Reinforcing bars have tensile strengths equal to approximately 100 times that of the usual concretes used.

The two materials bond together very well so there is little chance of slippage between the two; thus, they will act together as a unit in resisting forces. The excellent bond obtained is the result of the chemical adhesion between the two materials, the natural roughness of the bars, and the closely spaced rib-shaped deformations rolled onto the bars' surfaces.

Reinforcing bars are subject to corrosion, but the concrete surrounding them provides them with excellent protection. The strength of exposed steel subjected to the temperatures reached in fires of ordinary intensity is nil, but enclosing the reinforcing steel in concrete produces very satisfactory fire ratings. Finally, concrete and steel work well together in relation to temperature changes because their coefficients of thermal expansion are quite close. For steel, the coefficient is 0.0000065 per unit length per degree Fahrenheit, while it varies for concrete from about 0.000004 to 0.000007 (average value: 0.0000055).

1.7 Design Codes

The most important code in the United States for reinforced concrete design is the American Concrete Institute's *Building Code Requirements for Structural Concrete* (ACI 318-14).[10] This code, which is used primarily for the design of buildings, is followed for the majority of the numerical examples given in this text. Frequent references are made to this document, and section numbers are provided. Design requirements for various types of reinforced concrete members are presented in the code along with a commentary on those requirements. The commentary provides explanations, suggestions, and additional information concerning the design requirements. As a result, users will obtain a better background and understanding of the code.

The ACI Code is not in itself a legally enforceable document. It is merely a statement of current good practice in reinforced concrete design. It is, however, written in the form of a code or law so that various public bodies, such as city councils, can easily vote it into their local building codes, and then it becomes legally enforceable in that area. In this manner, the ACI Code has been incorporated into law by countless government organizations throughout the United States. The International Building Code (IBC), which was first published in 2000 by the International Code Council, has consolidated the three regional building codes (Building Officials and Code Administrators, International Conference of Building Officials, and Southern Building Code Congress International) into one national document. The IBC Code is updated every 3 years and refers to the most recent edition of ACI 318 for most of its provisions related to reinforced concrete design, with only a few modifications. It is expected that IBC 2015 will refer to ACI 318-14 for most of its reinforced concrete provisions. The ACI 318 Code is also widely accepted in Canada and Mexico and has had tremendous influence on the concrete codes of all countries throughout the world.

As more knowledge is obtained pertaining to the behavior of reinforced concrete, the ACI revises its code. The present objective is to make yearly changes in the code in the form of supplements and to provide major revisions of the entire code every 3 years.

Other well-known reinforced concrete specifications are those of the American Association of State Highway and Transportation Officials (AASHTO) and the American Railway Engineering Association (AREA).

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[10] American Concrete Institute, 2014, *Building Code Requirements for Structural Concrete* (ACI 318-14), Farmington Hills, Michigan.

1.8 Summary of 2014 ACI Code Changes

Because this is an introductory textbook on the subject of reinforced concrete design, most readers are seeing the material for the first time. For those readers, this section would not be of much interest. However, for readers who may be familiar with older versions of the ACI 318 Code, this section may be helpful in understanding the changes that have been introduced in the 2014 issue.

Reorganization

ACI 318-14 has been totally reorganized into six categories: General, Systems, Members, Joints and Connections, Toolbox, and Construction. Each category contains several chapters. For example, the Members chapters include separate chapters for One-Way Slabs, Two-Way Slabs, Beams, Columns, and Walls. The Toolbox chapters have provisions that are common to many types of elements including Strength Reduction Factors, Sectional Strength, Strut-and-Tie, Serviceability, and Reinforcement Details. An example of sectional strength is the nominal moment capacity of beam, which is determined in essentially the same way as that of a one-way slab, a two-way slab, a footing, or a wall.

New Chapters

ACI 318-14 has new chapters including Structural Systems (Chapter 4), Joints and Connections (Chapter 16), including beam/column and slab/column joints, connections between members and anchoring to concrete (formerly an appendix to the code). The new Construction chapter (Chapter 26) is written for the designer, not for the contractor, and includes what must be communicated via construction documents to the contractor. A new chapter on diaphragms (Chapter 12) is now included.

Tables

Many new tables have been added to ACI 318-14. Such tables are much clearer than verbiage in earlier codes.

Other Changes

Concrete mix design has been removed from ACI 318-14, and ACI 301 is simply referenced. The designer of a reinforced concrete structure is often not the same person who proportions that concrete mix. The essential requirements of the concrete, such as specified compressive strength and exposure classes, which the designer must specify, remain in the code.

Finite element analysis is now recognized and discussed in some detail. Guidance and limitations on the use of finite elements modeling are now included in the code.

1.9 SI Units and Shaded Areas

Most of this book is devoted to the design of reinforced concrete structures using U.S. customary units. The authors, however, feel that it is absolutely necessary for today's engineer to be able to design in either customary or SI units. Thus, SI equations, which are different from those in customary units, are presented herein, along with quite a few numerical examples using SI units. The equations are taken from the American Concrete Institute's metric version of *Building Code Requirements for Structural Concrete* (ACI 318M-14).[11]

[11] Ibid.

For many people it is rather distracting to read a book in which numbers, equations, and so on are presented in two sets of units. To try to reduce this annoyance, the authors have placed a shaded area around any items pertaining to SI units throughout the text.

If readers are working at a particular time with customary units, they can completely ignore the shaded areas. It is hoped, however, that the same shaded areas will enable a person working with SI units to easily find appropriate equations, examples, and so on.

1.10 Types of Portland Cement

Concretes made with normal portland cement require about 2 weeks to achieve a sufficient strength to permit the removal of forms and the application of moderate loads. Such concretes reach their design strengths after about 28 days and continue to gain strength at a slower rate thereafter.

On many occasions it is desirable to speed up construction by using *high-early-strength cements*, which, although more expensive, enable us to obtain desired strengths in 3 to 7 days rather than the normal 28 days. These cements are particularly useful for the fabrication of precast members, in which the concrete is placed in forms where it quickly gains desired strengths and is then removed from the forms, and the forms are used to produce more members. Obviously, the quicker the desired strength is obtained, the more efficient the operation. A similar case can be made for the forming of concrete buildings floor by floor. High-early-strength cements can also be used advantageously for emergency repairs of concrete and for *shotcreting* (where a mortar or concrete is blown through a hose at a high velocity onto a prepared surface).

There are other special types of portland cements available. The chemical process that occurs during the setting or hardening of concrete produces heat. For very massive concrete structures such as dams, mat foundations, and piers, the heat will dissipate very slowly and can cause serious problems. It will cause the concrete to expand during hydration. When cooling, the concrete will shrink and severe cracking will often occur.

Concrete may be used where it is exposed to various chlorides and/or sulfates. Such situations occur in seawater construction and for structures exposed to various types of soil. Some portland cements are manufactured that have lower heat of hydration, and others are manufactured with greater resistance to attack by chlorides and sulfates.

In the United States, the American Society for Testing and Materials (ASTM) recognizes five types of portland cement. These different cements are manufactured from just about the same raw materials, but their properties are changed by using various blends of those materials. Type I cement is the normal cement used for most construction, but four other types are useful for special situations in which high early strength or low heat or sulfate resistance is needed:

Type I—The common, all-purpose cement used for general construction work.

Type II—A modified cement that has a lower heat of hydration than does Type I cement and that can withstand some exposure to sulfate attack.

Type III—A high-early-strength cement that will produce in the first 24 hours a concrete with a strength about twice that of Type I cement. This cement does have a much higher heat of hydration.

Type IV—A low-heat cement that produces a concrete that generates heat very slowly. It is used for very large concrete structures.

Type V—A cement used for concretes that are to be exposed to high concentrations of sulfate.

Should the desired type of cement not be available, various admixtures may be purchased with which the properties of Type I cement can be modified to produce the desired effect.

Courtesy of Portland Cement Association.

One Peachtree Center in Atlanta, Georgia, is 854 ft high; built for the 1996 Olympics.

1.11 Admixtures

Materials added to concrete during or before mixing are referred to as admixtures. They are used to improve the performance of concrete in certain situations as well as to lower its cost. There is a rather well-known saying regarding admixtures, to the effect that they are to concrete as beauty aids are to the populace. Several of the most common types of admixtures are listed and briefly described below.

- *Air-entraining admixtures*, conforming to the requirements of ASTM C260 and C618, are used primarily to increase concrete's resistance to freezing and thawing and provide better resistance to the deteriorating action of deicing salts. The air-entraining agents cause the mixing water to foam, with the result that billions of closely spaced air bubbles are incorporated into the concrete. When concrete freezes, water moves into the air bubbles, relieving the pressure in the concrete. When the concrete thaws, the water can move out of the bubbles, with the result that there is less cracking than if air entrainment had not been used.
- The addition of *accelerating admixtures*, such as calcium chloride, to concrete will accelerate its early strength development. The results of such additions (particularly useful in cold climates) are reduced times required for curing and protection of the concrete and the earlier removal of forms.

(Section 26.4.1.4.1(c) of the ACI Code states that because of corrosion problems, calcium chloride may not be added to concretes with embedded aluminum, concretes cast against stay-in-place galvanized steel forms, or prestressed concretes.) Other accelerating admixtures that may be used include various soluble salts as well as some other organic compounds.

- *Retarding admixtures* are used to slow the setting of the concrete and to retard temperature increases. They consist of various acids or sugars or sugar derivatives. Some concrete truck drivers keep sacks of sugar on hand to throw into the concrete in case they get caught in traffic jams or are otherwise delayed. Retarding admixtures are particularly useful for large pours where significant temperature increases may occur. They also prolong the plasticity of the concrete, enabling better blending or bonding of successive pours. Retarders can also slow the hydration of cement on exposed concrete surfaces or formed surfaces to produce attractive exposed aggregate finishes.
- *Superplasticizers* are admixtures made from organic sulfonates. Their use enables engineers to reduce the water content in concretes substantially while at the same time increasing their slumps. Although superplasticizers can also be used to keep water–cement ratios constant while using less cement, they are more commonly used to produce workable concretes with considerably higher strengths while using the same amount of cement. (See Section 1.14.) A relatively new product, self-consolidating concrete, uses superplasticizers and modifications in mix designs to produce an extremely workable mix that requires no vibration, even for the most congested placement situations.
- *Waterproofing materials* usually are applied to hardened concrete surfaces, but they may be added to concrete mixes. These admixtures generally consist of some type of soap or petroleum products, as perhaps asphalt emulsions. They may help retard the penetration of water into porous concretes but probably don't help dense, well-cured concretes very much.

1.12 Properties of Concrete

A thorough knowledge of the properties of concrete is necessary for the student before he or she begins to design reinforced concrete structures. An introduction to several of these properties is presented in this section. The designer of a concrete structure is usually not the same person who designs the concrete mix, however. For this reason, a lot of information in ACI 318-11 related to concrete mix design is not included in ACI 318-14, and ACI 301[12] is cited instead.

Compressive Strength

The compressive strength of concrete is determined by testing to failure 28-day-old 6-in. diameter by 12-in. concrete cylinders at a specified rate of loading (4-in. diameter by 8-in. cylinders were first permitted in the 2008 code in lieu of the larger cylinders). For the 28-day period, the cylinders are usually kept under water or in a room with constant temperature and 100% humidity. Although concretes are available with 28-day ultimate strengths from 2500 psi up to as high as 10,000 psi to 20,000 psi, most of the concretes used fall into the 3000-psi to 7000-psi range. For ordinary applications, 3000-psi and 4000-psi concretes are used, whereas for prestressed construction, 5000-psi and 6000-psi strengths are common. For some applications, such as for the columns of the lower stories of high-rise buildings, concretes with strengths up to 9000 psi or 10,000 psi have been used and can be furnished by ready-mix companies. As a result, the use of such high-strength concretes is becoming increasingly common. At Two Union Square in Seattle, concrete with strengths up to 19,000 psi was used.

The values obtained for the compressive strength of concretes, as determined by testing, are to a considerable degree dependent on the sizes and shapes of the test units and the manner in which they

[12] 2010, Specifications for Structural Concrete, Farmington Hills, Michigan, 48331, USA, ISBN 978-0-87031-403-2.

are loaded. In many countries, the test specimens are cubes, 200 mm (7.87 in.) on each side. For the same batches of concrete, the testing of 6-in. by 12-in. cylinders provides compressive strengths only equal to about 80% of the values in psi determined with the cubes.

It is quite feasible to move from 3000-psi concrete to 5000-psi concrete without requiring excessive amounts of labor or cement. The approximate increase in material cost for such a strength increase is 15% to 20%. To move above 5000-psi or 6000-psi concrete, however, it requires very careful mix designs and considerable attention to such details as mixing, placing, and curing. These requirements cause relatively larger increases in cost.

Several comments are made throughout the text regarding the relative economy of using different strength concretes for different applications, such as those for beams, columns, footings, and prestressed members.

To ensure that the compressive strength of concrete in the structure is at least as strong as the specified value, f'_c, the design of the concrete mix must target a higher value, f'_{cr}. Section 4.2.3.3 of ACI 301 requires that the concrete compressive strengths used as a basis for selecting the concrete proportions exceed the specified 28-day strengths by fairly large values. For concrete production facilities that have field strength test records that comply with ACI 301-10 Section 4.2.3.4a, these mix proportions can be submitted for acceptance. For facilities that do not have sufficient records to calculate satisfactory standard deviations, ACI 301 Table 4.2.3.3b provides increases in required average design compressive strength (f'_{cr}) of 1000 psi for specified concrete strength (f'_c) of less than 3000 psi and appreciably higher increases for higher f'_c concretes.

In order to comply with the specified compressive strength, ACI 318-14, Section 26.12.3 requires that (1) the average compressive strength of any three consecutive tests equal or exceed f'_c and (2) no strength test falls below f'_c by more than 500 psi. For values of f'_c exceeding 5000 psi, no strength test shall fall below by more than $0.10 f'_c$.

The stress–strain curves of Figure 1.1 represent the results obtained from compression tests of sets of 28-day-old standard cylinders of varying strengths. You should carefully study these curves because they bring out several significant points:

(a) The curves are roughly straight while the load is increased from zero to about one-third to one-half the concrete's ultimate strength.

(b) Beyond this range the behavior of concrete is nonlinear. This lack of linearity of concrete stress–strain curves at higher stresses causes some problems in the structural analysis of concrete structures because their behavior is also nonlinear at higher stresses.

(c) Of particular importance is the fact that regardless of strengths, all the concretes reach their ultimate strengths at strains of about 0.002.

(d) Concrete does not have a definite yield strength; rather, the curves run smoothly on to the point of rupture at strains of from 0.003 to 0.004. It will be assumed for the purpose of future calculations in this text that concrete fails at 0.003 (ACI 318-14, Section 22.2.2.1). *The reader should note that this value, which is conservative for normal-strength concretes, may not be conservative for higher-strength concretes in the 8000-psi-and-above range*. The European code uses a different value for ultimate compressive strain for columns (0.002) than for beams and eccentrically loaded columns (0.0035).[13]

(e) Many tests have clearly shown that stress–strain curves of concrete cylinders are almost identical to those for the compression sides of beams.

(f) It should be further noticed that the weaker grades of concrete are less brittle than the stronger ones—that is, they will take larger strains before breaking.

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[13] MacGregor, J. G. and Wight, J. K., 2005, *Reinforced Concrete Mechanics and Design*, 4th ed. (Upper Saddle River, NJ: Pearson Prentice Hall), p. 111.

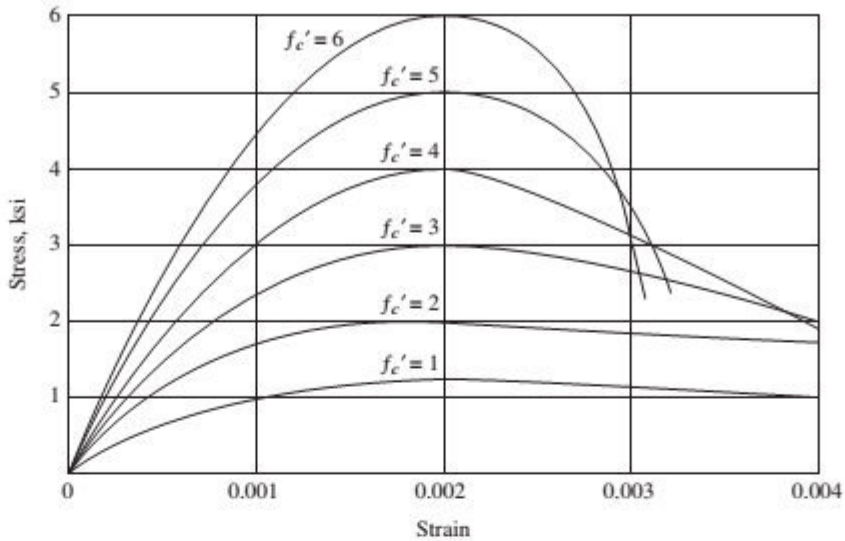

FIGURE 1.1 Typical concrete stress–strain curve, with short-term loading.

Static Modulus of Elasticity

Concrete has no clear-cut modulus of elasticity. Its value varies with different concrete strengths, concrete age, type of loading, and the characteristics and proportions of the cement and aggregates. Furthermore, there are several different definitions of the modulus:

(a) The *initial modulus* is the slope of the stress–strain diagram at the origin of the curve.

(b) The *tangent modulus* is the slope of a tangent to the curve at some point along the curve—for instance, at 50% of the ultimate strength of the concrete.

(c) The slope of a line drawn from the origin to a point on the curve somewhere between 25% and 50% of its ultimate compressive strength is referred to as a *secant modulus*.

(d) Another modulus, called the *apparent modulus* or the *long-term modulus*, is determined by using the stresses and strains obtained after the load has been applied for a certain length of time.

ACI 318-14, Section 19.2.2.1 states that the following expression can be used for calculating the modulus of elasticity of concretes weighing from 90 lb/ft^3 to 160 lb/ft^3:

$$E_c = w_c^{1.5} 33 \sqrt{f_c'}$$

In this expression, E_c is the modulus of elasticity in psi, w_c is the weight of the concrete in pounds per cubic foot, and f_c' is its specified 28-day compressive strength in psi. This is actually a secant modulus with the line (whose slope equals the modulus) drawn from the origin to a point on the stress–strain curve corresponding approximately to the stress ($0.45\, f_c'$) that would occur under the estimated dead and live loads the structure must support.

For normal-weight concrete weighing approximately 145 lb/ft^3, the ACI Code states that the following simplified version of the previous expression may be used to determine the modulus:

$$E_c = 57{,}000 \sqrt{f_c'}$$

Table A.1 (see Appendix A at the end of the book) shows values of E_c for different strength concretes having normal-weight aggregate. These values were calculated with the first of the preceding formulas assuming 145 lb/ft^3 concrete.

In SI units, $E_c = w_c^{1.5}(0.043)\sqrt{f'_c}$ with w_c varying from 1500 to 2600 kg/m^3 and with f'_c in N/mm^2 or MPa (megapascals). Should normal crushed stone or gravel concrete (with a mass of approximately 2320 kg/m^3) be used, $E_c = 4700\sqrt{f'_c}$. Table B.1 of Appendix B of this text provides moduli values for several different strength concretes.

The term *unit weight* is constantly used by structural engineers working with U.S. customary units. When using the SI system, however, this term should be replaced by the term *mass density*. A kilogram is not a force unit and only indicates the amount of matter in an object. The mass of a particular object is the same anywhere on Earth, whereas the weight of an object in our customary units varies depending on altitude because of the change in gravitational acceleration.

Concretes with strength above 6000 psi are referred to as high-strength concretes. Tests have indicated that the usual ACI equations for E_c when applied to high-strength concretes result in values that are too large. Based on studies at Cornell University, the expression to follow has been recommended for normal-weight concretes with f'_c values greater than 6000 psi and up to 12,000 psi and for lightweight concretes with f'_c greater than 6000 psi and up to 9000 psi.[14,15]

$$E_c \text{ (psi)} = \left[40{,}000\sqrt{f'_c} + 10^6\right]\left(\frac{w_c}{145}\right)^{1.5}$$

In SI units with f'_c in MPa and w_c in kg/m^3, the expression is

$$E_c \text{ (MPa)} = \left[3.32\sqrt{f'_c} + 6895\right]\left(\frac{w_c}{2320}\right)^{1.5}$$

Dynamic Modulus of Elasticity

The dynamic modulus of elasticity, which corresponds to very small instantaneous strains, is usually obtained by sonic tests. It is generally 20% to 40% higher than the static modulus and is approximately equal to the initial modulus. When structures are being analyzed for seismic or impact loads, the use of the dynamic modulus seems appropriate.

Poisson's Ratio

As a concrete cylinder is subjected to compressive loads, it not only shortens in length but also expands laterally. The ratio of this lateral expansion to the longitudinal shortening is referred to as *Poisson's ratio*. Its value varies from about 0.11 for the higher-strength concretes to as high as 0.21 for the weaker-grade concretes, with an average value of about 0.16. There does not seem to be any direct relationship between the value of the ratio and the values of items such as the water–cement ratio, amount of curing, aggregate size, and so on.

For most reinforced concrete designs, no consideration is given to the so-called Poisson effect. It may very well have to be considered, however, in the analysis and design of arch dams, tunnels, and some other statically indeterminate structures. Spiral reinforcing in columns takes advantage of Poisson's ratio and will be discussed in Chapter 9.

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[14] Nawy, E. G., 2006, *Prestressed Concrete: A Fundamental Approach*, 5th ed. (Upper Saddle River, NJ: Prentice-Hall), p. 38.

[15] Carrasquillol, R., Nilson, A., and Slate, F., 1981, "Properties of High-Strength Concrete Subject to Short-Term Loads," *Journal of ACI Proceedings*, 78(3), May–June.

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Concert at Naumburg Bandshell in Central Park, New York.

Shrinkage

When the materials for concrete are mixed, the paste consisting of cement and water fills the voids between the aggregate and bonds the aggregate together. This mixture needs to be sufficiently workable or fluid so that it can be made to flow in between the reinforcing bars and all through the forms. To achieve this desired workability, considerably more water (perhaps twice as much) is used than is necessary for the cement and water to react (called *hydration*).

After the concrete has been cured and begins to dry, the extra mixing water that was used begins to work its way out of the concrete to the surface, where it evaporates. As a result, the concrete shrinks and cracks. The resulting cracks may reduce the shear strength of the members and be detrimental to the appearance of the structure. In addition, the cracks may permit the reinforcing to be exposed to the atmosphere or chemicals, such as deicers, thereby increasing the possibility of corrosion. Shrinkage continues for many years, but under ordinary conditions probably about 90% of it occurs during the first year. The amount of moisture that is lost varies with the distance from the surface. Furthermore, the larger the surface area of a member in proportion to its volume, the larger the rate of shrinkage; that is, members with small cross sections shrink more proportionately than do those with large cross sections.

The amount of shrinkage is heavily dependent on the type of exposure. For instance, if concrete is subjected to a considerable amount of wind during curing, its shrinkage will be greater. In a related fashion, a humid atmosphere means less shrinkage, whereas a dry one means more.

It should also be realized that it is desirable to use low-absorptive aggregates such as those from granite and many limestones. When certain absorptive slates and sandstone aggregates are used, the result may be one and a half or even two times the shrinkage with other aggregates.

To minimize shrinkage it is desirable to (1) keep the amount of mixing water to a minimum; (2) cure the concrete well; (3) place the concrete for walls, floors, and other large items in small sections (thus allowing some of the shrinkage to take place before the next section is placed); (4) use construction joints to control the position of cracks; (5) use shrinkage reinforcement; and (6) use appropriate dense and nonporous aggregates.[16]

[16] Leet, K., 1991, *Reinforced Concrete Design*, 2nd ed. (New York: McGraw-Hill), p. 35.

Creep

Under sustained compressive loads, concrete will continue to deform for long periods of time. After the initial deformation occurs, the additional deformation is called *creep*, or *plastic flow*. If a compressive load is applied to a concrete member, an immediate or instantaneous elastic shortening occurs. If the load is left in place for a long time, the member will continue to shorten over a period of several years, and the final deformation will usually be two to three times the initial deformation. You will find in Chapter 6 that this means that long-term deflections may also be as much as two or three times initial deflections. Perhaps 75% of the total creep will occur during the first year.

Should the long-term load be removed, the member will recover most of its elastic strain and a little of its creep strain. If the load is replaced, both the elastic and creep strains will again develop.

The amount of creep is largely dependent on the amount of stress. It is almost directly proportional to stress as long as the sustained stress is not greater than about one-half of f'_c. Beyond this level, creep will increase rapidly.

Long-term loads not only cause creep but also can adversely affect the strength of the concrete. For loads maintained on concentrically loaded specimens for a year or longer, there may be a strength reduction of perhaps 15% to 25%. *Thus a member loaded with a sustained load of, say, 85% of its ultimate compression strength, f'_c, may very well be satisfactory for a while but may fail later.*[17]

Several other items affecting the amount of creep are the following:

- The longer the concrete cures before loads are applied, the smaller will be the creep. Steam curing, which causes quicker strengthening, will also reduce creep.
- Higher-strength concretes have less creep than do lower-strength concretes stressed at the same values. However, applied stresses for higher-strength concretes are, in all probability, higher than those for lower-strength concretes, and this fact tends to increase creep.
- Creep increases with higher temperatures. It is highest when the concrete is at about 150°F to 160°F.
- The higher the humidity, the smaller will be the free pore water that can escape from the concrete. Creep is almost twice as large at 50% humidity than at 100% humidity. It is obviously quite difficult to distinguish between shrinkage and creep.
- Concretes with the highest percentage of cement–water paste have the highest creep because the paste, not the aggregate, does the creeping. This is particularly true if a limestone aggregate is used.
- Obviously, the addition of reinforcing to the compression areas of concrete will greatly reduce creep because steel exhibits very little creep at ordinary stresses. Because creep tends to occur in the concrete, the reinforcing will block it and pick up more and more of the load.
- Large concrete members (i.e., those with large volume-to-surface area ratios) will creep proportionately less than smaller thin members where the free water has smaller distances to travel to escape.

Tensile Strength

The tensile strength of concrete varies from about 8% to 15% of its compressive strength. A major reason for this small strength is the fact that concrete is filled with fine cracks. The cracks have little effect when concrete is subjected to compression loads because the loads cause the cracks to close and permit compression transfer. Obviously, this is not the case for tensile loads.

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[17] Rüsch, H., 1960, "Researches Toward a General Flexure Theory for Structural Concrete," *Journal ACI*, 57(1), pp. 1–28.

Although tensile strength is normally neglected in design calculations, it is nevertheless an important property that affects the sizes and extent of the cracks that occur. Furthermore, the tensile strength of concrete members has a definite reduction effect on their deflections. (Because of the small tensile strength of concrete, little effort has been made to determine its tensile modulus of elasticity. Based on this limited information, however, it seems that its value is equal to its compression modulus.)

You might wonder why concrete is not assumed to resist a portion of the tension in a flexural member and the steel the remainder. The reason is that concrete cracks at such small tensile strains that the low stresses in the steel up to that time would make its use uneconomical. Once tensile cracking has occurred, concrete has no more tensile strength.

The tensile strength of concrete doesn't vary in direct proportion to its ultimate compression strength, f'_c. It does, however, vary approximately in proportion to the square root of f'_c. This strength is quite difficult to measure with direct axial tension loads because of problems in gripping test specimens so as to avoid stress concentrations and because of difficulties in aligning the loads. As a result of these problems, two indirect tests have been developed to measure concrete's tensile strength. These are the *modulus of rupture* and the *split-cylinder tests*.

The tensile strength of concrete in flexure is quite important when considering beam cracks and deflections. For these considerations, the tensile strengths obtained with the modulus of rupture test have long been used. The modulus of rupture (which is defined as the flexural tensile strength of concrete) is usually measured by loading a 6-in. × 6-in. × 30-in. plain (i.e., unreinforced) rectangular beam (with simple supports placed 24 in. on center) to failure with equal concentrated loads at its one-third points as per ASTM C78.[18] The load is increased until failure occurs by cracking on the tensile face of the beam. The modulus of rupture, f_r, is then determined from the flexure formula. In the following expressions, b is the beam width, h is its depth, and M is $PL/6$, which is the maximum computed moment:

$$f_r = \frac{Mc}{I} = \frac{M(h/2)}{\frac{1}{12}bh^3}$$

$$f_r = \text{modulus of rupture} = \frac{6M}{bh^2} = \frac{PL}{bh^2}$$

The stress determined in this manner is not very accurate because, in using the flexure formula, we are assuming that the concrete stresses vary in direct proportion to distances from the neutral axis. This assumption is not very good, and the test method has been withdrawn.

Based on hundreds of tests, the code (Section 19.2.3.1) provides a modulus of rupture f_r equal to $7.5\lambda\sqrt{f'_c}$, where f_r and f'_c are in units of psi.[19] The λ term reduces the modulus of rupture when lightweight aggregates are used (see Section 1.13).

The tensile strength of concrete may also be measured with the split-cylinder test.[20] A cylinder is placed on its side in the testing machine, and a compressive load is applied uniformly along the length of the cylinder, with support supplied along the bottom for the cylinder's full length (see Figure 1.2). The cylinder will split in half from end to end when its tensile strength is reached. The tensile strength at which splitting occurs is referred to as the *split-cylinder strength* and can be calculated with the following expression, in which P is the maximum compressive force, L is the length, and D is the diameter of the cylinder:

$$f_t = \frac{2P}{\pi LD}$$

Even though pads are used under the loads, some local stress concentrations occur during the tests. In addition, some stresses develop at right angles to the tension stresses. As a result, the tensile strengths obtained are not very accurate.

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[18] American Society for Testing and Materials, 2010, *Standard Test Method for Flexural Strength of Concrete (Using Simple Beam with Third-Point Loading)* (ASTM C78/C78M-10e), West Conshohocken, Pennsylvania.

[19] In SI units, $f_r = 0.7\lambda\sqrt{f'_c}$ MPa.

[20] American Society for Testing and Materials, *Standard Test Method*.

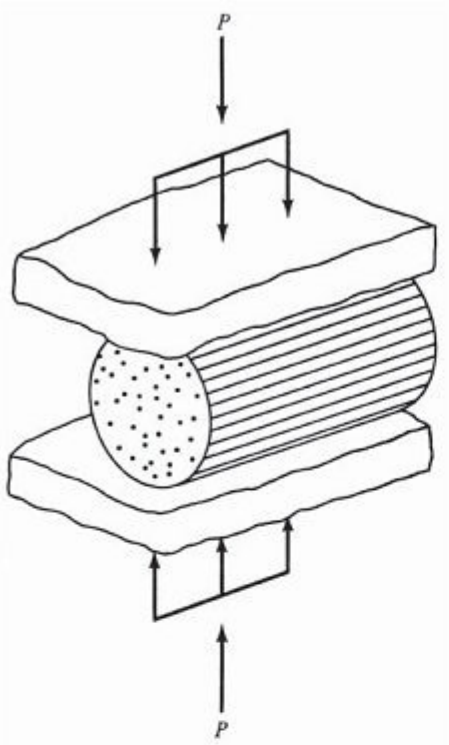


FIGURE 1.2 Split-cylinder test.

Shear Strength

It is extremely difficult in laboratory testing to obtain pure shear failures unaffected by other stresses. As a result, the tests of concrete shearing strengths through the years have yielded values all the way from one-third to four-fifths of the ultimate compressive strengths. You will learn in Chapter 8 that you do not have to worry about these inconsistent shear strength tests because design approaches are based on very conservative assumptions of that strength.

1.13 Aggregates

The aggregates used in concrete occupy about three-fourths of the concrete volume. Since they are less expensive than the cement, it is desirable to use as much of them as possible. Both fine aggregates (usually sand) and coarse aggregates (usually gravel or crushed stone) are used. Any aggregate that passes a No. 4 sieve (which has wires spaced $\frac{1}{4}$ in. on centers in each direction) is said to be fine aggregate. Material of a larger size is coarse aggregate.

The maximum-size aggregates that can be used in reinforced concrete are specified in Section 26.4.2.1(a) of ACI 318-14. *These limiting values are as follows: one-fifth of the narrowest dimensions between the sides of the forms, one-third of the depth of slabs, or three-quarters of the minimum clear spacing between reinforcing*. Larger sizes may be used if, in the judgment of the licensed design professional, the workability of the concrete and its method of consolidation are such that the aggregate used will not cause the development of honeycomb or voids.

Aggregates must be strong, durable, and clean. Should dust or other particles be present, they may interfere with the bond between the cement paste and the aggregate. The strength of the aggregate has an important effect on the strength of the concrete, and the aggregate properties greatly affect the concrete's durability.

Concretes that have 28-day strengths equal to or greater than 2500 psi and air-dry weights equal to or less than 115 lb/ft^3 are said to be *structural lightweight* concretes. The aggregates used for these

concretes are made from expanded shales of volcanic origin, fired clays, or slag. When lightweight aggregates are used for both fine and coarse aggregate, the result is called *all-lightweight* concrete. If sand is used for fine aggregate and if the coarse aggregate is replaced with lightweight aggregate, the result is referred to as *sand-lightweight* concrete. Concretes made with lightweight aggregates may not be as durable or tough as those made with normal-weight aggregates.

Some of the structural properties of concrete are affected by the use of lightweight aggregates. ACI 318-14 Section 19.2.4 requires that the modulus of rupture be reduced by the introduction of the term λ in the equation

$$f_r = 7.5\lambda\sqrt{f'_c} \qquad \text{(ACI Equation 19.2.3.1)}$$

or, in SI units with f'_c in N/mm^2, $f_r = 0.7\lambda\sqrt{f'_c}$

The value of λ depends on the aggregate that is replaced with lightweight material. If only the coarse aggregate is replaced (sand-lightweight concrete), λ is 0.85. If the sand is also replaced with lightweight material (all-lightweight concrete), λ is 0.75. Linear interpolation is permitted between the values of 0.85 and 1.0 as well as from 0.75 to 0.85 when partial replacement with lightweight material is used. Alternatively, if the average splitting tensile strength of lightweight concrete, f_{ct}, is specified, ACI 318-14 Section 19.2.4.3 defines λ as

$$\lambda = \frac{f_{ct}}{6.7\sqrt{f_{cm}}} \leq 1.0,$$

where f_{cm} is the measured average compressive strength of concrete in psi.

For normal-weight concrete and for concrete having normal-weight fine aggregate and a blend of lightweight and normal-weight coarse aggregate, $\lambda = 1.0$. Use of lightweight aggregate concrete can affect beam deflections, shear strength, coefficient of friction, development lengths of reinforcing bars and hooked bars, and prestressed concrete design.

1.14 High-Strength Concretes

Concretes with compression strengths exceeding 6000 psi are referred to as *high-strength concretes*. Another name sometimes given to them is *high-performance concretes* because they have other excellent characteristics besides just high strengths. For instance, the low permeability of such concretes causes them to be quite durable as regards the various physical and chemical agents acting on them that may cause the material to deteriorate.

Up until a few decades ago, structural designers felt that ready-mix companies could not deliver concretes with compressive strengths much higher than 4000 psi or 5000 psi. This situation, however, is no longer the case as these same companies can today deliver concretes with compressive strengths up to at least 9000 psi. Even stronger concretes than these have been used. At Two Union Square in Seattle, 19,000-psi concrete was obtained using ready-mix concrete delivered to the site. Furthermore, concretes have been produced in laboratories with strengths higher than 20,000 psi. Perhaps these latter concretes should be called *super-high-strength concretes* or *super-high-performance concretes*.

If we are going to use a very high-strength cement paste, we must not forget to use a coarse aggregate that is equally strong. If the planned concrete strength is, say, 15,000 psi to 20,000 psi, equally strong aggregate must be used, and such aggregate may very well not be available within reasonable distances. In addition to the strengths needed for the coarse aggregate, their sizes should be well graded, and their surfaces should be rough so that better bonding to the cement paste will be obtained. The rough surfaces of aggregates, however, may decrease the concrete's workability.

From an economical standpoint, you should realize that although concretes with 12,000-psi to 15,000-psi strengths cost approximately three times as much to produce as do 3000-psi concretes, their compressive strengths are four to five times as large.

High-strength concretes are sometimes used for both precast and prestressed members. They are particularly useful in the precast industry where their strength enables us to produce smaller and lighter members, with consequent savings in storage, handling, shipping, and erection costs. In addition, they have sometimes been used for offshore structures, but their common use has been for columns of tall reinforced concrete buildings, probably over 25 to 30 stories in height where the column loads are very large, say, 1000 kips or more. Actually, for such buildings, the columns for the upper floors, where the loads are relatively small, are probably constructed with conventional 4000-psi or 5000-psi concretes, while high-strength concretes are used for the lower heavily loaded columns. If conventional concretes were used for these lower columns, the columns could very well become so large that they would occupy excessive amounts of rentable floor space. High-strength concretes are also of advantage in constructing shear walls. (Shear walls are discussed in Chapter 18.)

To produce concretes with strengths above 6000 psi, it is first necessary to use more stringent quality control of the work and to exercise special care in the selection of the materials to be used. Strength increases can be made by using lower water–cement ratios, adding admixtures, and selecting clean and solid aggregates. The actual concrete strengths used by the designer for a particular job will depend on the size of the loads and the quality of the aggregate available.

In recent years, appreciable improvements have been made in the placing, vibrating, and finishing of concrete. These improvements have resulted in lower water–cement ratios and thus higher strengths. The most important factor affecting the strength of concrete is its porosity, which is controlled primarily by the water–cement ratio. This ratio should be kept as small as possible as long as adequate workability is maintained. In this regard, there are various water-reducing admixtures with which the ratios can be appreciably reduced, while at the same time maintaining suitable workability.

Concretes with strengths from 6000 psi to 10,000 psi or 12,000 psi can easily be obtained if admixtures such as silica fume and superplasticizers are used. Silica fume, which is more than 90% silicon dioxide, is an extraordinarily fine powder that varies in color from light to dark gray and can even be blue-green-gray. It is obtained from electric arc furnaces as a by-product during the production of metallic silicon and various other silicon alloys. It is available in both powder and liquid form. The amount of silica fume used in a mix varies from 5% to 30% of the weight of the cement.

Silica fume particles have diameters approximately 100 times smaller than the average cement particle, and their surface areas per unit weight are roughly 40 to 60 times larger than those of portland cement. As a result, they hold more water. (By the way, this increase of surface area causes the generation of more heat of hydration.) The water–cement ratios are smaller, and strengths are higher. Silica fume is a *pozzolan*: a siliceous material that by itself has no cementing quality, but when used in concrete mixes its extraordinarily fine particles react with the calcium hydroxide in the cement to produce a cementious compound. Quite a few pozzolans are available that can be used satisfactorily in concrete. Two of the most common ones are fly ash and silica fume. Here, only silica fume is discussed.

When silica fume is used, it causes increases in the density and strength of the concrete. These improvements are due to the fact that the ultrafine silica fume particles are dispersed between the cement particles. Unfortunately, this causes a reduction in the workability of the concrete, and it is necessary to add *superplasticizers* to the mix. Superplasticizers, also called *high-range water reducers*, are added to concretes to increase their workability. They are made by treating formaldehyde or napthalene with sulfuric acid. Such materials used as admixtures lower the viscosity or resistance to flow of the concrete. As a result, less water can be used, thus yielding lower water–cement ratios and higher strengths.

Organic polymers can be used to replace a part of the cement as the binder. An organic polymer is composed of molecules that have been formed by the union of thousands of molecules. The most commonly used polymers in concrete are latexes. Such additives improve concrete's strength, durability, and adhesion. In addition, the resulting concretes have excellent resistance to abrasion, freezing, thawing, and impact.

Another procedure that can increase the strength of concrete is *consolidation*. When precast concrete products are consolidated, excess water and air are squeezed out, thus producing concretes with optimum air contents. In a similar manner, the centrifugal forces caused by the spinning of concrete pipes during their manufacture consolidate the concrete and reduce the water and air contents. Not much work has been done in the consolidation area for cast-in-place concrete because of the difficulty of applying the squeezing forces. To squeeze such concretes, it is necessary to apply pressure to the forms. One major difficulty in doing this is that very special care must be used to prevent distortion of the wet concrete members.

1.15 Fiber-Reinforced Concretes

In recent years, a great deal of interest has been shown in fiber-reinforced concrete, and today there is much ongoing research on the subject. The fibers used are made from steel, plastics, glass, and other materials. Various experiments have shown that the addition of such fibers in convenient quantities (normally up to about 1% or 2% by volume) to conventional concretes can appreciably improve their characteristics.

The compressive strengths of fiber-reinforced concretes are not significantly greater than they would be if the same mixes were used without the fibers. The resulting concretes, however, are substantially tougher and have greater resistance to cracking and higher impact resistance. The use of fibers has increased the versatility of concrete by reducing its brittleness. The reader should note that a reinforcing bar provides reinforcing only in the direction of the bar, while randomly distributed fibers provide additional strength in all directions.

Steel is the most commonly used material for the fibers. The resulting concretes seem to be quite durable, at least as long as the fibers are covered and protected by the cement mortar. Concretes reinforced with steel fibers are most often used in pavements, thin shells, and precast products as well as in various patches and overlays. Glass fibers are more often used for spray-on applications as in shotcrete. It is necessary to realize that ordinary glass will deteriorate when in contact with cement paste. As a result, using alkali-resistant glass fibers is necessary.

The fibers used vary in length from about 0.25 in. up to about 3 in. while their diameters run from approximately 0.01 in. to 0.03 in. For improving the bond with the cement paste, the fibers can be hooked or crimped. In addition, the surface characteristics of the fibers can be chemically modified in order to increase bonding.

The improvement obtained in the toughness of the concrete (the total energy absorbed in breaking a member in flexure) by adding fibers is dependent on the fibers' *aspect ratio* (length/diameter). Typically, the aspect ratios used vary from about 25 up to as much as 150, with 100 being the average value. Other factors affecting toughness are the shape and texture of the fibers. ASTM C1018[21] is the test method for determining the toughness of fiber-reinforced concrete using the third-point beam-loading method described earlier.

When a crack opens up in a fiber-reinforced concrete member, the few fibers bridging the crack do not appreciably increase the strength of the concrete. They will, however, provide resistance to the opening up of the crack because of the considerable work that would be necessary to pull them out. As a result, the ductility and toughness of the concrete is increased. The use of fibers has been shown to increase the fatigue life of beams and lessen the widths of cracks when members are subject to fatigue loadings.

The use of fibers does significantly increase costs. It is probably for this reason that fiber-reinforced concretes have been used for overlays for highway pavements and airport runways rather than for whole concrete projects. Actually in the long run, if the increased service lives of

[21] American Society for Testing and Materials, 1997, *Standard Test Method for Flexural Toughness and First-Crack Strength of Fiber-Reinforced Concrete (Using Simple Beam with Third-Point Loading)* (ASTM C1018-1997), West Conshohocken, Pennsylvania.

fiber-reinforced concretes are considered, they may very well prove to be quite cost effective. For instance, many residential contractors use fiber-reinforced concrete to construct driveways instead of regular reinforced concrete.

Some people have the feeling that the addition of fibers to concrete reduces its slump and workability as well as its strength. Apparently, they feel this way because the concrete looks stiffer to them. Actually, the fibers do not reduce the slump unless the quantity is too great—that is, much above about one pound per cubic yard. The fibers only appear to cause a reduction in workability, but as a result concrete finishers will often add more water so that water–cement ratios are increased and strengths decreased. ASTM C1018 uses the third-point beam-loading method described earlier to measure the toughness and first-crack strength of fiber-reinforced concrete.

1.16 Concrete Durability

The exposure of concrete to such conditions as severe freezing and thawing, sulfate, water contact, or a corrosive environment increases the durability demands. ACI 318-14, Table 19.3.1.1 defines classes of exposure for various exposure conditions. Table 19.3.2.1 then imposes requirements for each exposure class. Examples of limitations are water–cement ratio, minimum f'_c, air content, cementitious materials, use of calcium chloride admixture, and maximum water-soluble chloride ion. The compressive strength of concrete may be dictated by exposure to freeze-thaw conditions or chemicals such as deicers or sulfates. These conditions may require a greater compressive strength or lower water–cement ratio than those required to carry the calculated loads. For concrete exposed to deicing chemicals, the amount of fly ash or other pozzolans is limited. Finally, the water–cement ratio is limited by exposure to sulfates as well. The designer is required to determine whether structural load-carrying requirements or durability requirements are more stringent and to specify the more restrictive requirements for f'_c, water–cement ratio, and air content.

1.17 Reinforcing Steel

The reinforcing used for concrete structures may be in the form of bars or welded wire fabric. Reinforcing bars are referred to as *plain* or *deformed*. The deformed bars, which have ribbed projections rolled onto their surfaces (patterns differing with different manufacturers) to provide better bonding between the concrete and the steel, are used for almost all applications. Instead of rolled-on deformations, deformed wire has indentations pressed into it. Plain bars are not used very often except for wrapping around longitudinal bars, primarily in columns.

Plain round bars are indicated by their diameters in fractions of an inch as $\frac{3 \text{ in.}}{8}\phi$, $\frac{1 \text{ in.}}{2}\phi$, and $\frac{5 \text{ in.}}{8}\phi$. Deformed bars are round and vary in sizes from #3 to #11, with two very large sizes, #14 and #18, also available. For bars up to and including #8, the number of the bar coincides with the bar diameter in eighths of an inch. For example, a #7 bar has a diameter of $\frac{7}{8}$ in. and a cross-sectional area of 0.60 in.^{2} (which is the area of a circle with a $\frac{7}{8}$-in. diameter). Bars were formerly manufactured in both round and square cross sections, but today all bars are round.

The #9, #10, and #11 bars have diameters that provide areas equal to the areas of the old 1-in. × 1-in. square bars, $1\frac{1}{8}$-in. × $1\frac{1}{8}$-in. square bars, and $1\frac{1}{4}$-in. × $1\frac{1}{4}$-in. square bars, respectively. Similarly, the #14 and #18 bars correspond to the old $1\frac{1}{2}$-in. × $1\frac{1}{2}$-in. square bars and 2-in. × 2-in. square bars, respectively. Table A.2 (see Appendix A) provides details about areas, diameters, and weights of reinforcing bars. Although #14 and #18 bars are shown in this table, the designer should check his or her suppliers to see if they have these very large sizes in stock. Reinforcing bars may be purchased in lengths up to 6 ft. Longer bars have to be specially ordered. In general, longer bars are too flexible and difficult to handle.

Courtesy of EFCO Corp.

A 10M gallon water reservoir required 144 30′ columns to support its roof in Oxford, United Kingdom.

Welded wire fabric is also frequently used for reinforcing slabs, pavements, and shells, and places where there is normally not sufficient room for providing the necessary concrete cover required for regular reinforcing bars. The mesh is made of cold-drawn wires running in both directions and welded together at the points of intersection. The size and spacing of the wires may be the same in both directions or may be different, depending on design requirements. Wire mesh is easily placed and has excellent bond with the concrete, and the spacing of the wires is well controlled.

Table A.3(A) in Appendix A provides information concerning certain styles of welded wire fabric that have been recommended by the Wire Reinforcement Institute as common stock styles (normally carried in stock at the mills or at warehousing points and, thus, usually immediately available). Table A.3(B) provides detailed information about diameters, areas, weights, and spacing of quite a few wire sizes normally used to manufacture welded wire fabric. Smooth and deformed wire fabric is made from wires whose diameters range from 0.134 in. to 0.628 in. for plain wire and from 0.225 in. to 0.628 in. for deformed wires.

Smooth wire is denoted by the letter W followed by a number that equals the cross-sectional area of the wire in hundredths of a square inch. Deformed wire is denoted by the letter D followed by a number giving the area. For instance, a D4 wire is a deformed wire with a cross-sectional area equal to 0.04 in.^{2} Smooth wire fabric is actually included within the ACI Code's definition of deformed reinforcement because of its mechanical bonding to the concrete caused by the wire intersections. Wire fabric that actually has deformations on the wire surfaces bonds even more to the concrete because of the deformations as well as the wire intersections. According to the code, deformed wire is not permitted to be larger than D31 or smaller than D4.

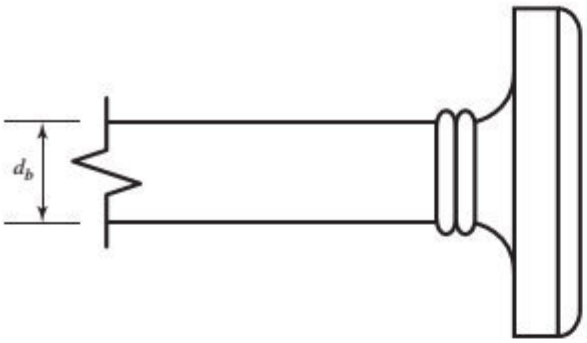


FIGURE 1.3 Headed deformed reinforcing bar.

Requirements for Headed Steel Bars for Concrete Reinforcement (ASTM A970/970M) are included in ACI 318 Section 25.4.4. Headed bars can be used instead of straight or hooked bars, with considerably less congestion in crowded areas such as beam–column intersections. The specification covers plain and deformed bars cut to lengths and having heads either forged or welded to one or both ends. Alternatively, heads may be connected to the bars by internal threads in the head mating to threads on the bar end or by a separate threaded nut to secure the head to the bar. Heads are forge formed, machined from bar stock, or cut from plate. Figure 1.3 illustrates a headed bar detail. The International Code Council has published acceptance criteria for headed ends of concrete reinforcement (ACC 347).

1.18 Grades of Reinforcing Steel

Reinforcing bars may be rolled from billet steel, axle steel, or rail steel. Only occasionally, however, are they rolled from old train rails or locomotive axles. These latter steels have been cold-worked for many years and are not as ductile as the billet steels.

There are several types of reinforcing bars, designated by the ASTM, which are listed after this paragraph. These steels are available in different grades as Grade 50, Grade 60, and so on, where Grade 50 means the steel has a specified yield point of 50,000 psi, Grade 60 means 60,000 psi, and so on.

- ASTM A615: Deformed and plain billet steel bars. These bars, which must be marked with the letter S (for type of steel), are the most widely used reinforcing bars in the United States. Bars are of four minimum yield strength levels: 40,000 psi (280 MPa); 60,000 psi (420 MPa); 75,000 psi (520 MPa); and 80,000 psi (550 MPa). Although the term "billet" appears in the title of the ASTM Specification, the word "carbon" is used in the document.
- ASTM A706: Low-alloy deformed and plain bars. These bars, which must be marked with the letter W (for type of steel), are to be used where controlled tensile properties and/or specially controlled chemical composition is required for welding purposes. They are available in two grades: 60,000 psi (420 MPa) and 80,000 psi (550 MPa), designated as Grade 60 (420) and Grade 80 (550), respectively.
- ASTM A996: Deformed rail steel or axle steel bars. They must be marked with the letter R (for type of steel).
- When deformed bars are produced to meet both the A615 and A706 specifications, they must be marked with both the letters S and W.
- ASTM A1035: Deformed and plain, low-carbon, chromium steel bars for concrete reinforcement. These bars are available in both Grades 100 and 120 (100,000 psi and 120,000 psi). Use of this high-strength reinforcing steel can result in using less steel, and the corrosion resistance is superior to most other steels. Larger development and lap splice lengths are a drawback, however.

Designers in almost all parts of the United States will probably never encounter rail or axle steel bars (A996) because they are available in such limited areas of the country. Of the 23 U.S. manufacturers of reinforcing bars listed by the Concrete Reinforcing Steel Institute,[22] only five manufacture rail steel bars, and not one manufactures axle bars.

Almost all reinforcing bars conform to the A615 specification, and a large proportion of the material used to make them is not new steel but is melted reclaimed steel, such as that from old car bodies. Bars conforming to the A706 specification are intended for certain uses when welding and/or bending are of particular importance. Bars conforming to this specification may not always be available from local suppliers.

There is only a small difference between the prices of reinforcing steel with yield strengths of 40 ksi and 60 ksi. As a result, the 60-ksi bars are the most commonly used in reinforced concrete design.

ACI 318 Section 20.2.1.2 allows two methods of determining the specified yield strength of non-prestressed bars and wires: (a) The offset method, using an offset of 0.2%, and (b) The yield point by the halt-of-force method, provided the non-prestressed bar or wire has a sharp-kneed or well-defined type of yield point. ACI 318 Section 20.2.2.4 has established an upper limit of 100 ksi on yield strengths permitted for design calculations for reinforced concrete. ACI 318, Table 20.2.2.4a provides limits on design yield strengths for different usages and applications. This table is a great improvement on the previous ACI 318 Code that provided similar information in text rather than in tabular form.

There has been gradually increasing demand through the years for Grade 75 and Grade 100 steel, particularly for use in high-rise buildings, where it is used in combination with high-strength concretes. The results are smaller columns, more rentable floor space, and smaller foundations for the resulting lighter buildings.

Grade 75 and Grade 100 steel are appreciably higher in cost, and the #14 and #18 bars are often unavailable from stock and will probably have to be specially ordered from the steel mills. This means that there may have to be a special rolling to supply the steel. As a result, its use may not be economically justified unless at least 50 or 60 tons are ordered.

Yield stresses above 60 ksi are also available in welded wire fabric, but the specified stresses must correspond to one of the two methods prescribed in ACI 318 Section 20.2.1.2. Smooth fabric must conform to ASTM A185, whereas deformed fabric cannot be smaller than size D4 and must conform to ASTM A496.

The modulus of elasticity for non-prestressed steels is considered to be equal to 29×10^6 psi. For prestressed steels, it varies somewhat from manufacturer to manufacturer, with a value of 27×10^6 psi being fairly common.

Stainless steel reinforcing (ASTM A955) was introduced in the 2008 code. It is highly resistant to corrosion, especially pitting and crevice corrosion from exposure to chloride-containing solutions such as deicing salts. While it is more expensive than normal carbon steel reinforcement, its life-cycle cost may be less when the costs of maintenance and repairs are considered.

1.19 SI Bar Sizes and Material Strengths

The metric version of the ACI Code 318M-14 makes use of the same reinforcing bars used for designs using U.S. customary units. The metric bar dimensions are merely soft conversions (i.e., almost equivalent) of the customary sizes. The SI concrete strengths (f'_c) and the minimum steel

[22] Concrete Reinforcing Steel Institute, 2001, *Manual of Standard Practice*, 28th ed., Chicago. Appendix A, pp. A-1 to A-5.

yield strengths (f_y) are converted from the customary values into metric units and rounded off a bit. A brief summary of metric bar sizes and material strengths is presented in the following paragraphs. These values are used for the SI examples and homework problems throughout the text.

1. The bar sizes used in the metric version of the code correspond to U.S. sizes #3 through #18 bars. They are numbered 10, 13, 16, 19, 22, 25, 29, 32, 36, 43, and 57. These numbers represent the U.S. customary bar diameters rounded to the nearest millimeter (mm). For instance, the metric #10 bar has a diameter equal to 9.5 mm, the metric #13 bar has a diameter equal to 12.7 mm, and so on. Detailed information concerning metric reinforcing bar diameters, cross-sectional areas, masses, and ASTM classifications is provided in Appendix B, Tables B.2 and B.3.
2. The steel reinforcing grades, or minimum steel yield strengths, referred to in the code are 300, 350, 420, and 520 MPa. These correspond, respectively, to 43,511, 50,763, 60,916, and 75,420 psi and, thus, correspond approximately to Grade 40, 50, 60, and 75 bars. Appendix B, Table B.3 provides ASTM numbers, steel grades, and bar sizes available in each grade.
3. The concrete strengths in metric units referred to in the code are 17, 21, 24, 28, 35, and 42 MPa. These correspond, respectively, to 2466, 3046, 3481, 4061, 5076, and 6092 psi, that is, to 2500-, 3000-, 3500-, 4000-, 5000-, and 6000-psi concretes.

In 1997, the producers of steel reinforcing bars in the United States began to produce soft metric bars. These are the same bars we have long called standard inch-pound bars, but they are marked with metric units. Today, the large proportion of metric bars manufactured in the United States are soft metric. By producing the exact same bars, the industry does not have to keep two different inventories (one set of inch-pound bar sizes and another set of different bar sizes in metric units). Table 1.1 shows the bar sizes given in both sets of units.

TABLE 1.1 Reinforcement Bar Sizes and Areas

Standard Inch-Pound Bars			Soft Metric Bars		
Bar No.	Diameter (in.)	Area (in.^{2})	Bar No.	Diameter (mm)	Area (mm^2)
3	0.375	0.11	10	9.5	71
4	0.500	0.20	13	12.7	129
5	0.625	0.31	16	15.9	199
6	0.750	0.44	19	19.1	284
7	0.875	0.60	22	22.2	387
8	1.000	0.79	25	25.4	510
9	1.128	1.00	29	28.7	645
10	1.270	1.27	32	32.3	819
11	1.410	1.56	36	35.8	1006
14	1.693	2.25	43	43.0	1452
18	2.257	4.00	57	57.3	2581

1.20 Corrosive Environments

When reinforced concrete is exposed to deicing salts, seawater, or spray from these substances, it is necessary to provide special corrosion protection for the reinforcing. The structures usually involved are bridge decks, parking garages, wastewater treatment plants, and various coastal structures. We must also consider structures exposed to occasional chemical spills that involve chlorides.

Should the reinforcement be insufficiently protected, it will corrode; as it corrodes, the resulting oxides occupy a volume far greater than that of the original metal. The results are large outward pressures that can lead to severe cracking and spalling of the concrete. This reduces the concrete protection, or *cover*, for the steel, and corrosion accelerates. Also, the *bond*, or sticking of the concrete to the steel, is reduced. The result of all of these factors is a decided reduction in the life of the structure.

ACI 318-14 Section 5.1.4 requires that for corrosive environments, more concrete cover must be provided for the reinforcing; it also requires that special concrete proportions or mixes be used.

The lives of such structures can be greatly increased if *epoxy-coated reinforcing* bars are used. Such bars need to be handled very carefully so as not to break off any of the coating. Furthermore, they do not bond as well to the concrete, and their embedment lengths will have to be increased somewhat for that reason, as you will learn in Chapter 7. A new type of bar coating, a dual coating of a zinc alloy and an epoxy coating, was introduced in the 2011 ACI 318 Code: ASTM A1055. Use of stainless steel reinforcing, as described in Section 1.15, can also significantly increase the service life of structures exposed to corrosive environments.

1.21 Identifying Marks on Reinforcing Bars

It is essential for people in the shop and the field to be able to identify at a glance the sizes and grades of reinforcing bars. If they are not able to do this, smaller and lower-grade bars other than those intended by the designer may be used. To prevent such mistakes, deformed bars have rolled-in identification markings on their surfaces. These markings are described in the following list and are illustrated in Figure 1.4.

1. The producing company is identified with a letter.
2. The bar size number (3 to 18) is given next.
3. Another letter is shown to identify the type of steel (S for carbon, R in addition to a rail sign for rail steel, A for axle, and W for low alloy).
4. Finally, the grade of the bars is shown either with numbers or with continuous lines. A Grade 60 bar has either the number 60 on it or a continuous longitudinal line in addition to its main ribs. A Grade 75 bar will have the number 75 on it or two continuous lines in addition to the main ribs.

1.22 Introduction to Loads

Perhaps the most important and most difficult task faced by the structural designer is the accurate estimation of the loads that may be applied to a structure during its life. No loads that may reasonably be expected to occur may be overlooked. After loads are estimated, the next problem is to decide the worst possible combinations of these loads that might occur at one time. For instance, would a highway bridge completely covered with ice and snow be simultaneously subjected to fast-moving lines of heavily loaded trailer trucks in every lane and to a 90-mile lateral wind, or is some lesser combination of these loads more reasonable?

The next few sections of this chapter provide a brief introduction to the types of loads with which the structural designer must be familiar. The purpose of these sections is not to discuss loads in great detail but rather to give the reader a feel for the subject. As will be seen, loads are classed as being dead, live, or environmental.

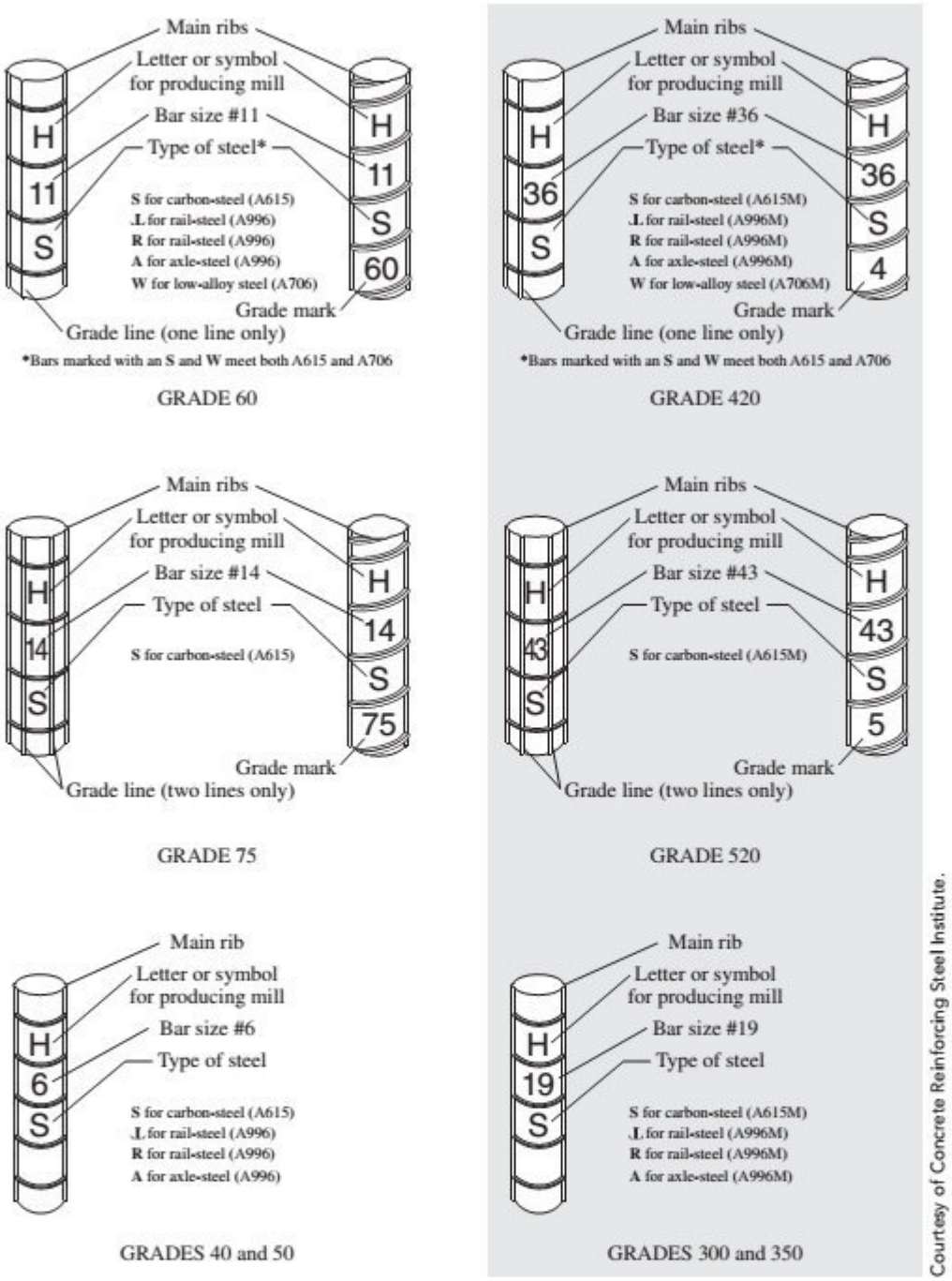


FIGURE 1.4 Identification marks for ASTM standard bars.

1.23 Dead Loads

Dead loads are loads of constant magnitude that remain in one position. They include the weight of the structure under consideration as well as any fixtures that are permanently attached to it. For a reinforced concrete building, some dead loads are the frames, walls, floors, ceilings, stairways, roofs, and plumbing.

To design a structure, it is necessary for the weights or dead loads of the various parts to be estimated for use in the analysis. The exact sizes and weights of the parts are not known until the structural

TABLE 1.2 Weights of Some Common Building Materials

Reinforced concrete (12 in.)	150 psf	2 × 12 @ 16-in. double wood floor	7 psf
Acoustical ceiling tile	1 psf	Linoleum or asphalt tile	1 psf
Suspended ceiling	2 psf	Hardwood flooring ($\frac{7}{8}$ in.)	4 psf
Plaster on concrete	5 psf	1-in. cement on stone-concrete fill	32 psf
Asphalt shingles	2 psf	Movable steel partitions	4 psf
3-ply ready roofing	1 psf	Wood studs with $\frac{1}{2}$-in. gypsum	8 psf
Mechanical duct allowance	4 psf	Clay brick wythes (4 in.)	39 psf

analysis is made and the members of the structure are selected. The weights that are determined from the actual design must be compared with the estimated weights. If large discrepancies are present, it will be necessary to repeat the analysis and design using better estimated weights.

Reasonable estimates of structure weights may be obtained by referring to similar structures or to various formulas and tables available in most civil engineering handbooks. An experienced designer can estimate very closely the weights of most structures and will spend little time repeating designs because of poor estimates.

The approximate weights of some common materials used for floors, walls, roofs, and the like are given in Table 1.2.

1.24 Live Loads

Live loads are loads that can change in magnitude and position. They include occupancy loads, warehouse materials, construction loads, overhead service cranes, equipment operating loads, and many others. In general, they are induced by gravity.

Some typical floor live loads that act on building structures are presented in Table 1.3. These loads, which are taken from Table 4-1 in ASCE 7-10,[23] act downward and are distributed uniformly over an entire floor. By contrast, roof live loads are 20 psf (pounds per square feet) maximum distributed uniformly over the entire roof.

Among the many other types of live loads are the following:

Traffic loads for bridges—Bridges are subjected to a series of concentrated loads of varying magnitudes caused by groups of truck or train wheels.

Impact loads—Impact loads are caused by the vibration of moving or movable loads. It is obvious that a crate dropped on the floor of a warehouse or a truck bouncing on uneven pavement of a bridge causes greater forces than that would result if the loads were applied gently and gradually. Impact loads are equal to the difference between the magnitude of the loads actually caused and the magnitude of the loads had they been dead loads.

Longitudinal loads—Longitudinal loads also need to be considered in designing some structures. Stopping a train on a railroad bridge or a truck on a highway bridge causes longitudinal forces to be applied. It is not difficult to imagine the tremendous longitudinal force developed when the driver of a 40-ton trailer truck traveling at 60 mph suddenly has to apply the brakes while crossing a highway bridge. There are other longitudinal load situations, such as ships running into docks and the movement of traveling cranes that are supported by building frames.

[23] American Society of Civil Engineers, 2010, *Minimum Design Loads for Buildings and Other Structures*, ASCE 7-10 (Reston, VA: American Society of Civil Engineers), pp. 17–19.

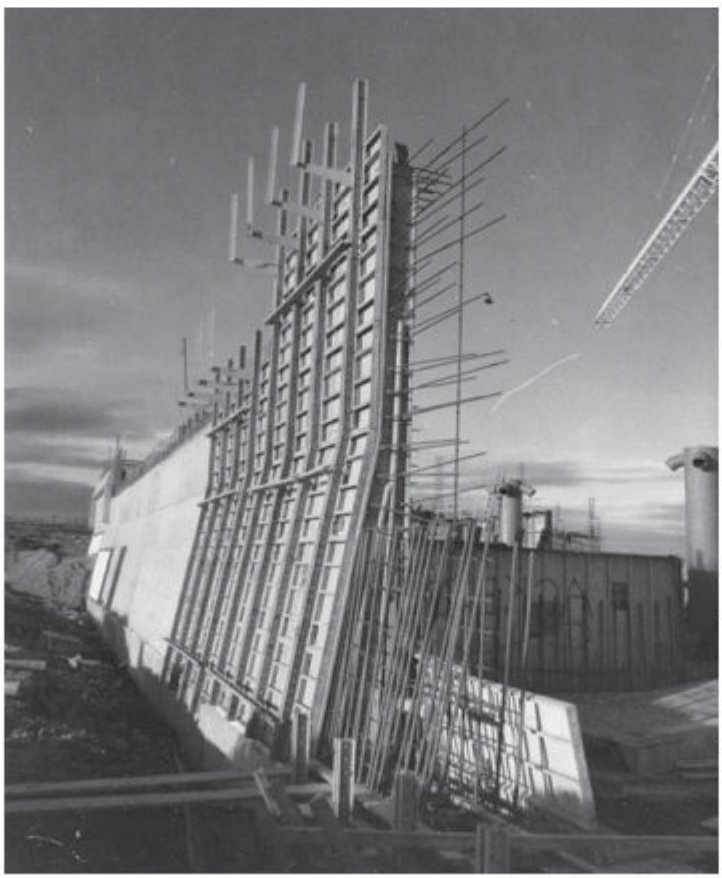
TABLE 1.3 Some Typical Uniformly Distributed Live Loads

Lobbies of assembly areas	100 psf	Classrooms in schools	40 psf
Dance hall and ballrooms	100 psf	Upper-floor corridors in schools	80 psf
Library reading rooms	60 psf	Stairs and exitways	100 psf
Library stack rooms	150 psf	Heavy storage warehouse	250 psf
Light manufacturing	125 psf	Retail stores—first floor	100 psf
Offices in office buildings	50 psf	Retail stores—upper floors	75 psf
Residential dwelling areas	40 psf	Walkways and elevated platforms	60 psf

psf = pounds per square foot

Miscellaneous loads—Among the other types of live loads with which the structural designer will have to contend are *soil pressures* (such as the exertion of lateral earth pressures on walls or upward pressures on foundations), *hydrostatic pressures* (such as water pressure on dams, inertia forces of large bodies of water during earthquakes, and uplift pressures on tanks and basement structures), *blast loads* (caused by explosions, sonic booms, and military weapons), and *centrifugal forces* (such as those caused on curved bridges by trucks and trains or similar effects on roller coasters).

Live load reductions are permitted, according to Section 4.8 of ASCE 7, because is it unlikely that the entire structure will be subjected to its full design live load over its entire floor area all at one time. This reduction can significantly reduce the total design live load on a structure, resulting in much lower column loads at lower floors and footing loads.



Courtesy of Aluma Systems/Brand Energy & Infrastructure Services.

Sewage treatment plant, Redwood City, California.

1.25 Environmental Loads

Environmental loads are loads caused by the environment in which the structure is located. For buildings, they are caused by rain, snow, wind, temperature change, and earthquake. Strictly speaking, these are also live loads, but they are the result of the environment in which the structure is located. Although they do vary with time, they are not all caused by gravity or operating conditions, as is typical with other live loads. In the next few paragraphs, a few comments are made about the various kinds of environmental loads.

1. *Snow and ice.* In the colder states, snow and ice loads are often quite important. One inch of snow is equivalent to approximately 0.5 psf, but it may be higher at lower elevations where snow is denser. For roof designs, snow loads of from 10 psf to 40 psf are used, the magnitude depending primarily on the slope of the roof and to a lesser degree on the character of the roof surface. The larger values are used for flat roofs, the smaller ones for sloped roofs. Snow tends to slide off sloped roofs, particularly those with metal or slate surfaces. A load of approximately 10 psf might be used for 45° slopes, and a 40-psf load might be used for flat roofs. Studies of snowfall records in areas with severe winters may indicate the occurrence of snow loads much greater than 40 psf, with values as high as 80 psf in northern Maine.

Snow is a variable load, which may cover an entire roof or only part of it. There may be drifts against walls or buildup in valleys or between parapets. Snow may slide off one roof and onto a lower one. The wind may blow it off one side of a sloping roof, or the snow may crust over and remain in position even during very heavy winds. The snow loads that are applied to a structure are dependent upon many factors, including geographic location, the pitch of the roof, sheltering, and the shape of the roof.

2. *Rain.* Although snow loads are a more severe problem than rain loads for the usual roof, the situation may be reversed for flat roofs—particularly those in warmer climates. If water on a flat roof accumulates faster than it runs off, the result is called *ponding* because the increased load causes the roof to deflect into a dish shape that can hold more water, which causes greater deflections, and so on. This process continues until equilibrium is reached or until collapse occurs. Ponding is a serious matter, as illustrated by the large number of flat-roof failures that occur as a result of ponding every year in the United States. It has been claimed that almost 50% of the lawsuits faced by building designers are concerned with roofing systems.[24] Ponding is one of the common subjects of such litigation.

3. *Wind.* A survey of engineering literature for the past 150 years reveals many references to structural failures caused by wind. Perhaps the most infamous of these have been bridge failures such as those of the Tay Bridge in Scotland in 1879 (which caused the death of 75 persons) and the Tacoma Narrows Bridge (Tacoma, Washington) in 1940. There have also been some disastrous building failures from wind during the same period, such as that of the Union Carbide Building in Toronto in 1958. It is important to realize that a large percentage of building failures from wind have occurred during the buildings' erection.[25]

A great deal of research has been conducted in recent years on the subject of wind loads. Nevertheless, more study is needed because the estimation of wind forces can by no means be classified as an exact science. The magnitude and duration of wind loads vary with geographical locations, the heights of structures aboveground, the types of terrain around the structures, the proximity of other buildings, the location within the structure, and the character of the wind itself.

Chapters 26 to 31 of the ASCE 7-10 specification provide a rather lengthy procedure for estimating the wind pressures applied to buildings. The procedure involves several factors with which we attempt to account for the terrain around the building, the importance of the building regarding human

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[24] Van Ryzin, Gary, 1980, "Roof Design: Avoid Ponding by Sloping to Drain," *Civil Engineering* (January), pp. 77–81.

[25] Task Committee on Wind Forces, Committee on Loads and Stresses, Structural Division, ASCE, 1961, "Wind Forces on Structures," Final Report, *Transactions ASCE* 126, Part II, pp. 1124–1125.

life and welfare, and of course the wind speed at the building site. Although use of the equations is rather complex, the work can be greatly simplified with the tables presented in the specification. The reader is cautioned, however, that the tables presented are for buildings of regular shapes. If a building having an irregular or unusual geometry is being considered, wind tunnel studies may be necessary.

The basic form of the equation presented in the specification is

$$p = qCG$$

In this equation, p is the estimated wind load (in psf) acting on the structure. This wind load will vary with height above the ground and with the location on the structure. The quantity, q, is the reference velocity pressure. It varies with height and with exposure to the wind. The aerodynamic shape factor, C, is dependent upon the shape and orientation of the building with respect to the direction from which the wind is blowing. Last, the gust response factor, G, is dependent upon the nature of the wind and the location of the building. Other considerations in determining design wind pressure include importance factor and surface roughness.

4. *Seismic loads.* Many areas of the world are in earthquake territory, and in those areas, it is necessary to consider seismic forces in design for all types of structures. Through the centuries, there have been catastrophic failures of buildings, bridges, and other structures during earthquakes. It has been estimated that as many as 50,000 people lost their lives in the 1988 earthquake in Armenia.[26] The 1989 Loma Prieta and 1994 Northridge earthquakes in California caused many billions of dollars of property damage as well as considerable loss of life. The 2008 earthquake in Sichuan Province, China, caused 69,000 fatalities and another 18,000 missing.

Recent earthquakes have clearly shown that the average building or bridge that has not been designed for earthquake forces can be destroyed by an earthquake that is not particularly severe. Most structures can be economically designed and constructed to withstand the forces caused during most earthquakes. The cost of providing seismic resistance to existing structures (called *retrofitting*), however, can be extremely high.

Some engineers seem to think that the seismic loads to be used in design are merely percentage increases of the wind loads. This assumption is incorrect, however, because seismic loads are different in their action and are not proportional to the exposed area of the building but rather are proportional to the distribution of the mass of the building above the particular level being considered.

Another factor to be considered in seismic design is the soil condition. Almost all of the structural damage and loss of life in the Loma Prieta earthquake occurred in areas that have soft clay soils. Apparently these soils amplified the motions of the underlying rock.[27]

It is well to understand that earthquakes load structures in an indirect fashion. The ground is displaced, and because the structures are connected to the ground, they are also displaced and vibrated. As a result, various deformations and stresses are caused throughout the structures.

From the preceding information, you can understand that no external forces are applied aboveground by earthquakes to structures. Procedures for estimating seismic forces such as the ones presented in Chapters 11 to 23 of ASCE 7-10 are very complicated. As a result, they usually are addressed in advanced structural analysis courses, such as structural dynamics or earthquake resistance design courses.

1.26 Selection of Design Loads

To assist the designer in estimating the magnitudes of live loads with which he or she should proportion structures, various records have been assembled through the years in the form of building codes and specifications. These publications provide conservative estimates of live-load magnitudes for various

[26] Fairweather, V., 1990, "The Next Earthquake," *Civil Engineering* (March), pp. 54–57.
[27] Ibid.

Courtesy of EFCO Corp.

Croke Park Stadium, Dublin, Ireland.

situations. One of the most widely used design-load specifications for buildings is that published by the American Society of Civil Engineers (ASCE).[28]

The designer is usually fairly well controlled in the design of live loads by the building code requirements in his or her particular area. Unfortunately, the values given in these various codes vary from city to city, and the designer must be sure to meet the requirements of a particular locality. In the absence of a governing code, the ASCE Code is an excellent one to follow.

Some other commonly used specifications are as follows:

- For railroad bridges, American Railway Engineering Association (AREA).[29]
- For highway bridges, American Association of State Highway and Transportation Officials (AASHTO).[30]
- For buildings, the International Building Code (IBC).[31]

These specifications will on many occasions clearly prescribe the loads for which structures are to be designed. Despite the availability of this information, the designer's ingenuity and knowledge of the situation are often needed to predict what loads a particular structure will have to support in years to come. Over the past several decades, insufficient estimates of future traffic loads by bridge designers have resulted in a great number of replacements with wider and stronger structures.

1.27 Calculation Accuracy

A most important point, which many students with their amazing computers and pocket calculators have difficulty in understanding, is that reinforced concrete design is not an exact science for which answers can be confidently calculated to six or eight places. The reasons for this statement should be

[28] American Society of Civil Engineers, 2010, *Minimum Design Loads for Buildings and Other Structures*, ASCE 7-10 (Reston, VA: American Society of Civil Engineers), 608 pages.
[29] American Railway Engineering Association (AREA), 2014, *Manual for Railway Engineering* (Chicago: AREA).
[30] *Standard Specifications for Highway Bridges*, 2002, 17th ed. (Washington, DC: American Association of State Highway and Transportation Officials [AASHTO]).
[31] International Building Code, 2015, International Code Council, Inc.

quite obvious: The analysis of structures is based on partly true assumptions; the strengths of materials used vary widely; structures are not built to the exact dimensions shown on the plans; and maximum loadings can only be approximated. With respect to this last sentence, how many users of this book could estimate to within 10% the maximum live load in pounds per square foot that will ever occur on the building floor they are now occupying? Calculations to more than two or three significant figures are obviously of little value and may actually mislead students into a false sense of accuracy.

1.28 Impact of Computers on Reinforced Concrete Design

The availability of personal computers has drastically changed the way in which reinforced concrete structures are analyzed and designed. In nearly every engineering school and office, computers are routinely used to handle structural design problems.

Many calculations are involved in reinforced concrete design, and many of these calculations are quite time consuming. With a computer, the designer can reduce the time required for these calculations tremendously and, thus, supposedly have time to consider alternative designs.

Although computers do increase design productivity, they do undoubtedly tend at the same time to reduce the designer's "feel" for structures. This can be a special problem for young engineers with little previous design experience. Unless designers have this "feel," computer usage, although expediting the work and reducing many errors, may occasionally result in large mistakes.

It is interesting to note that up to the present time, the feeling at most engineering schools has been that the best way to teach reinforced concrete design is with chalk and blackboard, supplemented with some computer examples.

Accompanying this text are several Excel spreadsheets that can be downloaded from this book's website at: www.wiley.com/college/mccormac.

These spreadsheets are intended to allow the student to consider multiple alternative designs and not as a tool to work basic homework problems.

PROBLEMS

Problem 1.1 Name several of the admixtures that are used in concrete mixes. What is the purpose of each?

Problem 1.2 What is Poisson's ratio, and where can it be of significance in concrete work?

Problem 1.3 What factors influence the creep of concrete?

Problem 1.4 What steps can be taken to reduce creep?

Problem 1.5 What is the effect of creep in reinforced concrete columns that are subjected to axial compression loads?

Problem 1.6 Why is silica fume used in high-strength concrete? What does it do?

Problem 1.7 Why do the surfaces of reinforcing bars have rolled-on deformations?

Problem 1.8 What are "soft metric" reinforcing bars?

Problem 1.9 What are the three factors that influence the magnitude of the earthquake load on a structure?

Problem 1.10 Why are epoxy-coated bars sometimes used in the construction of reinforced concrete?

Problem 1.11 What is the diameter and cross-sectional area of a #5 reinforcing bar?

CHAPTER 2 Flexural Analysis of Beams

2.1 Introduction

In this section, it is assumed that a small transverse load is placed on a concrete beam with tensile reinforcing and that the load is gradually increased in magnitude until the beam fails. As this takes place, the beam will go through three distinct stages before collapse occurs: (1) the uncracked concrete stage, (2) the concrete cracked–elastic stresses stage, and (3) the ultimate-strength stage. A relatively long beam is considered for this discussion so that shear will not have a large effect on its behavior.

Uncracked Concrete Stage

At small loads when the tensile stresses are less than the *modulus of rupture* (the bending tensile stress at which the concrete begins to crack), the entire cross section of the beam resists bending, with compression on one side and tension on the other. Figure 2.1 shows the variation of stresses and strains for these small loads; a numerical example of this type is presented in Section 2.2.

Concrete Cracked–Elastic Stresses Stage

As the load is increased after the modulus of rupture of the concrete is exceeded, cracks begin to develop in the bottom of the beam. The moment at which these cracks begin to form—that is, when the tensile stress in the bottom of the beam equals the modulus of rupture—is referred to as the *cracking moment*, M_{cr}. As the load is further increased, these cracks quickly spread up to the vicinity of the neutral axis, and then the neutral axis begins to move upward. The cracks occur at those places along the beam where the actual moment is greater than the cracking moment, as shown in Figure 2.2(a).

Now that the bottom has cracked, another stage is present because the concrete in the cracked zone obviously cannot resist tensile stresses—the steel must do it. This stage will continue as long as the compression stress in the top fibers is less than about one-half of the concrete's compression

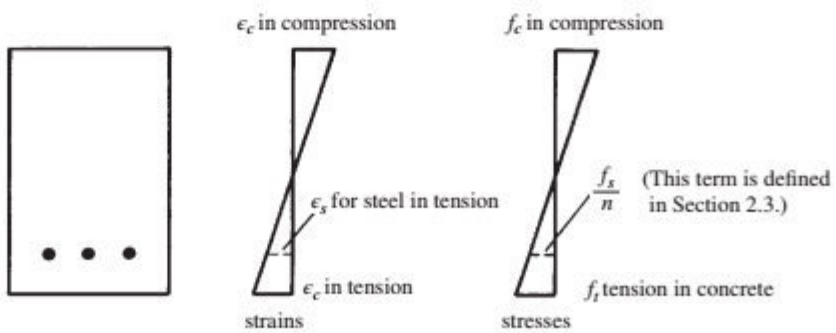


FIGURE 2.1 Uncracked concrete stage.

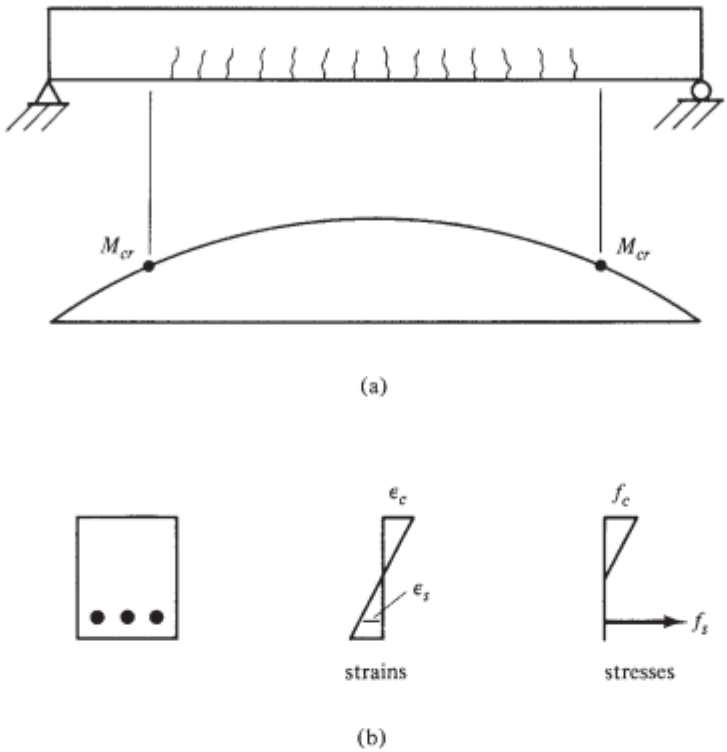


FIGURE 2.2 Concrete cracked–elastic stresses stage.

strength, f'_c, and as long as the steel stress is less than its yield stress. The stresses and strains for this range are shown in Figure 2.2(b). In this stage, the compressive stresses vary linearly with the distance from the neutral axis or as a straight line.

The straight-line stress–strain variation normally occurs in reinforced concrete beams under normal service-load conditions because at those loads, the stresses are generally less than $0.50f'_c$. To compute the concrete and steel stresses in this range, the transformed-area method (to be presented in Section 2.3) is used. The *service* or *working* loads are the loads that are assumed to actually occur when a structure is in use or service. Under these loads, moments develop that are considerably larger than the cracking moments. Obviously, the tensile side of the beam will be cracked. You will learn to estimate crack widths and methods of limiting their widths in Chapter 6.

Beam Failure—Ultimate-Strength Stage

As the load is increased further so that the compressive stresses are greater than $0.50f'_c$, the tensile cracks move farther upward, as does the neutral axis, and the concrete compression stresses begin to change appreciably from a straight line. For this initial discussion, it is assumed that the reinforcing bars have yielded. The stress variation is much like that shown in Figure 2.3. You should relate the information shown in this figure to that given in Figure 1.1 in Chapter 1 as to the changing ratio of stress to strain at different stress levels.

To further illustrate the three stages of beam behavior that have just been described, a moment–curvature diagram is shown in Figure 2.4.[1] For this diagram, θ is defined as the angle change

[1] MacGregor, J. G., 2005, *Reinforced Concrete Mechanics and Design*, 4th ed. (Upper Saddle River, NJ: Prentice Hall), p. 109.

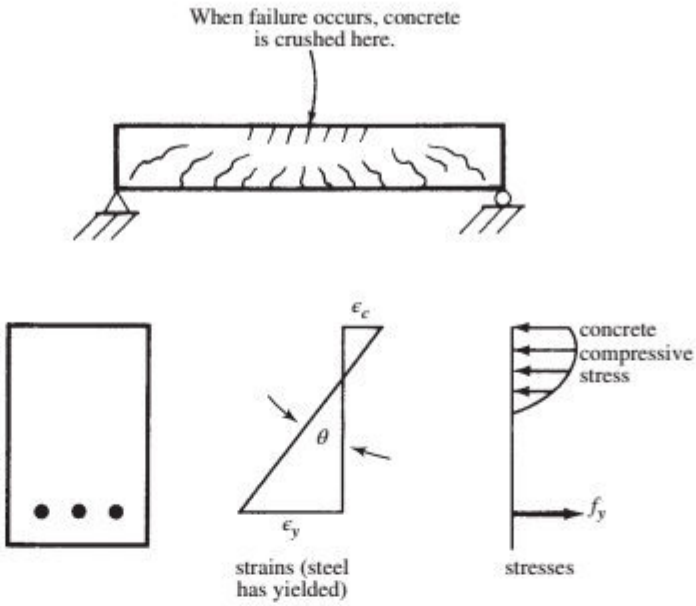


FIGURE 2.3 Ultimate-strength stage.

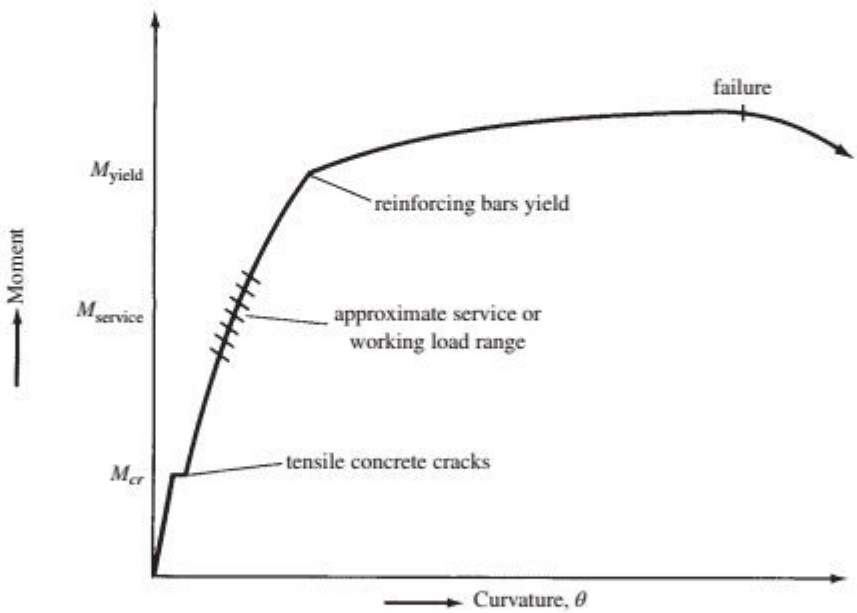


FIGURE 2.4 Moment–curvature diagram for reinforced concrete beam with tensile reinforcement only.

of the beam section over a certain length and is computed by the following expression in which ϵ is the strain in a beam fiber at some distance, y, from the neutral axis of the beam:

$$\theta = \frac{\epsilon}{y}$$

The first stage of the diagram is for small moments less than the cracking moment, M_{cr}, where the entire beam cross section is available to resist bending. In this range, the strains are small, and the diagram is nearly vertical and very close to a straight line. When the moment is increased beyond the cracking moment, the slope of the curve will decrease a little because the beam is not quite as stiff as it

Courtesy of EFCO Corp.

Mega beam concrete construction 100′ above grade. Arena ciudad de Mexico, Mexico City, Mexico.

was in the initial stage before the concrete cracked. The diagram will follow almost a straight line from M_{cr} to the point where the reinforcing is stressed to its yield point. Until the steel yields, a fairly large additional load is required to appreciably increase the beam's deflection.

After the steel yields, the beam has very little additional moment capacity, and only a small additional load is required to substantially increase rotations as well as deflections. The slope of the diagram is now very flat.

2.2 Cracking Moment

The area of reinforcing as a percentage of the total cross-sectional area of a beam is quite small (usually 2% or less), and its effect on the beam properties is almost negligible as long as the beam is uncracked. Therefore, an approximate calculation of the bending stresses in such a beam can be obtained based on the gross properties of the beam's cross section. The stress in the concrete at any point a distance y from the neutral axis of the cross section can be determined from the following flexure formula in which M is the bending moment equal to or less than the cracking moment of the section and I_g is the gross moment of inertia of the cross section:

$$f = \frac{My}{I_g}$$

Section 24.2.3.5 of the ACI 318 Code states that the cracking moment of a section may be determined with ACI Equation 24.2.3.5b, in which f_r is the modulus of rupture of the concrete and y_t is the distance from the centroidal axis of the section to its extreme fiber in tension. Section 19.2.3.1 of ACI 318 states that f_r may be taken equal to $7.5\lambda\sqrt{f'_c}$ with f'_c in psi.

Or in SI units with f'_c in N/mm^2 or MPa, $f_r = 0.7\lambda\sqrt{f'_c}$

The "lambda" term is 1.0 for normal-weight concrete and is less than 1.0 for lightweight concrete, as described in Section 1.12. The cracking moment is as follows:

$$M_{cr} = \frac{f_r I_g}{y_t} \qquad \text{(ACI Equation 24.2.3.5b)}$$

Example 2.1 presents calculations for a reinforced concrete beam where tensile stresses are less than its modulus of rupture. As a result, no tensile cracks are assumed to be present, and the stresses are similar to those occurring in a beam constructed with a homogeneous material.

Example 2.1

(a) Assuming the concrete is uncracked, compute the bending stresses in the extreme fibers of the beam of Figure 2.5 for a bending moment of 25 ft-k. The normal-weight concrete has an f'_c of 4000 psi and a modulus of rupture $f_r = 7.5\,(1.0)\sqrt{4000}$ psi = 474 psi.

(b) Determine the cracking moment of the section.

SOLUTION

(a) Bending Stresses

$$I_g = \frac{1}{12}bh^3 \text{ with } b = 12 \text{ in. and } h = 18 \text{ in.}$$

$$I_g = \left(\frac{1}{12}\right)(12 \text{ in.})\,(18 \text{ in.})^3 = 5832 \text{ in.}^4$$

$$f = \frac{My}{I_g} \text{ with } M = 25 \text{ ft-k} = 25{,}000 \text{ ft-lb}$$

Next, multiply 25,000 ft-lb by 12 in/ft to obtain in-lb as shown here:

$$f = \frac{(12 \text{ in/ft} \times 25{,}000 \text{ ft-lb})\,(9.00 \text{ in.})}{5832 \text{ in.}^4} = 463 \text{ psi}$$

Since this stress is less than the tensile strength or modulus of rupture of the concrete of 474 psi, the section is assumed not to have cracked.

(b) Cracking Moment

$$M_{cr} = \frac{f_r I_g}{y_t} = \frac{(474 \text{ psi})\,(5832 \text{ in.}^4)}{9.00 \text{ in.}} = 307{,}152 \text{ in-lb} = \underline{\underline{25.6 \text{ ft-k}}}$$

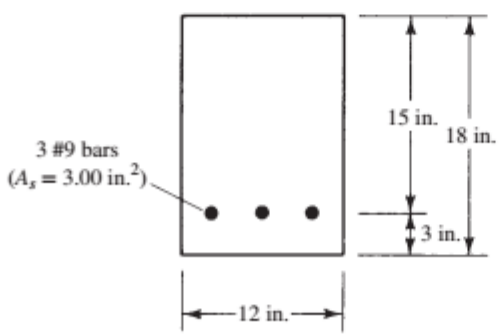


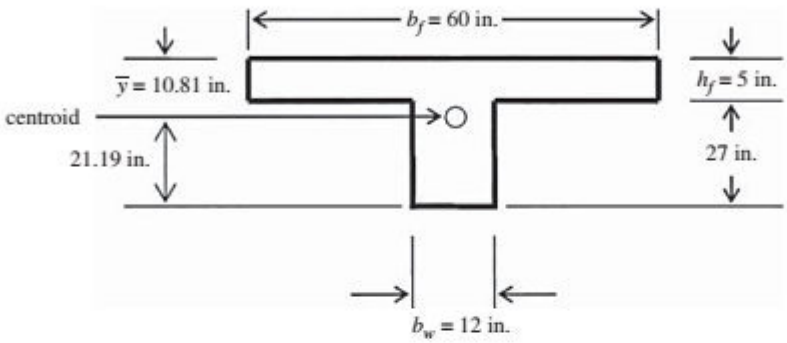
FIGURE 2.5 Beam cross section for Example 2.1.

Example 2.2

(a) If the T-beam shown is uncracked, calculate the stress in the concrete at the top and bottom extreme fibers under a positive bending moment of 80 ft-k.

(b) If $f'_c = 3000$ psi and normal-weight concrete is used, what is the maximum uniformly distributed load the beam can carry if it is used as a simple beam with 24-ft span without exceeding the modulus of rupture of the concrete?

(c) Repeat part (b) if the beam is inverted.



SOLUTION

(a) Locate the neutral axis with respect to the top of the section:

$$\bar{y} = \frac{b_f h_f \left(\frac{h_f}{2}\right) + (b_f)(h - h_f)\left(h_f - \frac{h - h_f}{2}\right)}{b_f h_f + (b_f)(h - h_f)}$$

$$= \frac{(60 \text{ in.})(5 \text{ in.})(2.5 \text{ in.}) + (12 \text{ in.})(27 \text{ in.})\left(5 \text{ in.} + \frac{27 \text{ in.}}{2}\right)}{(60 \text{ in.})(5 \text{ in.}) + (12 \text{ in.})(27 \text{ in.})} = 10.81 \text{ in.}$$

The moment of inertia is:

$$I_g = \frac{b_f h_f^3}{12} + b_f h_f \left[\left(\bar{y} - \frac{h_f}{2}\right)^2 + \frac{b_w (h - h_f)^3}{12} + b_w (h - h_f)\right]\left[\bar{y} - h_f - \frac{(h - h_f)}{2}\right]^2$$

$$= \frac{(60 \text{ in.})(5 \text{ in.})^3}{12} + (60 \text{ in.})(5 \text{ in.})\left(10.81 \text{ in.} - \frac{5 \text{ in.}}{2}\right)^2 + \frac{(12 \text{ in.})(32 \text{ in.} - 5 \text{ in.})^3}{12}$$

$$+ (12 \text{ in.})(32 \text{ in.} - 5 \text{ in.})\left(10.81 \text{ in.} - 5 \text{ in.} - \frac{27 \text{ in.}}{2}\right)^2$$

$$= 60{,}185 \text{ in.}^4$$

The stress in the bottom fiber under the given moment of 80 ft-k is:

$$f_{top} = \frac{Mc}{I} = \frac{(80 \text{ ft-k})(12 \text{ in/ft})(32 \text{ in.} - 10.81 \text{ in.})}{60{,}185 \text{ in.}^4} = 0.338 \text{ k/in.}^2 = 338 \text{ lb/in.}^2$$

The stress in the top fiber is:

$$f_{top} = \frac{Mc}{I} = \frac{(80 \text{ ft-k})(12 \text{ in/ft})(10.81 \text{ in.})}{60{,}185 \text{ in.}^4} = 0.172 \text{ k/in.}^2 = 172 \text{ lb/in.}^2$$

(b) The modulus of rupture, f_r, of normal-weight concrete with $f'_c = 3000$ psi is:

$$f_r = 7.5\lambda\sqrt{f'_c} = 7.5\,(1.0)\sqrt{3000} = 411 \text{ lb/in.}^2$$

The moment that causes a stress equal to the modulus of rupture is:

$$M_{cr} = \frac{f_r I_g}{c} = \frac{(411 \text{ lb/in.}^2)\,(60{,}185 \text{ in.}^4)}{(32 \text{ in.} - 10.81 \text{ in.})} = 1167.344 \text{ in-lb} = 97.28 \text{ ft-k}$$

The uniformly distributed load on a simple span that causes this much moment is:

$$w = \frac{8M}{l^2} = \frac{8\,(97.28 \text{ ft-k})}{(24 \text{ ft})^2} = 1.351 \text{ k/ft} = 1351 \text{ lb/ft}$$

(c) If the beam is inverted, then the c term used to calculate M_{cr} is 10.81 in. instead of 21.19 in., hence:

$$M_{cr} = \frac{f_r I_g}{c} = \frac{(411 \text{ lb/in.}^2)\,(60{,}185 \text{ in.}^4)}{(10.81 \text{ in.})} = 2{,}288{,}255 \text{ in-lb} = 190.69 \text{ ft-k}$$

The uniformly distributed load on a simple span that causes this much moment is:

$$w = \frac{8M}{l^2} = \frac{8\,(190.69 \text{ ft-k})}{(24 \text{ ft})^2} = 2.648 \text{ k/ft} = 2648 \text{ lb/ft}$$

This is almost double the load that the beam can carry if oriented the opposite way. Don't get the impression that this is the best orientation for a T beam, however. In the next section, when we examine reinforced sections, the opposite will be true.

2.3 Elastic Stresses—Concrete Cracked

When the bending moment is sufficiently large to cause the tensile stress in the extreme fibers to be greater than the modulus of rupture, it is assumed that all of the concrete on the tensile side of the beam is cracked and must be neglected in the flexure calculations.

The cracking moment of a beam is normally quite small compared to the service load moment. Thus, when the service loads are applied, the bottom of the beam cracks. The cracking of the beam does not necessarily mean that the beam is going to fail. The reinforcing bars on the tensile side begin to pick up the tension caused by the applied moment.

On the tensile side of the beam, an assumption of perfect bond is made between the reinforcing bars and the concrete. Thus, the strain in the concrete and in the steel will be equal at equal distances from the neutral axis. If the strains in the two materials at a particular point are the same, however, their stresses cannot be the same since they have different moduli of elasticity. Thus, their stresses are in proportion to the ratio of their moduli of elasticity. The ratio of the steel modulus to the concrete modulus is called the *modular ratio, n*:

$$n = \frac{E_s}{E_c}$$

If the modular ratio for a particular beam is 10, the stress in the steel will be 10 times the stress in the concrete at the same distance from the neutral axis. Another way of saying this is that when $n = 10$, 1 in.^{2} of steel will carry the same total force as 10 in.^{2} of concrete.

For the beam of Figure 2.6, the steel bars are replaced with an equivalent area of fictitious concrete (nA_s), which supposedly can resist tension. This area is referred to as the *transformed area*. The resulting revised cross section or transformed section is handled by the usual methods for elastic homogeneous beams. Also shown in the figure is a diagram showing the stress variation in the beam. On the tensile side, a dashed line is shown because the diagram is discontinuous. There, the concrete is assumed to

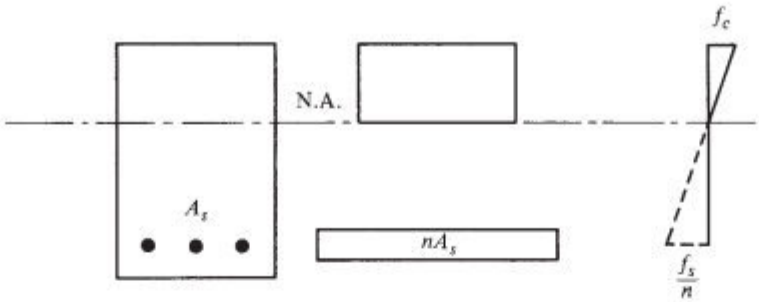


FIGURE 2.6 Cracked, transformed section.

be cracked and unable to resist tension. The value shown opposite the steel is the fictitious stress in the concrete if it could carry tension. This value is shown as f_s/n because it must be multiplied by n to give the steel stress f_s.

Examples 2.3 to 2.5 are transformed-area problems that illustrate the calculations necessary for determining the stresses and resisting moments for reinforced concrete beams. The first step to be taken in each of these problems is to locate the neutral axis, which is assumed to be located a distance x from the compression surface of the beam. The first moment of the compression area of the beam cross section about the neutral axis must equal the first moment of the tensile area about the neutral axis. The resulting quadratic equation can be solved by completing the squares or by using the quadratic formula.

Donovan Reese/Getty Images, Inc.

Bridge construction on an expressway interchange.

After the neutral axis is located, the moment of inertia of the transformed section is calculated, and the stresses in the concrete and the steel are computed with the flexure formula.

Example 2.3

Calculate the bending stresses in the beam shown in Figure 2.7 by using the transformed-area method, $f'_c = 3000$ psi, $n = 9$, and $M = 70$ ft-k.

SOLUTION

Taking Moments about Neutral Axis (Referring to Figure 2.8)

$$(12 \text{ in.})\,(x)\left(\frac{x}{2}\right) = (9)\,(3.00 \text{ in.})\,(17 \text{ in.} - x)$$
$$6x^2 = 459 - 27.00x$$

Solving by Completing the Square

$$6x^2 + 27.00x = 459$$
$$x^2 + 4.50x = 76.5$$
$$(x + 2.25)\,(x + 2.25) = 76.5 + (2.25)^2$$
$$x = 2.25 + \sqrt{76.5 + (2.25)^2}$$
$$x = 6.780 \text{ in.}$$

Moment of Inertia

$$I = \left(\frac{1}{3}\right)(12 \text{ in.})\,(6.78 \text{ in.})^3 + (9)\,(3.00 \text{ in.}^2)\,(10.22 \text{ in.})^2 = 4067 \text{ in.}^4$$

Bending Stresses

$$f_c = \frac{My}{I} = \frac{(12)\,(70{,}000 \text{ ft-lb})\,(6.78 \text{ in.})}{4067 \text{ in.}^4} = \underline{\underline{1400 \text{ psi}}}$$

$$f_s = n\frac{My}{I} = (9)\frac{(12)\,(70{,}000 \text{ ft-lb})\,(10.22 \text{ in.})}{4067 \text{ in.}^4} = \underline{\underline{18{,}998 \text{ psi}}}$$

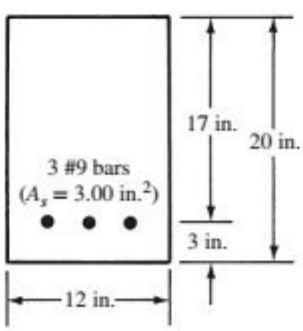


FIGURE 2.7 Beam cross section for Example 2.3.

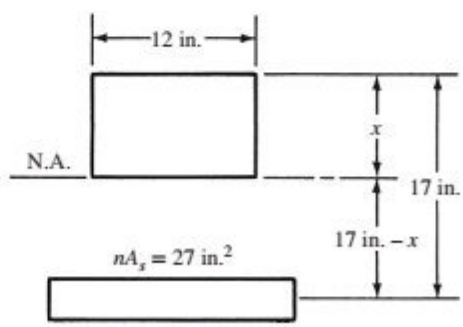


FIGURE 2.8 Cracked, transformed section for Example 2.3.

Example 2.4

Determine the allowable resisting moment of the beam of Example 2.3, if the allowable stresses are $f_c = 1350$ psi and $f_s = 20{,}000$ psi.

SOLUTION

$$M_c = \frac{f_c I}{y} = \frac{(1350 \text{ psi})\,(4067 \text{ in.}^4)}{6.78 \text{ in.}} = 809{,}800 \text{ in-lb} = \underline{\underline{67.5 \text{ ft-k}}} \leftarrow$$

$$M_s = \frac{f_s I}{ny} = \frac{(20{,}000 \text{ psi})\,(4067 \text{ in.}^4)}{(9)\,(10.22 \text{ in.})} = 884{,}323 \text{ in-lb} = \underline{\underline{73.7 \text{ ft-k}}}$$

Discussion

For a given beam, the concrete and steel will not usually reach their maximum allowable stresses at exactly the same bending moments. Such is the case for this example beam, where the concrete reaches its maximum permissible stress at 67.5 ft-k, while the steel does not reach its maximum value until 73.7 ft-k is applied. The resisting moment of the section is 67.5 ft-k because if that value is exceeded, the concrete becomes overstressed even though the steel stress is less than its allowable stress.

Example 2.5

Compute the bending stresses in the beam shown in Figure 2.9 by using the transformed-area method; $n = 8$ and $M = 110$ ft-k.

SOLUTION

Locating Neutral Axis (Assuming Neutral Axis below Hole)

$$(18 \text{ in.})\,(x)\left(\frac{x}{2}\right) - (6 \text{ in.})\,(6 \text{ in.})\,(x - 3 \text{ in.}) = (8)\,(5.06 \text{ in.}^2)\,(23 \text{ in.} - x)$$

$$9x^2 - 36x + 108 = 931 - 40.48x$$

$$9x^2 + 4.48x = 823$$

$$x^2 + 0.50x = 91.44$$

$$(x + 0.25)\,(x + 0.25) = 91.44 + (0.25)^2 = 91.50$$

$$x + 0.25 = \sqrt{91.50} = 9.57$$

$$x = 9.32 \text{ in.} > 6 \text{ in.} \qquad \therefore \text{ N.A. below hole as assumed}$$

Moment of Inertia

$$I = \left(\frac{1}{3}\right)(6 \text{ in.})\,(9.32 \text{ in.})^3(2) + \left(\frac{1}{3}\right)(6 \text{ in.})\,(3.32 \text{ in.})^3 + (8)\,(5.06 \text{ in.}^2)\,(13.68 \text{ in.})^2 = 10{,}887 \text{ in.}^4$$

Computing Stresses

$$f_c = \frac{(12)\,(110{,}000 \text{ ft-lb})\,(9.32 \text{ in.})}{10{,}887 \text{ in.}^4} = \underline{\underline{1130 \text{ psi}}}$$

$$f_s = (8)\frac{(12)\,(110{,}000 \text{ ft-lb})\,(13.68 \text{ in.})}{10{,}887 \text{ in.}^4} = \underline{\underline{13{,}269 \text{ psi}}}$$

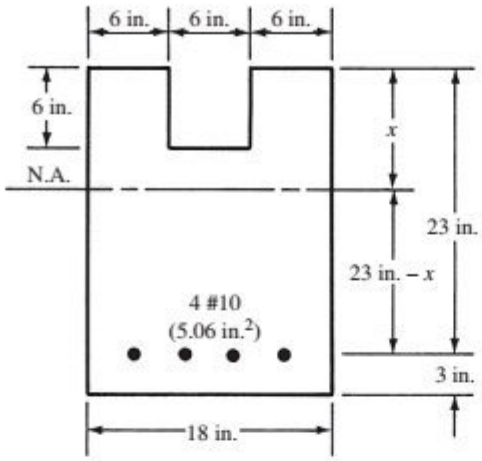
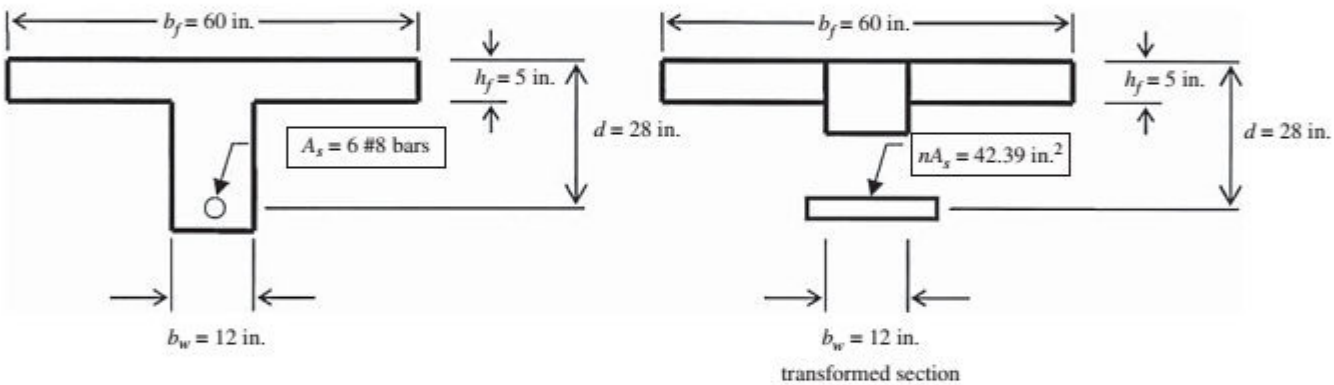


FIGURE 2.9 Beam cross section for Example 2.5.

Example 2.6

Calculate the bending stresses in the concrete and the reinforcing steel, using the transformed-area method: $f'_c = 3000$ psi, normal-weight concrete, $n = 9$, $M = 250$ ft-k.



SOLUTION

Assume the neutral axis is in the web, and take moments about the neutral axis of the transformed section for this example:

$$(b_f - b_w)h_f\left(x - \frac{h_f}{2}\right) + \frac{b_w x^2}{2} = nA_s(d - x)$$

$$(60\text{ in.} - 12\text{ in.})(5\text{ in.})\left(x - \frac{5\text{ in.}}{2}\right) + \frac{(12\text{ in.})(x)^2}{2} = (9)(4.71\text{ in.}^2)(28\text{ in.} - x)$$

Using a calculator with a solver for quadratic equations results in $x = 5.65$ in. Because this value of x exceeds h_f of 5 in., the assumption that the neutral axis is in the web is valid. If x had been smaller than 5 in., then the value we obtained would not have been valid, and the preceding equations would have to be rewritten and solved assuming $x < h_f$.

$$I_{cr} = \frac{(b_f - b_w)h_f^3}{12} + (b_f - b_w)h_f\left(x - \frac{h_f}{2}\right)^2 + \frac{b_w x^3}{12} + b_w x\left(\frac{x}{2}\right) + nA_s(d - x)^2$$

$$= \frac{(60\text{ in.} - 12\text{ in.})(5\text{ in.})^3}{12} + (60\text{ in.} - 12\text{ in.})(5\text{ in.})\left(5.65\text{ in.} - \frac{5\text{ in.}}{2}\right)^2$$

$$+ \frac{(12\text{ in.})(5.65\text{ in.})^3}{3} + (9)(4.71\text{ in.}^2)(28\text{ in.} - 5.65\text{ in.})$$

$$= 24{,}778\text{ in.}^4$$

The T-shaped part of the transformed section could be divided into rectangles in other ways besides the one shown. The resulting answer would still be the same.

The stress in the concrete can now be calculated:

$$f_c = \frac{Mx}{I_{cr}} = \frac{(250 \text{ ft-k})\,(5.65 \text{ in.})\,(12 \text{ in/ft})}{24{,}778 \text{ in.}^4} = 0.684 \text{ k/in.}^2 = 684 \text{ lb/in.}^2$$

This concrete stress is well below the allowable values that were once in the ACI Code. They used to be $0.45f'_c = (0.45)\,(3000 \text{ lb/in.}^2) = 1350 \text{ lb/in.}^2$.

The stress in the reinforcing steel can now be calculated:

$$f_s = \frac{nM(d-x)}{I_{cr}} = \frac{(9)\,(250 \text{ ft-k})\,(28 \text{ in.} - 5.65 \text{ in.})\,(12 \text{ in/ft})}{24{,}778 \text{ in.}^4} = 24.354 \text{ k/in.}^2 = 24{,}354 \text{ lb/in.}^2$$

This reinforcing steel stress is slightly greater than the allowable values that were once in the ACI Code. They were formerly 24,000 lb/in.^{2} for Grade 60 reinforcing steel. This is about a 1.5% overstress in the steel, and many engineers would accept this much overstress as being within the accuracy of their other assumptions. This beam would be called "tension controlled" because the moment capacity is controlled by the steel, not the concrete. This same beam could be compression controlled if a lot more steel were used. Tension-controlled beams are preferable to compression-controlled ones, as will be discussed later in this text.

Example 2.7 illustrates the analysis of a doubly reinforced concrete beam—that is, one that has compression steel as well as tensile steel. Compression steel is generally thought to be uneconomical, but occasionally its use is quite advantageous.

Compression steel will permit the use of appreciably smaller beams than those that make use of tensile steel only. Reduced sizes can be very important where space or architectural requirements limit the sizes of beams. Compression steel is quite helpful in reducing long-term deflections, and such steel is useful for positioning stirrups or shear reinforcing, a subject to be discussed in Chapter 8. A detailed discussion of doubly reinforced beams is presented in Chapter 5.

The creep or plastic flow of concrete was described in Section 1.12. Should the compression side of a beam be reinforced, the long-term stresses in that reinforcing will be greatly affected by the creep in the concrete. As time goes by, the compression concrete will compact more tightly, leaving the reinforcing bars (which themselves have negligible creep) to carry more and more of the load.

As a consequence of this creep in the concrete, the stresses in the compression bars computed by the transformed-area method are assumed to double as time goes by. In Example 2.7, the transformed area of the compression bars is assumed to equal $2n$ times their area, A'_s.

On the subject of "hairsplitting," it will be noted in the example that the compression steel area is really multiplied by $2n - 1$. The transformed area of the compression side equals the gross compression area of the concrete plus $2nA'_s$ minus the area of the holes in the concrete ($1A'_s$), which theoretically should not have been included in the concrete part. This equals the compression concrete area plus $(2n - 1)A'_s$. Similarly, $2n - 1$ is used in the moment of inertia calculations. The stresses in the compression bars are determined by multiplying $2n$ times the stresses in the concrete located at the same distance from the neutral axis.

Example 2.7

Compute the bending stresses in the beam shown in Figure 2.10; $n = 10$ and $M = 118$ ft-k.

SOLUTION

Locating Neutral Axis

$$(14 \text{ in.})(x)\left(\frac{x}{2}\right) + (20-1)(2.00 \text{ in.}^2)(x - 2.5 \text{ in.}) = (10)(4.00 \text{ in.}^2)(17.5 \text{ in.} - x)$$

$$7x^2 + 38x - 95 = 700 - 40x$$

$$7x^2 + 78x = 795$$

$$x^2 + 11.14x = 113.57$$

$$x + 5.57 = \sqrt{113.57 + (5.57)^2} = 12.02$$

$$x = 6.45 \text{ in.}$$

Moment of Inertia

$$I = \left(\frac{1}{3}\right)(14 \text{ in.})(6.45 \text{ in.})^3 + (20-1)(2.00 \text{ in.}^2)(3.95 \text{ in.})^2 + (10)(4.00 \text{ in.}^2)(11.05 \text{ in.})^2$$

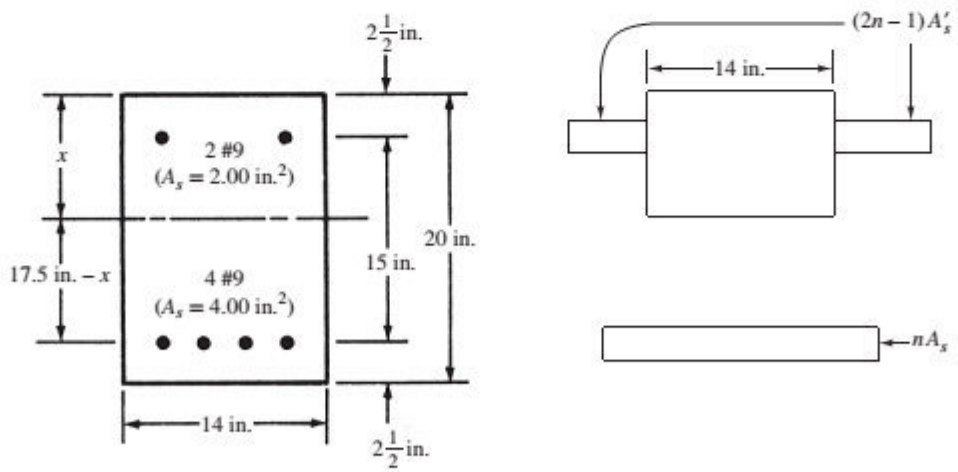
$$= 6729 \text{ in.}^4$$

Bending Stresses

$$f_c = \frac{(12)(118{,}000 \text{ ft-lb})(6.45 \text{ in.})}{6729 \text{ in.}^4} = \underline{\underline{1357 \text{ psi}}}$$

$$f'_s = 2n\frac{My}{I} = (2)(10)\frac{(12)(118{,}000 \text{ ft-lb})(3.95 \text{ in.})}{6729 \text{ in.}^4} = \underline{\underline{16{,}624 \text{ psi}}}$$

$$f_s = (10)\frac{(12)(118{,}000 \text{ ft-lb})(11.05 \text{ in.})}{6729 \text{ in.}^4} = \underline{\underline{23{,}253 \text{ psi}}}$$



(a) Actual section (b) Transformed section

FIGURE 2.10 Beam cross section for Example 2.7.

2.4 Ultimate or Nominal Flexural Moments

In this section, a very brief introduction to the calculation of the ultimate or nominal flexural strength of beams is presented. This topic is continued at considerable length in the next chapter, where formulas, limitations, designs, and other matters are presented. For this discussion, it is assumed that the tensile reinforcing bars are stressed to their yield point before the concrete on the compressive side of the beam is crushed. You will learn in Chapter 3 that the ACI Code requires all beam designs to fall into this category.

After the concrete compression stresses exceed about $0.50f'_c$, they no longer vary directly as the distance from the neutral axis or as a straight line. Rather, they vary much as shown in Figure 2.11(b). It is assumed for the purpose of this discussion that the curved compression diagram is replaced with a rectangular one with a constant stress of $0.85f'_c$, as shown in part (c) of the figure. The rectangular diagram of depth a is assumed to have the same c.g. (center of gravity) and total magnitude as the curved diagram. (In Section 3.4 of Chapter 3 of this text, you will learn that this distance a is set equal to $\beta_1 c$, where β_1 is a value determined by testing and specified by the code.) These assumptions will enable us to easily calculate the theoretical or nominal flexural strength of reinforced concrete beams. Experimental tests show that with the assumptions used here, accurate flexural strengths are determined.

To obtain the nominal or theoretical moment strength of a beam, the simple steps to follow are illustrated in Figure 2.11 and Example 2.8.

1. Compute total tensile force $T = A_s f_y$.
2. Equate total compression force $C = 0.85f'_c ab$ to $A_s f_y$ and solve for a. In this expression, ab is the assumed area stressed in compression at $0.85f'_c$. The compression force C and the tensile force T must be equal to maintain equilibrium at the section.
3. Calculate the distance between the centers of gravity of T and C. (For a rectangular beam cross section, it equals $d - a/2$.)
4. Determine M_n, which equals T or C times the distance between their centers of gravity.

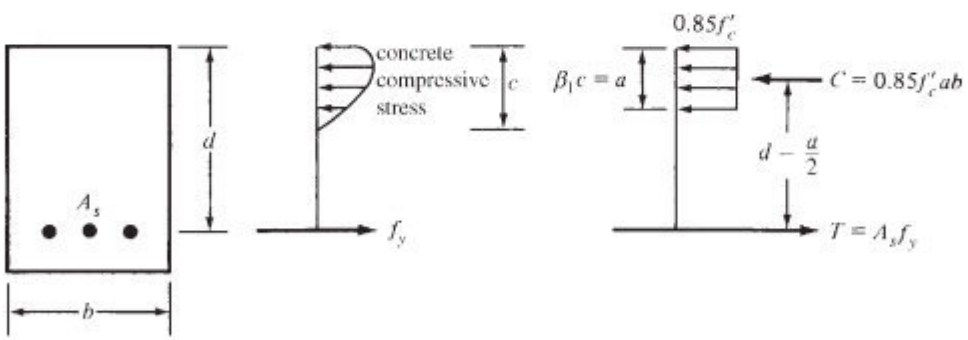


FIGURE 2.11 Compression and tension couple at nominal moment.

Example 2.8

Determine M_n, the nominal or theoretical ultimate moment strength of the beam section shown in Figure 2.12, if $f_y = 60{,}000$ psi and $f'_c = 3000$ psi.

SOLUTION

Computing Tensile and Compressive Forces *T* and *C*

$$T = A_s f_y = (3.00 \text{ in.}^2)\ (60 \text{ k/in.}^2) = 180 \text{ k}$$

$$C = 0.85 f'_c ab = (0.85)\ (3 \text{ k/in.}^2)\ (a)\ (14 \text{ in.}) = 35.7a$$

Equating *T* and *C* and Solving for *a*

$$T = C \text{ for equilibrium}$$

$$180 \text{ k} = 35.7a$$

$$a = 5.04 \text{ in.}$$

Computing the Internal Moment Arm and Nominal Moment Capacity

$$d - \frac{a}{2} = 21 \text{ in.} - \frac{5.04 \text{ in.}}{2} = 18.48 \text{ in.}$$

$$M_n = (180 \text{ k})\ (18.48 \text{ in.}) = 3326.4 \text{ in-k} = \underline{\underline{277.2 \text{ ft-k}}}$$

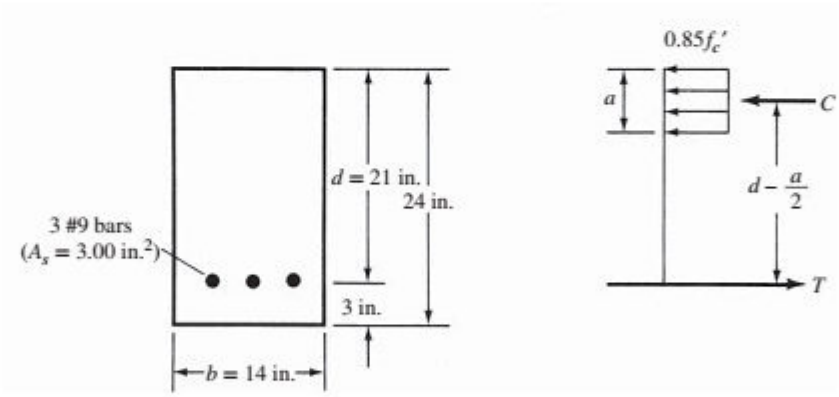


FIGURE 2.12 Beam cross section for Example 2.8.

In Example 2.9, the nominal moment capacity of another beam is determined much as it was in Example 2.8. The only difference is that the cross section of the compression area (A_c) stressed at $0.85f'_c$ is not rectangular. As a result, once this area is determined, we need to locate its center of gravity. The c.g. for the beam of Figure 2.13 is shown as being a distance $\bar{y}$ from the top of the beam in Figure 2.14. The lever arm from C to T is equal to $d - \bar{y}$ (which corresponds to $d - a/2$ in Example 2.8) and M_n equals $A_s f_y (d - \bar{y})$.

With this very simple procedure, values of M_n can be computed for tensilely reinforced beams of any cross section.

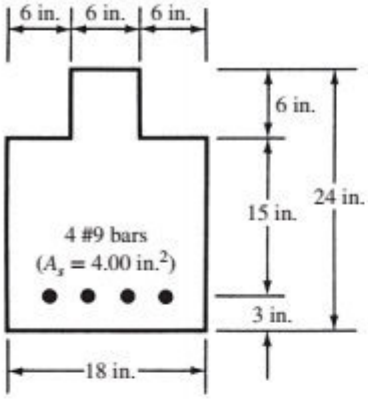


FIGURE 2.13 Beam cross section for Example 2.9.

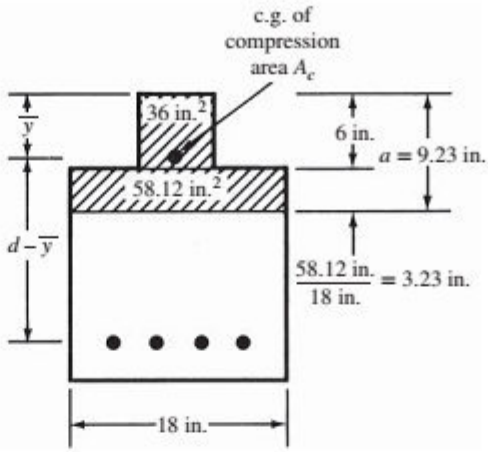


FIGURE 2.14 Area under compression stress block for Example 2.9.

Example 2.9

Calculate the nominal or theoretical ultimate moment strength of the beam section shown in Figure 2.13, if $f_y = 60{,}000$ psi and $f'_c = 3000$ psi. The 6-in.-wide ledges on top are needed for the support of precast concrete slabs.

SOLUTION

$$T = A_s f_y = (4.00 \text{ in.}^2)\ (60 \text{ k/in.}^2) = 240 \text{ k}$$

$$C = (0.85 f'_c)\ (\text{area of concrete } A_c \text{ stressed to } 0.85 f'_c)$$

$$= 0.85 f'_c A_c$$

Courtesy of EFCO Corp.

Finger piers for U.S. Coast Guard base, Boston, Massachusetts.

Equating *T* and *C* and Solving for A_c

$$A_c = \frac{T}{0.85f'_c} = \frac{240\text{ k}}{(0.85)\,(3\text{ k/in.}^2)} = 94.12\text{ in.}^2$$

The top 94.12 in.^{2} of the beam in Figure 2.14 is stressed in compression to $0.85f'_c$. This area can be shown to extend 9.23 in. down from the top of the beam. Its c.g. is located by taking moments at the top of the beam as follows:

$$\bar{y} = \frac{(36\text{ in.}^2)\,(3\text{ in.}) + (58.12\text{ in.}^2)\left(6\text{ in.} + \dfrac{3.23\text{ in.}}{2}\right)}{94.12} = 5.85\text{ in.}$$

$$d - \bar{y} = 21\text{ in.} - 5.85\text{ in.} = 15.15\text{ in.}$$

$$M_n = (240\text{ k})\,(15.15\text{ in.}) = 3636\text{ in-k} = \underline{\underline{303\text{ ft-k}}}$$

2.5 SI Example

In Example 2.10, the nominal moment strength of a beam is computed using SI units. Appendix B, Tables B.1 to B.9 provide information concerning various concrete and steel grades, as well as bar diameters, areas, and so on, all given in SI units.

Example 2.10

Determine the nominal moment strength of the beam shown in Figure 2.15 if $f'_c = 28$ MPa and $f_y = 420$ MPa.

SOLUTION

$$T = C$$

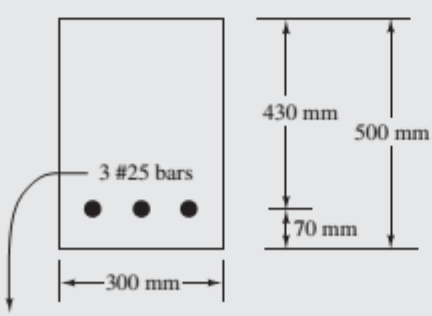
$$A_s f_y = 0.85 f'_c a b$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(1530\text{ mm}^2)\,(420\text{ MPa})}{(0.85)\,(28\text{ MPa})\,(300\text{ mm})} = 90\text{ mm}$$

$$M_n = T\left(d - \frac{a}{2}\right) = C\left(d - \frac{a}{2}\right) = A_s f_y\left(d - \frac{a}{2}\right)$$

$$= (1530\text{ mm}^2)\,(420\text{ MPa})\left(430\text{ mm} - \frac{90\text{ mm}}{2}\right)$$

$$= 2.474 \times 10^8\text{ N·mm} = \underline{\underline{247.4\text{ kN·m}}}$$



($A_s = 1530$ mm^2 from Appendix B, Table B.4)

FIGURE 2.15 Beam cross section for Example 2.10.

2.6 Computer Examples

On the John Wiley website for this textbook (www.wiley.com/college/mccormac), several spreadsheets have been provided for the student to use in assisting in the solution of problems. They are categorized by chapter. Note that most of the spreadsheets have multiple worksheets indicated by tabs at the bottom. The three worksheets available for Chapter 2 include (1) calculation of cracking moment, (2) stresses in singly reinforced rectangular beams, and (3) nominal strength of singly reinforced rectangular beams.

Example 2.11

Repeat Example 2.1 using the spreadsheet provided for Chapter 2.

SOLUTION

Open the Chapter 2 spreadsheet and select the worksheet called Cracking Moment. Input only the cells highlighted in yellow (only in the Excel spreadsheets, not in the printed example), the first six values below.

f'_c	=	4000	psi
M	=	25	ft-k
b	=	12	in.
h	=	18	in.
γ_c	=	145	pcf
λ	=	1.00	
I_g	$= bh^3/12 =$	5832	$in.^4$
f_r	$= 7.5\lambda\ SQRT(f'_c) =$	474	psi
f	=	463	psi
M_{cr}	=	307,373	in-lb
M_{cr}	=	25.6	ft-k

The last five values are the same as calculated in Example 2.1.

Example 2.12

Repeat Example 2.3 using the spreadsheet provided for Chapter 2.

SOLUTION

Open the Chapter 2 spreadsheet and select the worksheet called Elastic Stresses. Input only the cells highlighted in yellow, the first seven values below.

b	=	12	in.
d	=	17	in.
n	=	9	
A_s	=	3	in.^{2}
M	=	70	ft-k
f'_c	=	3000	psi
γ_c	=	145	pcf
E_c	=	3,155,924	psi
n	=	9.19	
$n\rho$	=	0.132	
x	=	6.78	in.
I_{cr}	=	4067	in.^{4}
f_c	$= Mx/I =$	1401	psi
f_s	$= nM(d-x)/I =$	18,996	psi

The last four values are the same (within a small round off) as calculated in Example 2.2.

Example 2.13

Repeat Example 2.8 using the spreadsheet provided for Chapter 2.

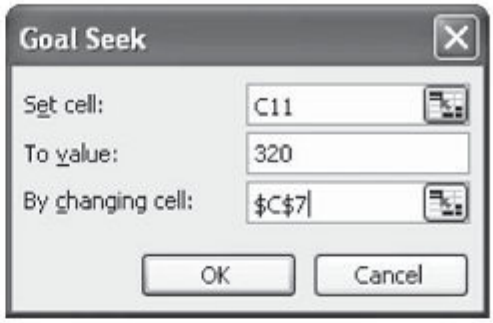
SOLUTION

Open the Chapter 2 spreadsheet, and select the worksheet called Nominal Moment Strength. Input only the cells highlighted in yellow, the first five values below.

f'_c	=	3000	psi
b	=	14	in.
d	=	21	in.
A_s	=	3	in.^{2}
f_y	=	60	ksi
a	=	5.04	
M_n	=	3326.2	in-k
M_n	=	277.2	ft-k

The third worksheet, called Nominal Moment Strength, can be used to easily work Example 2.8. In this case, enter the first five values, and the results are the same as in the example. The process can be reversed if "goal seek" is used. Suppose that you would like to know how much reinforcing steel, A_s, is needed to resist a moment, M_n, of 320 ft-k for the beam shown in Example 2.8. Highlight the

cell where M_n is calculated in ft-k (cell C11), then go to "Data" at the top of the Excel window and select "What-If Analysis" and "Goal seek … ". The Goal Seek window shown will open. In "Set cell," C11 appears because it was highlighted when you selected "Goal seek … ". In "To value," type 320 because that is the moment you are seeking. Finally, for "By changing cell," insert C7 because the area of reinforcing steel is what you want to vary to produce a moment of 320 ft-k. Click OK, and the value of A_s will change to 3.55. This means that a steel area of 3.55 in.^{2} is required to produce a moment capacity M_n of 320 ft-k. The Goal Seek feature can be used in a similar manner for most of the spreadsheets provided in this text.

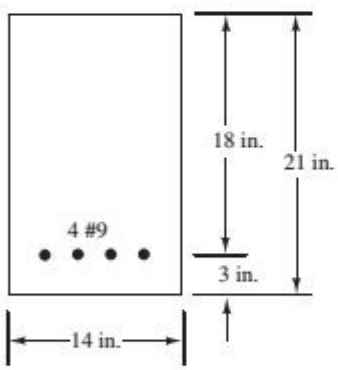


PROBLEMS

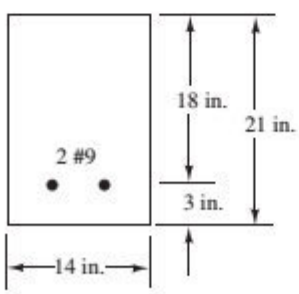
Cracking Moments

For Problems 2.1 to 2.5, determine the cracking moments for the sections shown if $f'_c = 4000$ psi and $f_r = 7.5\lambda\sqrt{f'_c}$, normal-weight concrete, $\lambda = 1.0$.

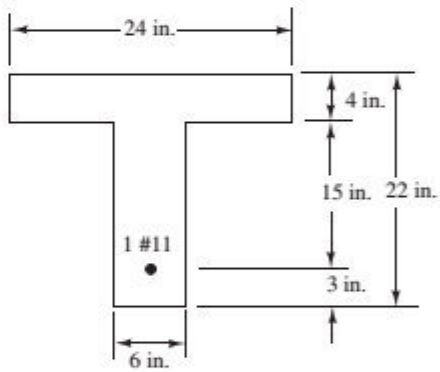
Problem 2.1 (*Ans*. 40.7 ft-k)



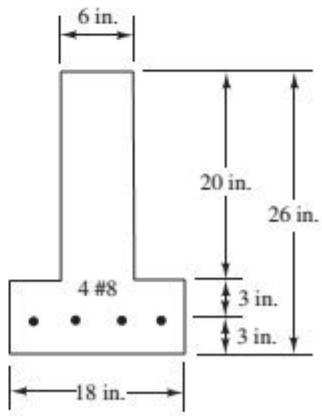
Problem 2.2



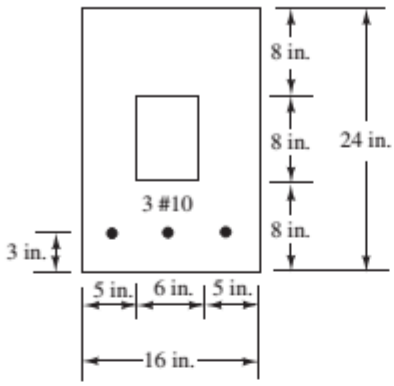
Problem 2.3 (*Ans*. 25.6 ft-k)



Problem 2.4

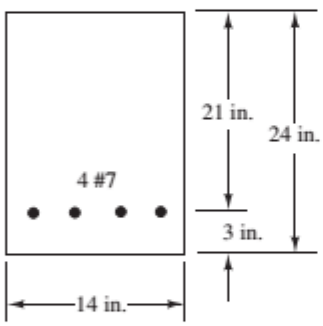


Problem 2.5 (*Ans.* 59.8 ft-k)

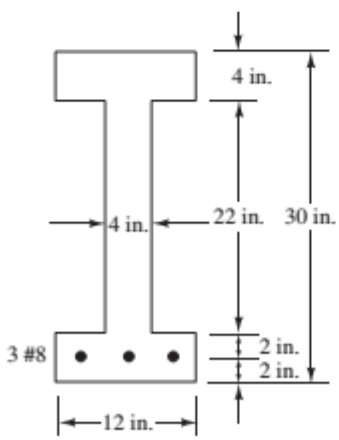


For Problems 2.6 and 2.7, calculate the uniform load (in addition to the beam weight) that will cause the sections to begin to crack if they are used for 28-ft simple spans. $f'_c = 4000$ psi, $f_r = 7.5\lambda\sqrt{f'_c}$, normal-weight concrete, $\lambda = 1.0$, and reinforced concrete weight = 150 lb/ft^3.

Problem 2.6



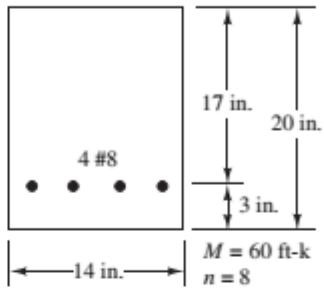
Problem 2.7 (*Ans.* 0.343 k/ft)



Transformed-Area Method

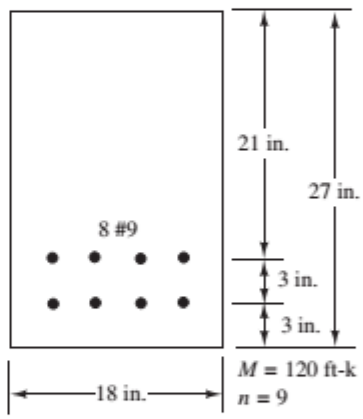
For Problems 2.8 to 2.14, assume the sections have cracked and use the transformed-area method to compute their flexural stresses for the loads or moments given.

Problem 2.8

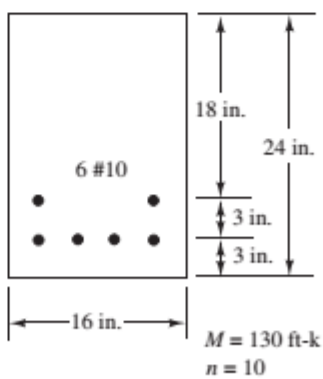


Problem 2.9 Repeat Problem 2.8 if four #7 bars are used. (*Ans.* $f_c = 1214$ psi; $f_s = 19{,}826$ psi)

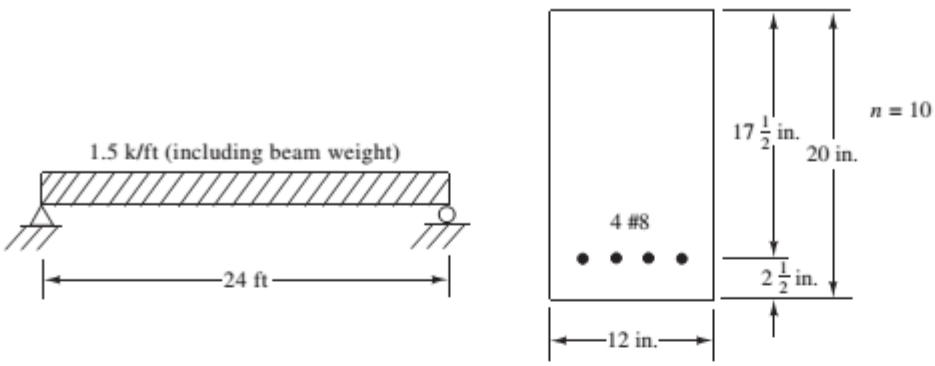
Problem 2.10



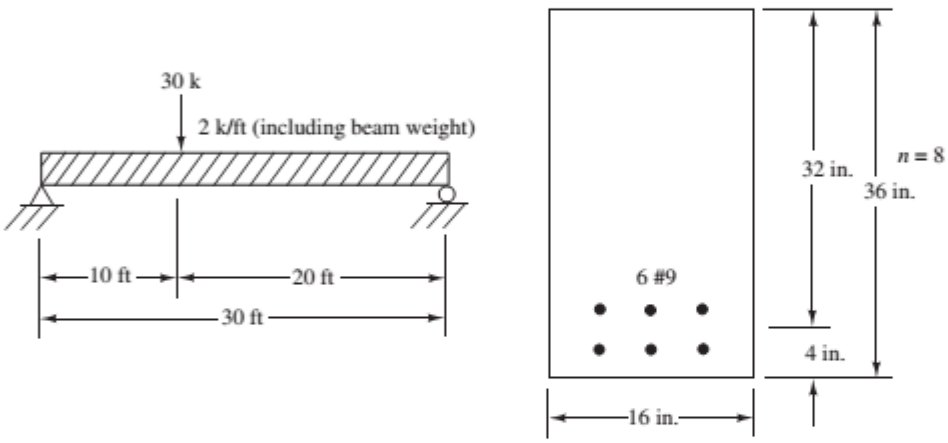
Problem 2.11 (*Ans.* $f_c = 1187$ psi; $f_s = 13{,}497$ psi in bottom layer; $f_s = 12{,}289$ psi at steel centroid)



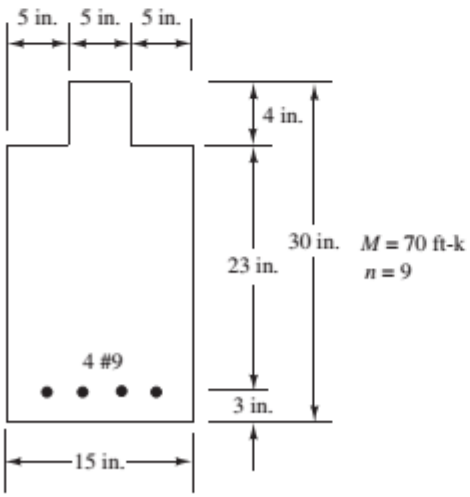
Problem 2.12



Problem 2.13 (*Ans.* $f_c = 1762$ psi; $f_s = 28{,}635$ psi at the steel centroid)



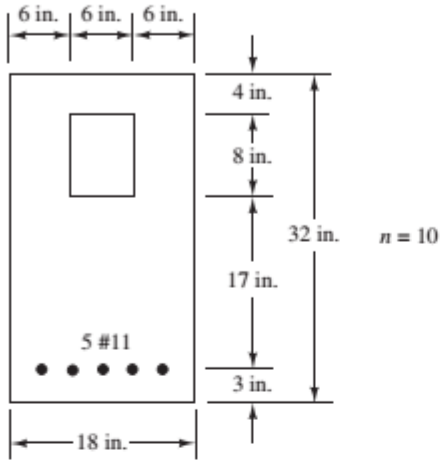
Problem 2.14



Problem 2.15 Using the transformed-area method, compute the allowable resisting moment of the beam of Problem 2.10 if allowable stresses are $f_s = 24{,}000$ psi and $f_c = 1800$ psi. (*Ans.* 258.8 ft-k)

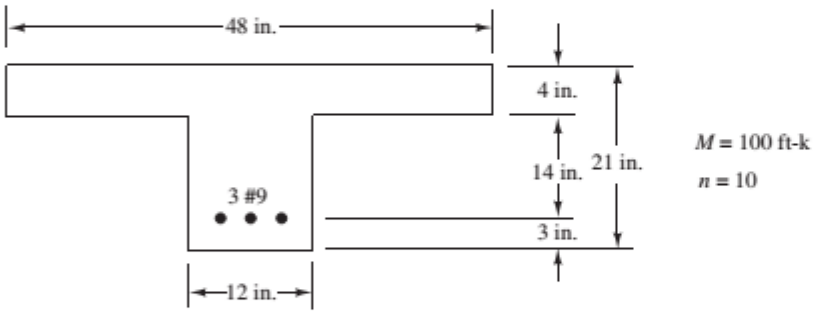
Problem 2.16 Compute the resisting moment of the beam of Problem 2.13 if eight #10 bars are used and $n = 10$, $f_s = 20{,}000$ psi, and $f_c = 1125$ psi. Use the transformed-area method.

Problem 2.17 Using the transformed area method, what allowable uniform load can this beam support in addition to its own weight for a 28-ft simple span? Concrete weight = 150 lb/ft^3, f_s = 24,000 psi, and f_c = 1800 psi. (*Ans.* 3.11 k/ft)

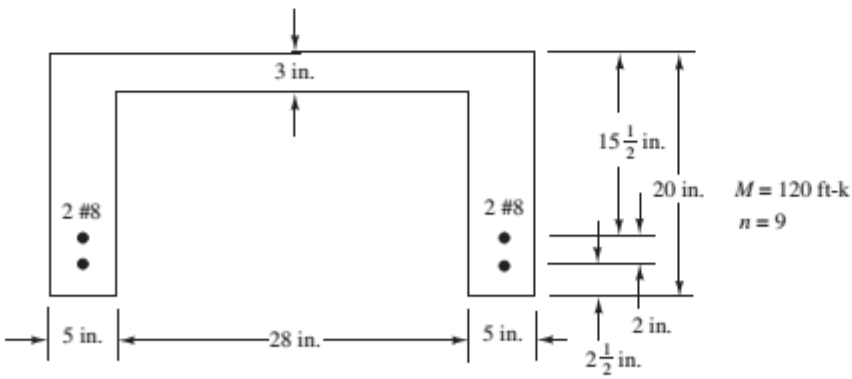


For Problems 2.18 to 2.21, determine the flexural stresses in these members using the transformed-area method.

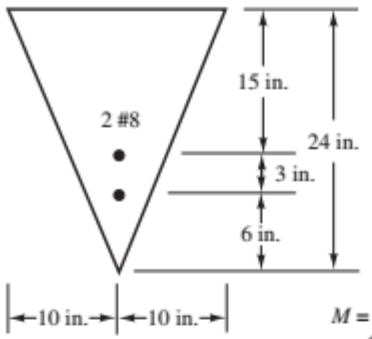
Problem 2.18



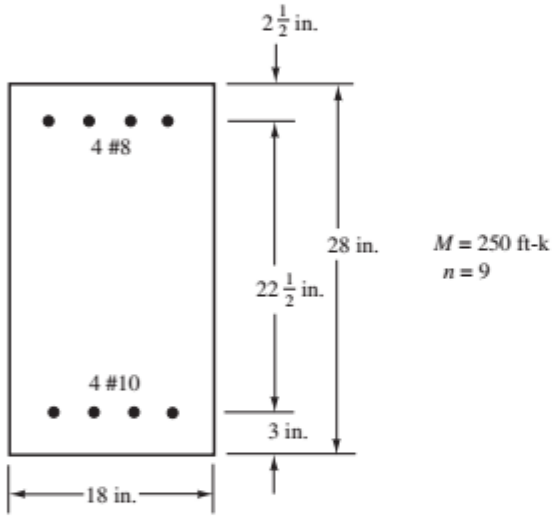
Problem 2.19 (*Ans.* f_c = 1223 psi; f_s = 30,185 psi at the steel centroid)



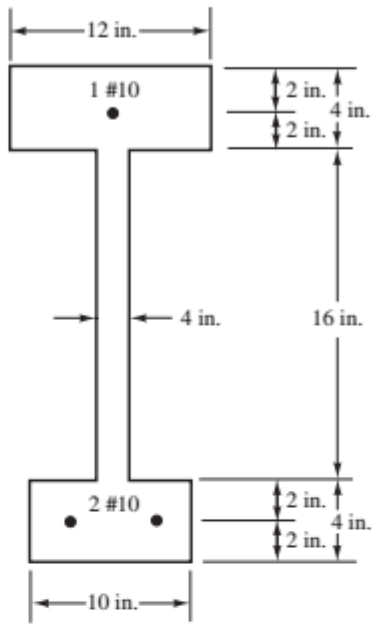
Problem 2.20



Problem 2.21 (*Ans.* $f_c = 1280$ psi; $f'_s = 15{,}463$ psi; $f_s = 26{,}378$ psi)

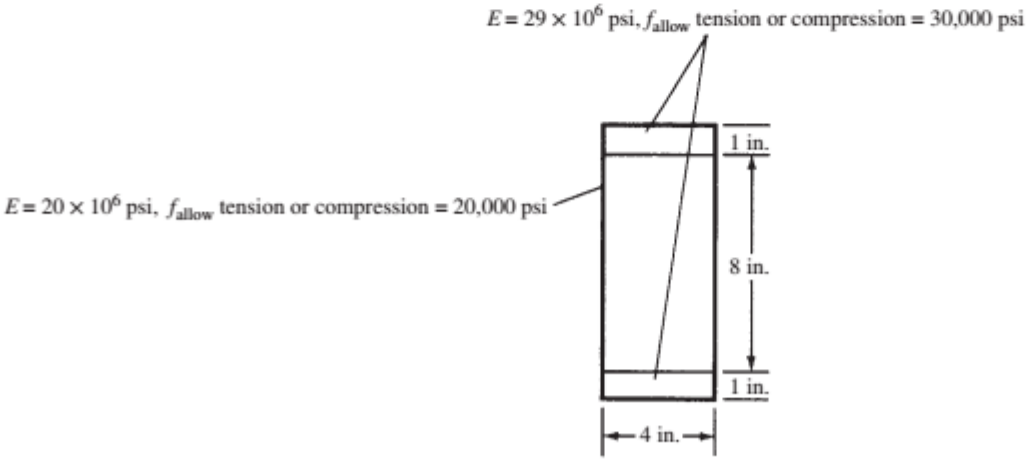


Problem 2.22 Compute the allowable resisting moment of the section shown using the transformed area method if allowable stresses are $f_c = 1800$ psi, $f_s = f'_s = 24{,}000$ psi, and $n = 8$.

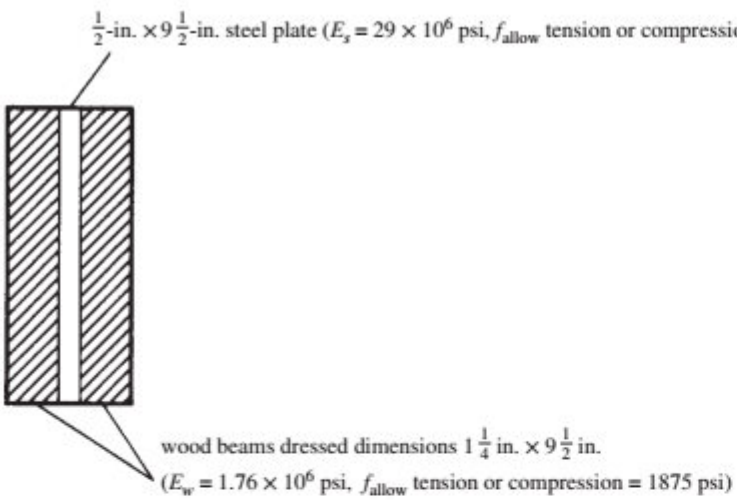


For Problems 2.23 to 2.25, using the transformed-area method, determine the allowable resisting moments of the sections shown.

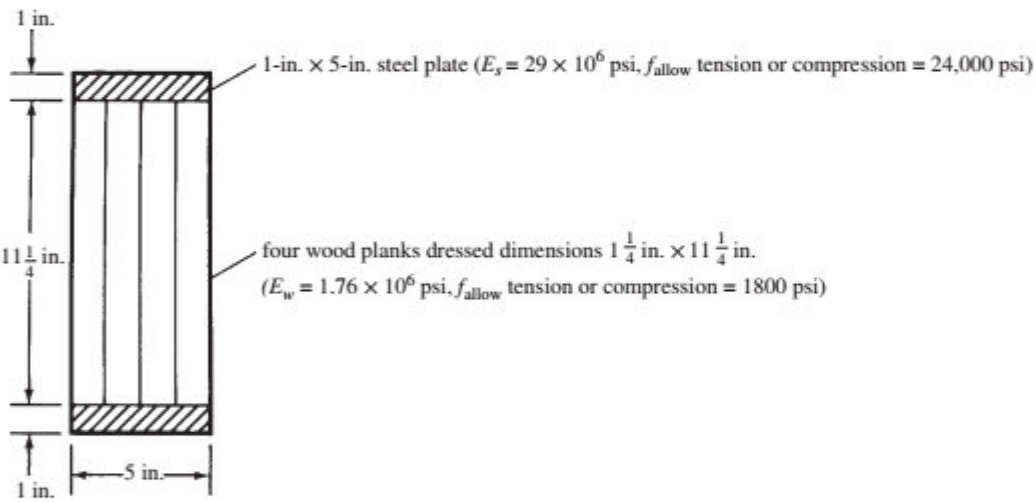
Problem 2.23 (*Ans.* 140.18 ft-k)



Problem 2.24


$\frac{1}{2}$-in. × 9 $\frac{1}{2}$-in. steel plate ($E_s = 29 \times 10^6$ psi, f_{allow} tension or compression = 24,000 psi)
wood beams dressed dimensions 1 $\frac{1}{4}$ in. × 9 $\frac{1}{2}$ in.
($E_w = 1.76 \times 10^6$ psi, f_{allow} tension or compression = 1875 psi)

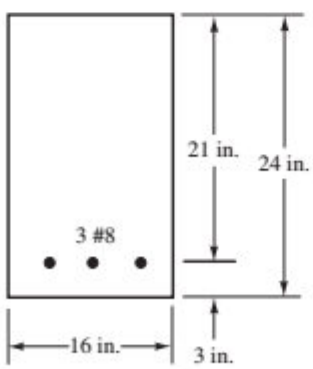
Problem 2.25 (*Ans.* 124.4 ft-k)


1 in.
1-in. × 5-in. steel plate ($E_s = 29 \times 10^6$ psi, f_{allow} tension or compression = 24,000 psi)
11 $\frac{1}{4}$ in.
four wood planks dressed dimensions 1 $\frac{1}{4}$ in. × 11 $\frac{1}{4}$ in.
($E_w = 1.76 \times 10^6$ psi, f_{allow} tension or compression = 1800 psi)
5 in.
1 in.

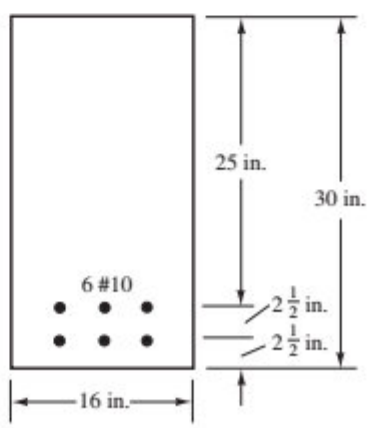
Nominal Strength Analysis

For Problems 2.26 to 2.29, determine the nominal or theoretical moment capacity M_n of each beam if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.

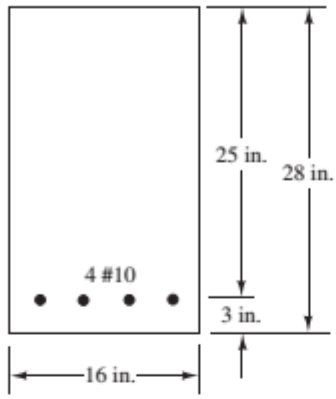
Problem 2.26



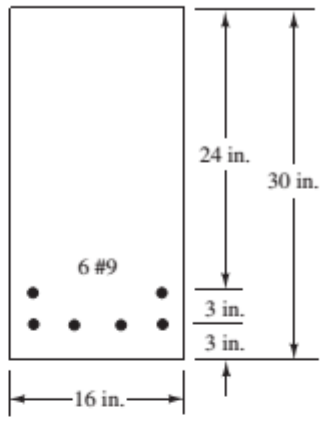
Problem 2.27 (*Ans.* 837.3 ft-k)



Problem 2.28



Problem 2.29 (*Ans.* 680.7 ft-k)

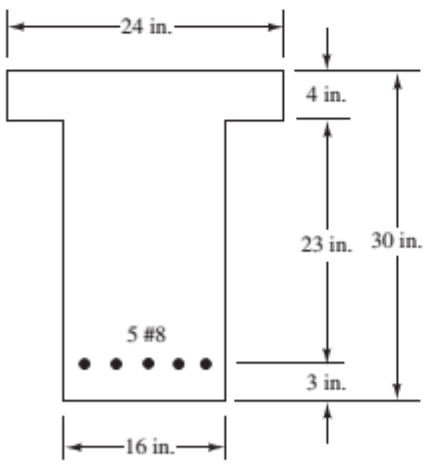


For Problems 2.30 to 2.34, determine the nominal moment capacity M_n for each of the rectangular beams.

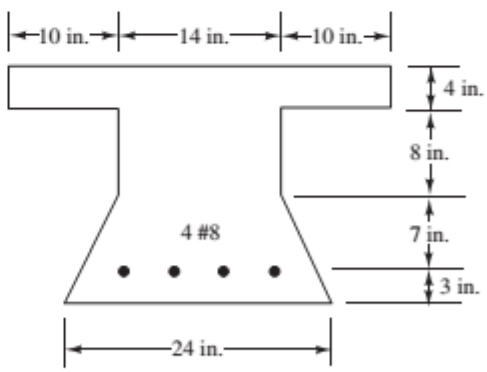
Problem No.	b (in.)	d (in.)	Bars	f'_c (ksi)	f_y (ksi)	Ans.
2.30	14	21	3 #9	4.0	60	—
2.31	18	28.5	8 #10	4.0	60	1191.1 ft-k
2.32	14	20.5	4 #10	5.0	60	—
2.33	18	25.5	4 #11	5.0	75	876.4 ft-k
2.34	22	36	6 #11	3.0	60	—

For Problems 2.35 to 2.39, determine M_n if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.

Problem 2.35 (*Ans.* 502.2 ft-k)



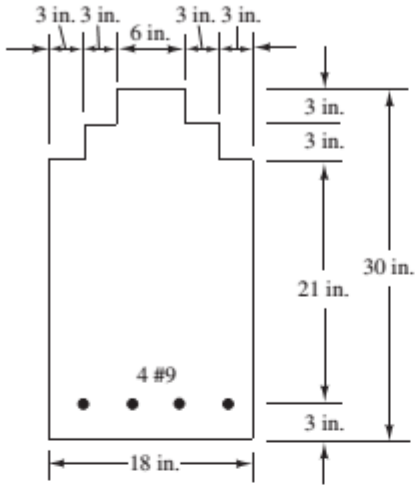
Problem 2.36



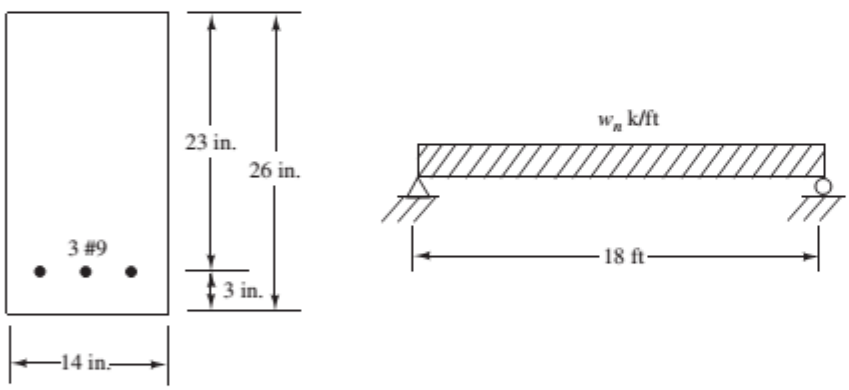
Problem 2.37 Repeat Problem 2.35 if 4 #11 bars are used. (*Ans.* 771.3 ft-k)

Problem 2.38 Compute M_n for the beam of Problem 2.36 if 6 #8 bars are used.

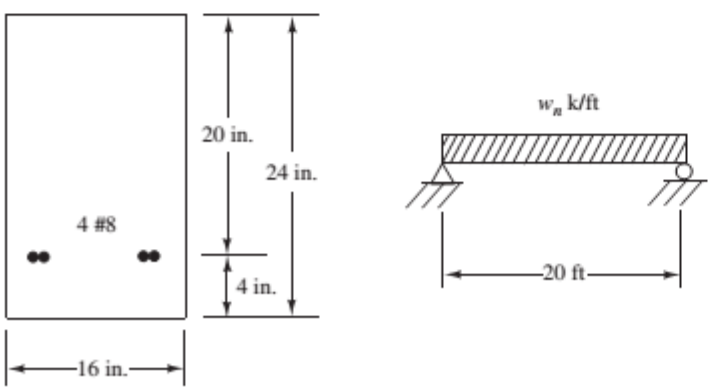
Problem 2.39 (*Ans.* 456 ft-k)



Problem 2.40 Determine the nominal uniform load w_n (including beam weight) that will cause a bending moment equal to M_n. $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.



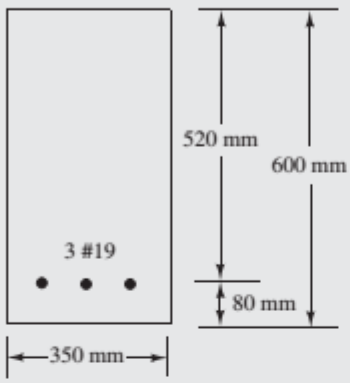
Problem 2.41 Determine the nominal uniform load w_n (including beam weight) that will cause a bending moment equal to M_n. $f'_c = 3000$ psi and $f_y = 60{,}000$ psi. (*Ans.* 5.74 k/ft)



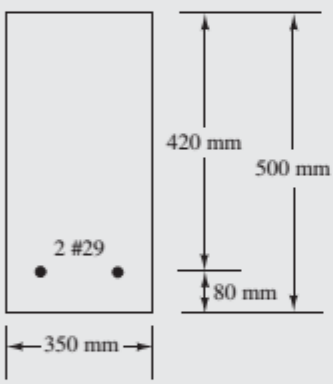
Problems in SI Units

For Problems 2.42 to 2.44, determine the cracking moments for the sections shown if $f'_c = 28$ MPa and the modulus of rupture is $f_r = 0.7\sqrt{f'_c}$ with f'_c in MPa.

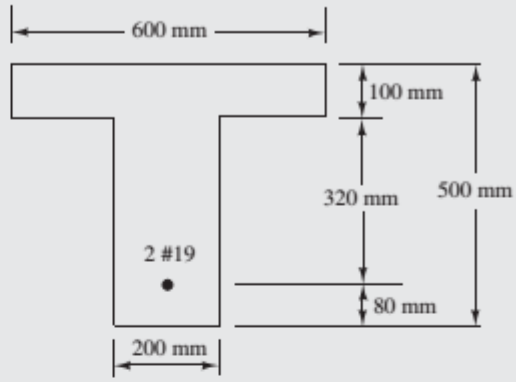
Problem 2.42



Problem 2.43 (*Ans.* 54.0 kN·m)

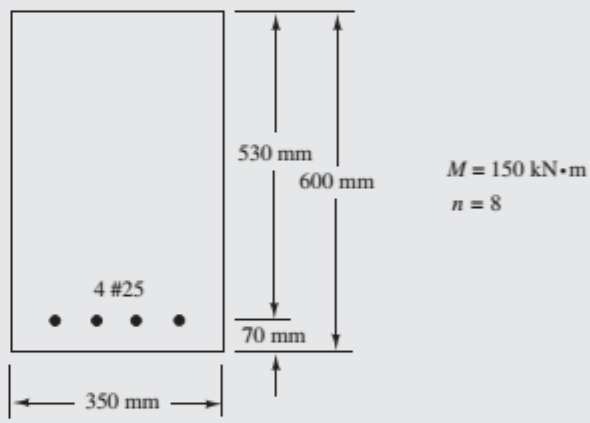


Problem 2.44

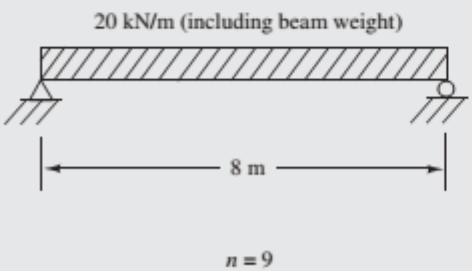


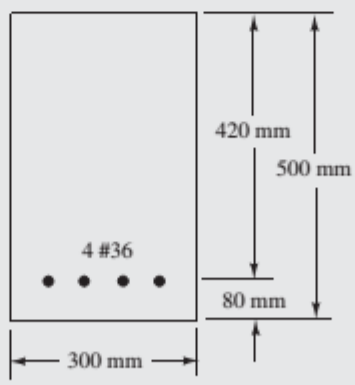
For Problems 2.45 to 2.47, compute the flexural stresses in the concrete and steel for the beams shown using the transformed-area method.

Problem 2.45 (*Ans.* $f_c = 10.07$ MPa; $f_s = 155.9$ MPa)

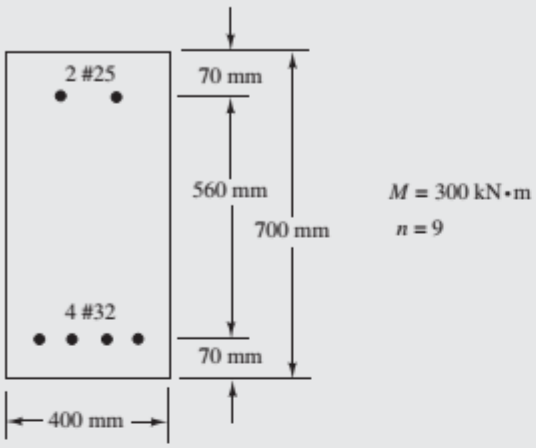


Problem 2.46





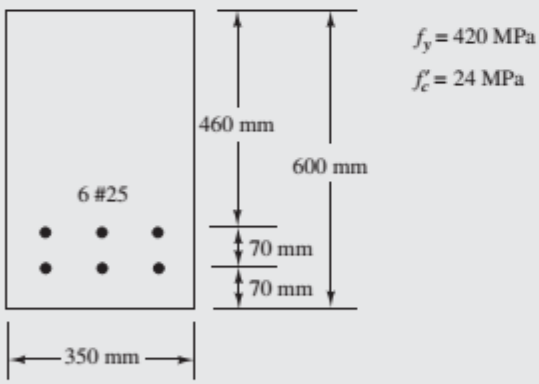
Problem 2.47 (*Ans.* $f_c = 9.68$ MPa; $f'_s = 118.4$ MPa; $f_s = 164.1$ MPa)



For Problems 2.48 to 2.55, compute M_n values.

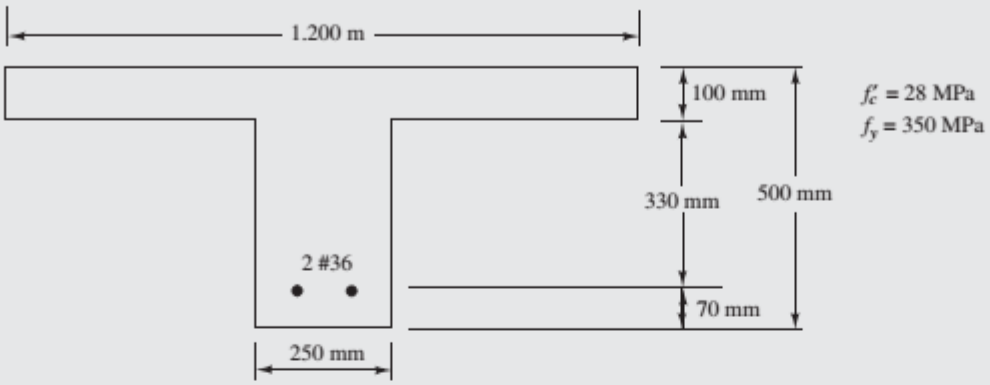
Problem No.	b (mm)	d (mm)	Bars	f'_c (MPa)	f_y (MPa)	Ans.
2.48	300	600	3 #36	35	350	—
2.49	320	600	3 #36	28	350	560.5 kN · m
2.50	350	530	3 #25	24	420	—
2.51	400	660	3 #32	42	420	644 kN · m

Problem 2.52

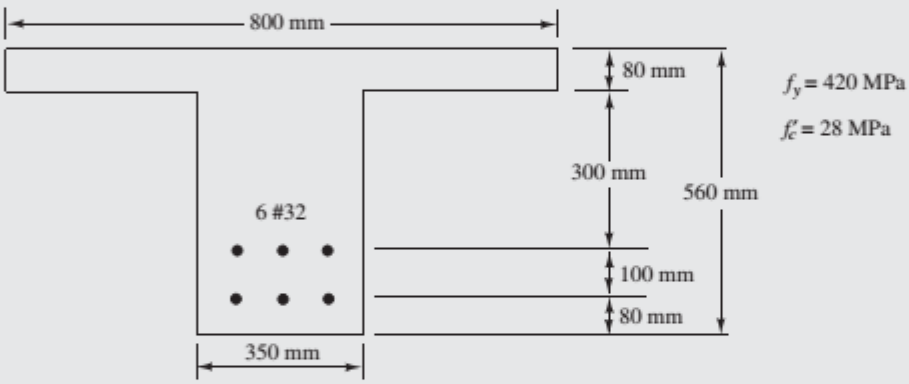


Problem 2.53 Repeat Problem 2.48 if four #36 bars are used.
(*Ans.* 734 kN·m)

Problem 2.54



Problem 2.55 (*Ans*. 766 kN•m)



Problem 2.56 Repeat Problem 2.27 using Chapter 2 spreadsheets.

Problem 2.57 Repeat Problem 2.28 using Chapter 2 spreadsheets. (*Ans*. 561.9 ft-k)

Problem 2.58 Prepare a flowchart for the determination of M_n for a rectangular tensilely reinforced concrete beam.

CHAPTER 3 Strength Analysis of Beams According to ACI Code

3.1 Design Methods

From the early 1900s until the early 1960s, nearly all reinforced concrete design in the United States was performed by the WSD method (also called *allowable-stress design* or *straight-line design*). In this method, frequently referred to as WSD, the dead and live loads to be supported, called *working loads* or *service loads*, were first estimated. Then the members of the structure were proportioned so that stresses calculated by a transformed area method did not exceed certain permissible or allowable values.

After 1963, the ultimate-strength design method rapidly gained popularity because (1) it makes use of a more rational approach than does WSD, (2) it uses a more realistic consideration of safety, and (3) it provides more economical designs. With this method (now called *strength design*), the working dead and live loads are multiplied by certain load factors (equivalent to safety factors), and the resulting values are called *factored loads*. The members are then selected so they will theoretically just fail under the factored loads.

Even though almost all of the reinforced concrete structures the reader will encounter will be designed by the strength design method, it is still useful to be familiar with WSD for several reasons:

1. Some designers use WSD for proportioning fluid-containing structures (such as water tanks and various sanitary structures). When these structures are designed by WSD, stresses are kept at fairly low levels, with the result that there is appreciably less cracking and less consequent leakage. (If the designer uses strength design and makes use of proper crack control methods, as described in Chapter 6, there should be few leakage problems.)
2. The ACI method for calculating the moments of inertia to be used for deflection calculations requires some knowledge of the working-stress procedure.
3. The design of prestressed concrete members is based not only on the strength method but also on elastic stress calculations at service load conditions.

The reader should realize that WSD has several disadvantages. When using the method, the designer has little knowledge about the magnitudes of safety factors against collapse; no consideration is given to the fact that different safety factors are desirable for dead and live loads; the method does not account for variations in resistances and loads, nor does it account for the possibility that as loads are increased, some increase at different rates than others.

In 1956, the ACI Code for the first time included ultimate-strength design, as an appendix, although the concrete codes of several other countries had been based on such considerations for several decades. In 1963, the code gave ultimate-strength design equal status with WSD; the 1971 code made the ultimate-strength design the predominant method and only briefly mentioned the working-stress method. From 1971 until 1999, each issue of the code permitted designers to use WSD and set out certain provisions for its application. *Beginning with the 2002 code, however, permission is not included for using the WSD method.*

Today's design method was called *ultimate-strength design* for several decades, but, as mentioned, the code now uses the term *strength design*. The strength for a particular reinforced concrete member

is a value given by the code and is not necessarily the true ultimate strength of the member. Therefore, the more general term *strength design* is used whether beam strength, column strength, shear strength, or others are being considered.

3.2 Advantages of Strength Design

Among the several advantages of the strength design method as compared to the no-longer-permitted WSD method are the following:

1. The derivation of the strength design expressions takes into account the nonlinear shape of the stress–strain diagram. When the resulting equations are applied, decidedly better estimates of load-carrying ability are obtained.
2. With strength design, a more consistent theory is used throughout the designs of reinforced concrete structures. For instance, with WSD the transformed-area or straight-line method was used for beam design, and a strength design procedure was used for columns.
3. A more realistic factor of safety is used in strength design. The designer can certainly estimate the magnitudes of the dead loads that a structure will have to support more accurately than he or she can estimate the live and environmental loads. With WSD, the same safety factor was used for dead, live, and environmental loads. This is not the case for strength design. For this reason, use of different load or safety factors in strength design for the different types of loads is a definite improvement.
4. A structure designed by the strength method will have a more uniform safety factor against collapse throughout. The strength method takes considerable advantage of higher-strength steels, whereas WSD did so only partly. The result is better economy for strength design.
5. The strength method permits more flexible designs than did the working-stress method. For instance, the percentage of steel may be varied quite a bit. As a result, large sections may be used with small percentages of steel, or small sections may be used with large percentages of steel. Such variations were not the case in the relatively fixed working-stress method. If the same amount of steel is used in strength design for a particular beam as would have been used with WSD, a smaller section will result. If the same size section is used as required by WSD, a smaller amount of steel will be required.

3.3 Structural Safety

The structural safety of a reinforced concrete structure can be calculated with two methods. The first method involves calculations of the stresses caused by the working or service loads and their comparison with certain allowable stresses. Usually the safety factor against collapse when the working-stress method was used was said to equal the smaller of f'_c/f_c or f'_y/f_s.

The second approach to structural safety is the one used in strength design in which uncertainty is considered. The working loads are multiplied by certain load factors that are larger than 1. The resulting larger or factored loads are used for designing the structure. The values of the load factors vary depending on the type and combination of the loads.

To accurately estimate the ultimate strength of a structure, it is necessary to take into account the uncertainties in material strengths, dimensions, and workmanship. This is done by multiplying the theoretical ultimate strength (called the *nominal strength* herein) of each member by the *strength reduction factor*, ϕ, which is less than 1. These values generally vary from 0.90 for bending down to 0.65 for some columns.

In summary, the strength design approach to safety is to select a member whose computed ultimate load capacity multiplied by its strength reduction factor will at least equal the sum of the service loads multiplied by their respective load factors.

Courtesy of Dayton Superior.

Water Tower Place, Chicago, Illinois, tallest reinforced concrete building in the United States (74 stories, 859 ft).

Member capacities obtained with the strength method are appreciably more accurate than member capacities predicted with the working-stress method.

3.4 Derivation of Beam Expressions

Tests of reinforced concrete beams confirm that strains vary in proportion to distances from the neutral axis even on the tension sides and even near ultimate loads. Compression stresses vary approximately in a straight line until the maximum stress equals about $0.50f'_c$. This is not the case, however, after stresses go higher. When the ultimate load is reached, the strain and stress variations are approximately as shown in Figure 3.1.

The compressive stresses vary from zero at the neutral axis to a maximum value at or near the extreme fiber. The actual stress variation and the actual location of the neutral axis vary somewhat from

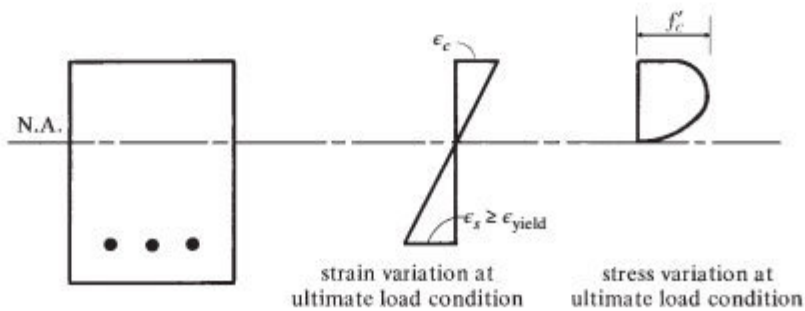


FIGURE 3.1 Nonlinear stress distribution at ultimate conditions.

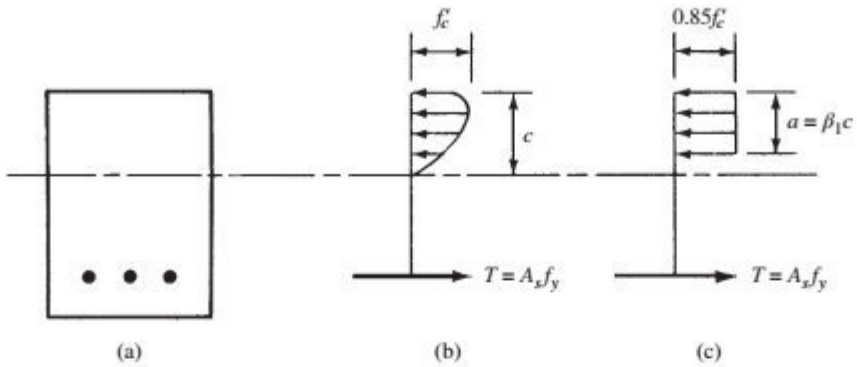


FIGURE 3.2 Some possible stress distribution shapes.

beam to beam, depending on such variables as the magnitude and history of past loadings, shrinkage and creep of the concrete, size and spacing of tension cracks, speed of loading, and so on.

If the shape of the stress diagram were the same for every beam, it would be possible to derive a single rational set of expressions for flexural behavior. Because of these stress variations, however, it is necessary to base the strength design on a combination of theory and test results.

Although the actual stress distribution given in Figure 3.2(b) may seem to be important, in practice any assumed shape (rectangular, parabolic, trapezoidal, etc.) can be used if the resulting equations compare favorably with test results. The most common shapes proposed are the rectangle, parabola, and trapezoid, with the rectangular shape [used in this text as shown in Figure 3.2(c)] being the most common one.

If the concrete is assumed to crush at a strain of about 0.003 (which is a little conservative for most concretes) and the steel to yield at f_y, it is possible to make a reasonable derivation of beam formulas without knowing the exact stress distribution. However, it is necessary to know the value of the total compression force and its centroid.

Whitney[1] replaced the curved stress block [Figure 3.2(b)] with an equivalent rectangular block of intensity $0.85f'_c$ and depth $a = \beta_1 c$, as shown in Figure 3.2(c). The area of this rectangular block should equal that of the curved stress block, and the centroids of the two blocks should coincide. Sufficient test results are available for concrete beams to provide the depths of the equivalent rectangular stress blocks. The values of β_1 given by the code (Table 22.2.2.4.3) are intended to give this result. For f'_c values of 4000 psi or less, $\beta_1 = 0.85$, and it is to be reduced continuously at a rate of 0.05 for each 1000-psi increase in f'_c above 4000 psi. Their value may not be less than 0.65. The values of β_1 are reduced for high-strength concretes primarily because of the shapes of their stress–strain curves (see Figure 1.1 in Chapter 1).

For concretes with $f'_c > 4000$ psi, β_1 can be determined with the following formula:

$$\beta_1 = 0.85 - \left(\frac{f'_c - 4000 \text{ psi}}{1000}\right)(0.05) \geq 0.65$$

In SI units, β_1 is to be taken equal to 0.85 for concrete strengths up to and including 30 MPa. For strengths above 30 MPa, β_1 is to be reduced continuously at a rate of 0.05 for each 7 MPa of strength in excess of 30 MPa but shall not be taken less than 0.65.

For concretes with $f'_c > 30$ MPa, β_1 can be determined with the following expression:

$$\beta_1 = 0.85 - 0.008\,(f'_c - 30 \text{ MPa}) \geq 0.65$$

[1] Whitney, C. S., 1942, "Plastic Theory of Reinforced Concrete Design," *Transactions ASCE*, 107, pp. 251–326.

Based on these assumptions regarding the stress block, equations of statics can easily be written for the sum of the horizontal forces and for the resisting moment produced by the internal couple. These expressions can then be solved separately for a and for the moment, M_n.

A very clear statement should be made here regarding the term M_n because it otherwise can be confusing to the reader. M_n is defined as the theoretical or nominal resisting moment of a section. In Section 3.3, it was stated that the usable strength of a member equals its theoretical strength times the strength reduction factor, or, in this case, ϕM_n. The usable flexural strength of a member, ϕM_n, must at least be equal to the calculated factored moment, M_u, caused by the factored loads

$$\phi M_n \geq M_u$$

For writing the beam expressions, reference is made to Figure 3.3. Equating the horizontal forces C and T and solving for a, we obtain

$$0.85 f'_c ab = A_s f_y$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{\rho f_y d}{0.85 f'_c}, \text{ where } \rho = \frac{A_s}{bd} = \text{percentage of tensile steel}$$

Because the reinforcing steel is limited to an amount such that it will yield well before the concrete reaches its ultimate strength, the value of the nominal moment, M_n, can be written as

$$M_n = T\left(d - \frac{a}{2}\right) = A_s f_y \left(d - \frac{a}{2}\right)$$

and the usable flexural strength is

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) \qquad \text{(Equation 3-1)}$$

If we substitute into this expression the value previously obtained for a (it was $\rho f_y d/0.85 f'_c$), replace A_s with ρbd, and equate ϕM_n to M_u, we obtain the following expression:

$$\phi M_n = M_u = \phi b d^2 f_y \rho \left(1 - \frac{\rho f_y}{1.7 f'_c}\right) \qquad \text{(Equation 3-2)}$$

Replacing A_s with ρbd and letting $R_n = M_u/\phi bd^2$, we can solve this expression for ρ (the percentage of steel required for a particular beam) with the following results:

$$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right) \qquad \text{(Equation 3-3)}$$

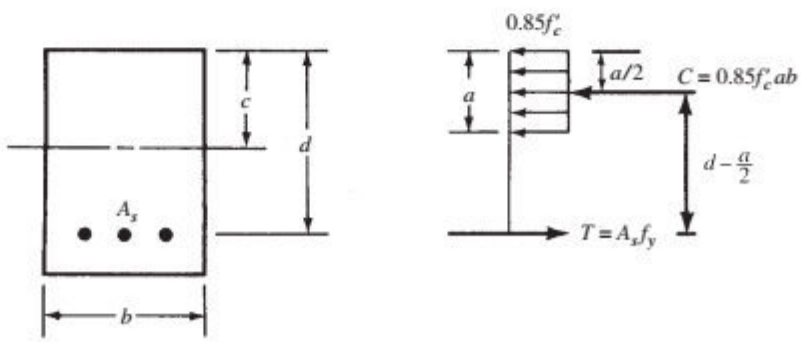


FIGURE 3.3 Beam internal forces at ultimate conditions.

Instead of substituting into this equation for ρ when rectangular sections are involved, the reader will find Tables A.8 to A.13 in Appendix A of this text to be quite convenient. (For SI units, refer to Tables B.8 and B.9 in Appendix B.) Another way to obtain the same information is to refer to Graph 1 in Appendix A. The user, however, will have some difficulty in reading this small-scale graph accurately. This expression for ρ is also very useful for tensilely reinforced rectangular sections having only tension reinforcing steel that do not fall into the tables. An iterative technique for determination of reinforcing steel area is also presented later in this chapter.

3.5 Strains in Flexural Members

As previously mentioned, Section 22.2.1.2 of the code states that the strains in concrete members and their reinforcement are to be assumed to vary directly with distances from their neutral axes. (This assumption is not applicable to deep flexural members whose depths over their clear spans are greater than 0.25.) Furthermore, in Section 22.2.2.1 the code states that the maximum usable strain in the extreme compression fibers of a flexural member is to be 0.003. Finally, Section 21.2.2.1 states that for Grade 60 reinforcement and for all prestressed reinforcement we may set the strain in the steel equal to 0.002 at the balanced condition. (Theoretically, for 60,000-psi steel, it equals $f_y/E_s = 60{,}000 \text{ psi}/29 \times 10^6 \text{ psi} = 0.00207$.)

In Section 3.4, a value was derived for a, the depth of the equivalent stress block of a beam. It can be related to c with the factor β_1 also given in that section:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \beta_1 c$$

Then the distance c from the extreme concrete compression fibers to the neutral axis is

$$c = \frac{a}{\beta_1}$$

In Example 3.1, the values of a and c are determined for the beam previously considered in Example 2.8, and by straight-line proportions the strain in the reinforcing ϵ_t is computed.

Example 3.1

Determine the values of a, c, and ϵ_t for the beam shown in Figure 3.4. $f_y = 60{,}000$ psi and $f'_c = 3000$ psi.

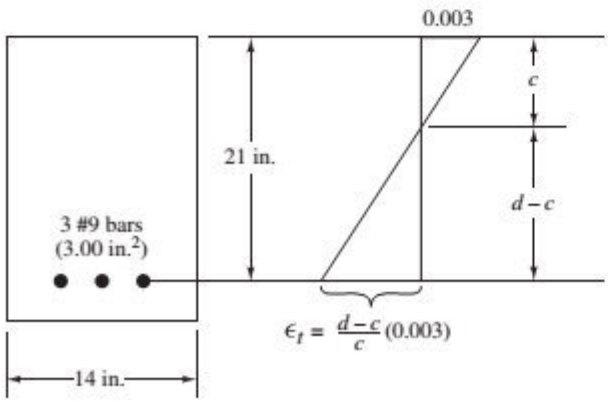


FIGURE 3.4 Beam cross section for Example 3.1.

SOLUTION

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(3.00 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(3 \text{ ksi})(14 \text{ in.})} = \underline{\underline{5.04 \text{ in.}}}$$

$\beta_1 = 0.85$ for 3000-psi concrete

$$c = \frac{a}{\beta_1} = \frac{5.04 \text{ in.}}{0.85} = \underline{\underline{5.93 \text{ in.}}}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \left(\frac{21 \text{ in.} - 5.93 \text{ in.}}{5.93 \text{ in.}}\right)(0.003) = \underline{\underline{0.00762}}$$

This value of strain is much greater than the yield strain of 0.002. This is an indication of ductile behavior of the beam, because the steel is well into its yield plateau before concrete crushes.

3.6 Balanced Sections, Tension-Controlled Sections, and Compression-Controlled or Brittle Sections

A beam that has a *balanced steel ratio* is one for which the tensile steel will theoretically just reach its yield point at the same time the extreme compression concrete fibers attain a strain equal to 0.003. Should a flexural member be so designed that it has a balanced steel ratio or be a member whose compression side controls (i.e., if its compression strain reaches 0.003 before the steel yields), the member can suddenly fail without warning. As the load on such a member is increased, its deflections will usually not be particularly noticeable, even though the concrete is highly stressed in compression and failure will probably occur without warning to users of the structure. These members are *compression controlled* and are referred to as *brittle members*. Obviously, such members must be avoided.

The code, in Section 21.2.2, states that members whose computed tensile strains are equal to or greater than 0.0050 and at the same time whose concrete strains are 0.003 are to be referred to as *tension-controlled sections*. For such members, the steel will yield before the compression side crushes and deflections will be large, giving users warning of impending failure. Furthermore, members with $\epsilon_t \geq 0.005$ are considered to be fully ductile. The ACI chose the 0.005 value for ϵ_t to apply to all types of steel permitted by the code, whether regular or prestressed. The code further states that members that have net steel strains or ϵ_t values between ϵ_{ty} and 0.005 are in a transition region between compression-controlled and tension-controlled sections. For Grade 60 reinforcing steel, which is quite common, ϵ_{ty} may be approximated by 0.002 (ACI 318, Section 21.2.2.1).

3.7 Strength Reduction or ϕ Factors

Strength reduction factors are used to take into account the uncertainties of material strengths, inaccuracies in the design equations, approximations in analysis, possible variations in dimensions of the concrete sections and placement of reinforcement, the importance of members in the structures of which they are part, and so on. The ACI 318 Code (21.2.1) prescribes ϕ values or strength reduction factors for most situations. Among these values are the following:

0.90 for tension-controlled beams and slabs

0.75 for shear and torsion in beams

0.65 or 0.75 for columns

0.65 or 0.75 to 0.9 for columns supporting very small axial loads

0.65 for bearing on concrete

The sizes of these factors are rather good indications of our knowledge of the subject in question. For instance, calculated nominal moment capacities in reinforced concrete members seem to be quite accurate, whereas computed bearing capacities are more questionable.

For ductile or tension-controlled beams and slabs where $\epsilon_t \geq 0.005$, the value of ϕ for bending is 0.90. Should ϵ_t be less than 0.005, it is still possible to use the sections if ϵ_t is not less than certain values. This situation is shown in Figure 3.5, which is similar to Figure R21.2.2(b) in the Commentary to the ACI 318-14 Code.

Members subject to axial loads equal to or less than $0.10f'_cA_g$ may be used only when ϵ_t is no lower than 0.004 (ACI 318, Section 9.3.3.1). An important implication of this limit is that reinforced concrete beams must have a tension strain of at least 0.004. Should the members be subject to axial loads $\geq 0.10f'_cA_g$, then ϵ_t is not limited. When ϵ_t values fall between ϵ_{yt} and 0.005, they are said to be in the transition range between tension-controlled and compression-controlled sections. In this range, ϕ values will fall between 0.65 or 0.75 and 0.90, as shown in Figure 3.5. When $\epsilon_t \leq \epsilon_{yt}$, the member is compression controlled, and the column ϕ factors apply.

The procedure for determining ϕ values in the transition range is described later in this section. *You must clearly understand that the use of flexural members in this range is usually uneconomical, and it is probably better, if the situation permits, to increase member depths and/or decrease steel percentages until ϵ_t is equal to or larger than 0.005.* If this is done, not only will ϕ values equal 0.9 but also steel percentages will not be so large as to cause crowding of reinforcing bars. The net result will be slightly larger concrete sections, with consequent smaller deflections. Furthermore, as you will learn in subsequent chapters, the bond of the reinforcing to the concrete will be increased as compared to cases where higher percentages of steel are used.

We have computed values of steel percentages for different grades of concrete and steel for which ϵ_t will exactly equal 0.005 and present them in Appendix Tables A.7 and B.7 of this textbook. It is desirable, under ordinary conditions, to design beams with steel percentages that are no larger than these values, and we have shown them as suggested maximum percentages to be used.

The resulting general equations for ϕ in the range $\epsilon_{yt} < \epsilon_t < 0.005$ are

$$\phi = 0.75 + (\epsilon_t - \epsilon_{yt})\frac{0.15}{(0.005 - \epsilon_{yt})} \qquad \textit{for spiral members}$$

and

$$\phi = 0.65 + (\epsilon_t - \epsilon_{yt})\frac{0.25}{(0.005 - \epsilon_{yt})} \qquad \textit{for other members}$$

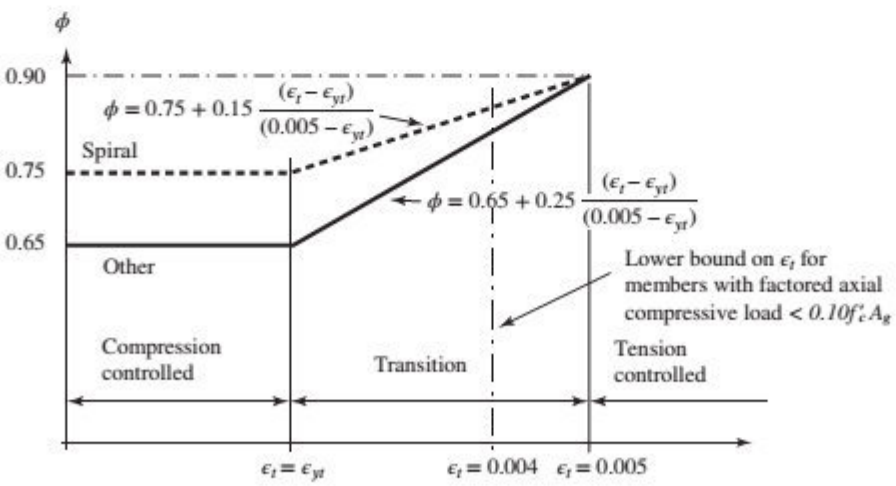


FIGURE 3.5 Variation of ϕ with net tensile strain, ϵ_t.

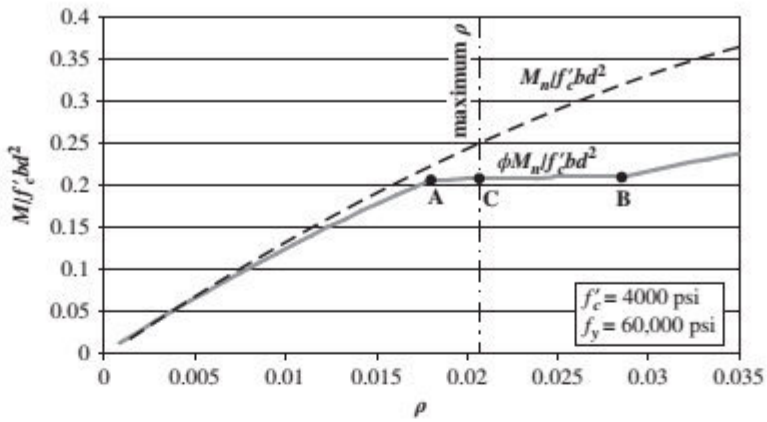


FIGURE 3.6 Moment capacity versus ρ.

The impact of the variable ϕ factor on moment capacity is shown in Figure 3.6. The two curves show the moment capacity with and without the application of the ϕ factor. Point A corresponds to a tensile strain, ϵ_t, of 0.005 and $\rho = 0.0181$ (Appendix A, Table A.7). This is the largest value of ρ for $\phi = 0.9$. Above this value of ρ, ϕ decreases to as low as 0.65 as shown by point B, which corresponds to ϵ_t of ϵ_{yt}. ACI 9.3.3.1 requires ϵ_t not be less than 0.004 for flexural members with compressive axial loads less than $0.10 f'_m A_g$. This situation corresponds to point C in Figure 3.6. The only allowable range for ρ is left of point C. From the figure, it is clear that little moment capacity is gained in adding steel area above point A. The variable ϕ factor provisions essentially permit a constant value of ϕM_n when ϵ_t is less than 0.005. It is important for the designer to know this because often actual bar selections result in more steel area than theoretically required. If the slope between points A and C were negative, the designer could not use a larger area. Knowing the slope is slightly positive, the designer can use the larger bar area with confidence that the design capacity is not reduced.

For values of f_y of 75 ksi and higher, the slope between point A and B in Figure 3.6 can actually be negative. It is therefore especially important, when using high-strength reinforcing steel, to verify your final design to be sure the bars you have selected do not result in a moment capacity less than the design value.

Continuing our consideration of Figure 3.5, you can see that when ϵ_t is less than 0.005, the values of ϕ will vary along a straight line from their 0.90 value for ductile sections to 0.65 at balanced conditions where ϵ_t is ϵ_{yt}. Later you will learn that ϕ can equal 0.75 rather than 0.65 at this latter strain situation if spirally reinforced sections are being considered.

3.8 Minimum Percentage of Steel

A brief discussion of the modes of failure that occur for various reinforced beams was presented in Section 3.6. Sometimes, because of architectural or functional requirements, beam dimensions are selected that are much larger than are required for bending alone. Such members theoretically require very small amounts of reinforcing.

Actually, another mode of failure can occur in very lightly reinforced beams. If the ultimate resisting moment of the section is less than its cracking moment, the section will fail immediately when a crack occurs. This type of failure may occur without warning. To prevent such a possibility, ACI 318 (9.6.1.2) specifies a certain minimum amount of reinforcing that must be used at every section of flexural members where tensile reinforcing is required by analysis, whether for positive or negative moments. In the following equations, b_w represents the web width of beams. In the case of a statically determinate beam with a flange in tension, the value of b_w is the lesser of b_f and $2b_w$. *The effective flange width, b_f,*

as defined in ACI 318 (6.3.2.1) applies to T beams and includes the width of the web plus the effective overhanding flange width.

$$A_{s\,\min} = \frac{3\sqrt{f'_c}}{f_y} b_w d$$

$$\text{nor less than } \frac{200 b_w d}{f_y}$$

In SI units, these expressions are $\left(\frac{\sqrt{f'_c}}{4 f_y}\right) b_w d$ and $\left(\frac{1.4 b_w d}{f_y}\right)$, respectively.

The $(200 b_w d)/f_y$ value was obtained by calculating the cracking moment of a plain concrete section and equating it to the strength of a reinforced concrete section of the same size, applying a safety factor of 2.5 and solving for the steel required. It has been found, however, that when f'_c is higher than about 5000 psi, this value may not be sufficient. Thus, the $(3\sqrt{f'_c}/f_y) b_w d$ value is also required to be met, and it will actually control when f'_c is greater than 4440 psi.

This ACI equation for the minimum area of flexural reinforcing can be written as a minimum reinforcement ratio, as follows:

$$\rho_{\min} \text{ for flexure} = \frac{3\sqrt{f'_c}}{f_y} \geq \frac{200}{f_y}$$

Values of $\rho_{\min}$ for flexure have been calculated by the authors and are shown for several grades of concrete and steel in Appendix A, Table A.7 of this text. They are also included in Tables A.8 to A.13. (For SI units, the appropriate tables are in Appendix B, Tables B.7 to B.9.)

Courtesy of EFCO Corp.

Wastewater treatment plant, Fountain Hills, Arizona.

Section 9.6.1.3 of the code states that the preceding minimums do not have to be met if the area of the tensile reinforcing furnished at every section is at least one-third greater than the area required by moment. A new provision in ACI 318-14 provides another exception to the minimum required reinforcement provisions. Section 9.6.2.2 exempts beams if both flexural and shear design strength are at least twice the required strength. Furthermore, ACI Section 7.6.1.1 states that for non-prestressed one-way slabs, the minimum area of tensile reinforcing in the direction of the span is that specified in ACI Section 24.4.3.2 for shrinkage and temperature steel, which is much lower. When slabs are overloaded in certain areas, there is a tendency for those loads to be distributed laterally to other parts of the slab, thus substantially reducing the chances of sudden failure. This explains why a reduction of the minimum reinforcing percentage is permitted in slabs of uniform thickness. Supported slabs, such as slabs on grade, are not considered to be structural slabs in this section unless they transmit vertical loads from other parts of the structure to the underlying soil.

3.9 Balanced Steel Percentage

In this section, an expression is derived for ρ_b, the reinforcement ratio required for a balanced design. At ultimate load for such a beam, the concrete will theoretically reach its limiting compressive strain of 0.003, and the steel will simultaneously yield (see Figure 3.7).

The neutral axis is located by the triangular strain relationships that follow, noting that $E_s = 29 \times 10^6$ psi for the reinforcing bars:

$$\frac{c_b}{d} = \frac{0.00300}{0.00300 + (f_y/E_s)} = \frac{0.00300}{0.003 + (f_y/29 \times 10^6 \text{ psi})}$$

This expression is rearranged and simplified, giving

$$c_b = \frac{87{,}000}{87{,}000 + f_y} d$$

where f_y must be in units of psi.

In Section 3.4 of this chapter, an expression was derived for depth of the compression stress block, a, by equating the values of C and T. This value can be converted to the neutral axis depth, c, by dividing it by β_1:

$$a = \frac{\rho f_y d}{0.85 f_c'}$$

$$c = \frac{a}{\beta_1} = \frac{\rho f_y d}{0.85 \beta_1 f_c'}$$

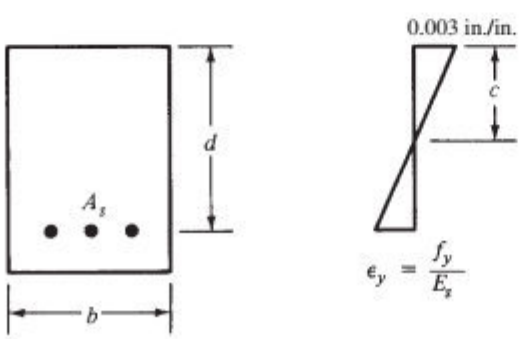


FIGURE 3.7 Balanced conditions.

Two expressions are now available for c, and they are equated to each other and solved for the percentage of steel. This is the balanced percentage, ρ_b:

$$\frac{\rho_b f_y d}{0.85\beta_1 f'_c} = \frac{87{,}000}{87{,}000 + f_y} d$$

$$\rho_b = \left(\frac{0.85\beta_1 f'_c}{f_y}\right)\left(\frac{87{,}000}{87{,}000 + f_y}\right)$$

where f'_c and f_y are in units of psi.

or in SI units $\left(\dfrac{0.85\beta_1 f'_c}{f_y}\right)\left(\dfrac{600}{600 + f_y}\right)$

where f'_c and f_y are in units of MPa.

Values of ρ_b can easily be calculated for different values of f'_c and f_y and tabulated for U.S. customary units as shown in Appendix A, Table A.7. For SI units, it's Appendix B, Table B.7.

Previous codes (1963–1999) limited flexural members to 75% of the balanced steel ratio, ρ_b. However, this approach was changed in the 2002 code to the new philosophy explained in Section 3.7, whereby the member capacity is penalized by reducing the ϕ factor when the strain in the reinforcing steel at ultimate is less than 0.005.

3.10 Example Problems

Examples 3.2 to 3.4 demonstrate the computation of the design moment capacities of three beams using the ACI Code limitations. Remember that, according to the ACI 318 (9.3.3.1), beams whose axial load is less than $0.10 f'_c A_g$ may not, when loaded to their nominal strengths, have calculated net tensile strains less than 0.004.

Example 3.2

Determine the ACI design moment capacity, ϕM_n, of the beam shown in Figure 3.8 if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

Checking Steel Percentage

$$\rho = \frac{A_s}{bd} = \frac{4.00 \text{ in.}^2}{(15 \text{ in.})\,(24 \text{ in.})} = 0.0111$$

$$\left.\begin{aligned} &> \rho_{min} = 0.0033 \\ &< \rho_{max} = 0.0181 \end{aligned}\right\} \quad \begin{aligned} &\text{both from} \\ &\text{Appendix A, Table A.7} \end{aligned}$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(4.00 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.85)(4000 \text{ psi})\,(15 \text{ in.})} = 4.71 \text{ in.}$$

$\beta_1 = 0.85$ for 4000-psi concrete

$$c = \frac{a}{\beta_1} = \frac{4.71 \text{ in.}}{0.85} = 5.54 \text{ in.}$$

Drawing Strain Diagram (Figure 3.9)

$$\epsilon_t = \frac{d-c}{c}(0.003) = \frac{18.46 \text{ in.}}{5.54 \text{ in.}}(0.003) = 0.0100$$

$$> 0.005 \quad \therefore \text{ tension controlled}$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = (4.00 \text{ in.}^2)(60 \text{ ksi})\left(24 \text{ in.} - \frac{4.71 \text{ in.}}{2}\right)$$

$$= 5194.8 \text{ in-k} = 432.9 \text{ ft-k}$$

$$\phi M_n = (0.9)(432.9 \text{ ft-k}) = \underline{\underline{389.6 \text{ ft-k}}}$$

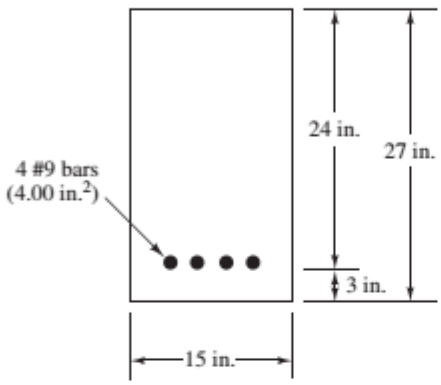


FIGURE 3.8 Beam cross section for Example 3.2.

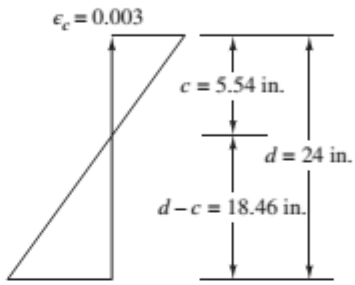


FIGURE 3.9 Neutral axis location for Example 3.2.

Example 3.3

Determine the ACI design moment capacity, ϕM_n, of the beam shown in Figure 3.10 if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

Checking Steel Percentage

$$\rho = \frac{A_s}{bd} = \frac{4.68 \text{ in.}^2}{(12 \text{ in.})(15 \text{ in.})} = 0.026 > \rho_{\min} = 0.0033$$

$> \rho_{\max} = 0.0181$ (from Appendix A, Table A.7). As a result, we know that ϵ_t will be < 0.005.

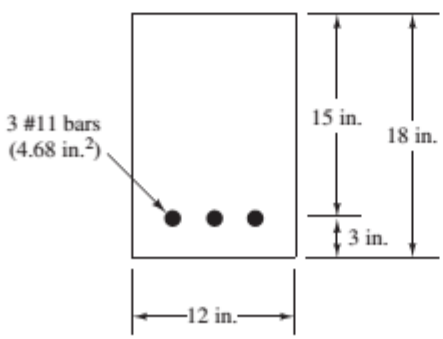


FIGURE 3.10 Beam cross section for Example 3.3.

Computing Value of ϵ_t

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(4.68 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.85)(4000 \text{ psi})(12 \text{ in.})} = 6.88 \text{ in.}$$

$$\beta_1 = 0.85 \text{ for 4000-psi concrete}$$

$$c = \frac{a}{\beta_1} = \frac{6.88 \text{ in.}}{0.85} = 8.09 \text{ in.}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \frac{15 \text{ in.} - 8.09 \text{ in.}}{8.09 \text{ in.}}(0.003)$$
$$= 0.00256 < 0.004$$

∴ Section is not ductile and may not be used as per ACI Section 9.3.3.1.

Example 3.4

Determine the ACI design moment capacity, ϕM_n, for the beam of Figure 3.11 if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

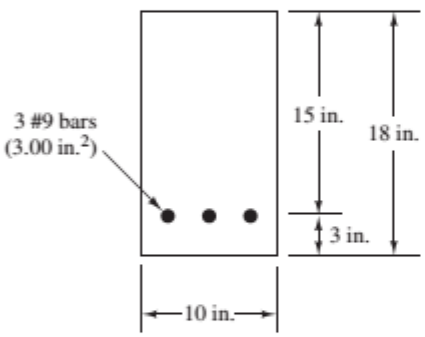


FIGURE 3.11 Beam cross section for Example 3.4.

SOLUTION

Checking Steel Percentage

$$\rho = \frac{A_s}{bd} = \frac{3.00 \text{ in.}^2}{(10 \text{ in.})(15 \text{ in.})} = 0.020 > \rho_{\min} = 0.0033$$
$$\text{but also} < \rho_{\max} = 0.0181 (\text{for } \epsilon_t = 0.005)$$

Computing Value of ϵ_t

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(3.00 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.85)(4000 \text{ psi})(10 \text{ in.})} = 5.29 \text{ in.}$$

$$\beta_1 = 0.85 \text{ for 4000-psi concrete}$$

$$c = \frac{a}{\beta_1} = \frac{5.29 \text{ in.}}{0.85} = 6.22 \text{ in.}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \left(\frac{15 \text{ in.} - 6.22 \text{ in.}}{6.22 \text{ in.}}\right)(0.003) = 0.00423 > 0.004 \text{ and} < 0.005$$

∴ Beam is in transition zone and

$$\phi \text{ (from Figure 3.5)} = 0.65 + (0.00423 - 0.002)\left(\frac{250}{3}\right) = 0.836$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = (3.00 \text{ in.}^2)(60 \text{ ksi})\left(15 \text{ in.} - \frac{5.29 \text{ in.}}{2}\right) = 2223.9 \text{ in-k} = 185.3 \text{ ft-k}$$

$$\phi M_n = (0.836)(185.3 \text{ ft-k}) = \underline{\underline{154.9 \text{ ft-k}}}$$

Example 3.5

Determine the ACI design moment capacity ϕM_n for the beam of Figure 3.12 if $f'_c = 28$ MPa and $f_y = 420$ MPa.

SOLUTION

Checking Steel Percentage

$$A_s = 3276 \text{ mm}^2$$

$$\rho = \frac{A_s}{bd} = \frac{3276 \text{ mm}^2}{(350 \text{ mm})(625 \text{ mm})} = 0.015$$

Computing Value of ϵ_t

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(3276 \text{ mm}^2)(420 \text{ MPa})}{(0.85)(28 \text{ MPa})(350 \text{ mm})} = 165.2 \text{ mm}$$

$$\beta_1 = 0.85 \text{ for 28 MPa concrete}$$

$$c = \frac{a}{\beta_1} = \frac{165.2 \text{ mm}}{0.85} = 194.3 \text{ mm}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \left(\frac{625 \text{ mm} - 194.3 \text{ mm}}{194.3 \text{ mm}}\right)(0.003) = 0.00665 > 0.005$$

∴ Beam is in tension zone and

$$\phi = 0.9$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = (3276 \text{ mm}^2)(420 \text{ MPa})\left(625 \text{ mm} - \frac{165.2 \text{ mm}}{2}\right) = 746\,300 \text{ N} \cdot \text{m}$$

$$\phi M_n = (0.9)(746\,300 \text{ N} \cdot \text{m}) = \underline{\underline{672 \text{ kN} \cdot \text{m}}}$$

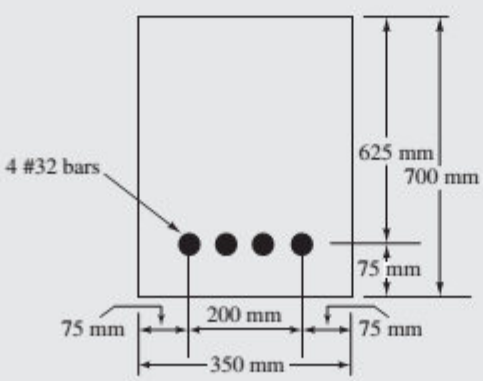


FIGURE 3.12 Beam cross section for Example 3.5.

3.11 Computer Examples

Example 3.6

Repeat Example 3.2 using the Excel spreadsheet provided for Chapter 3.

SOLUTION

Open the Chapter 3 spreadsheet, and open the Rectangular Beam worksheet. Enter values only in the cells highlighted yellow (only in the Excel spreadsheet, not the printed example). The final result is $\phi M_n = 389.6$ ft-k (same answer as for Example 3.2).

Example 3.7

Repeat Example 3.3 using the Excel spreadsheet provided for Chapter 3.

SOLUTION

Open the Chapter 3 spreadsheet and the Rectangular Beam worksheet. Enter values only in the cells highlighted yellow. The spreadsheet displays a message, "code violation … too much steel." This is an indication that the beam violates ACI Section 9.3.3.1 and is not ductile. This beam is not allowed by the ACI Code.

Example 3.8

Repeat Example 3.4 using the Excel spreadsheet provided for Chapter 3.

SOLUTION

Open the Chapter 3 spreadsheet and the Rectangular Beam worksheet. Enter values only in the cells highlighted yellow. The final result is $\phi M_n = 154.5$ ft-k (nearly the same answer as for Example 3.4). The ϕ factor is also nearly the same as in Example 3.4 (0.0834 compared with 0.0836). The difference is the result of the spreadsheet using the more general value for ϵ_y of $f_y/E_s = 0.00207$ instead of the approximate value of 0.002 permitted by the code for Grade 60 reinforcing steel. A difference of this magnitude is not important, as discussed in Section 1.26, "Calculation Accuracy."

PROBLEMS

Problem 3.1 What are the advantages of the strength design method as compared to the allowable-stress or alternate design method?

Problem 3.2 What is the purpose of strength reduction factors? Why are they smaller for columns than for beams?

Problem 3.3 What are the basic assumptions of the strength design theory?

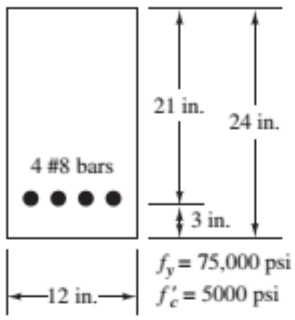
Problem 3.4 Why does the ACI Code specify that a certain minimum percentage of reinforcing be used in beams?

Problem 3.5 Distinguish between tension-controlled and compression-controlled beams.

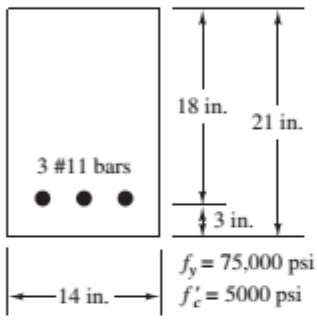
Problem 3.6 Explain the purpose of the minimum cover requirements for reinforcing specified by the ACI Code.

For Problems 3.7 to 3.10, determine the values of ϵ_t, ϕ, and ϕM_n for the sections shown.

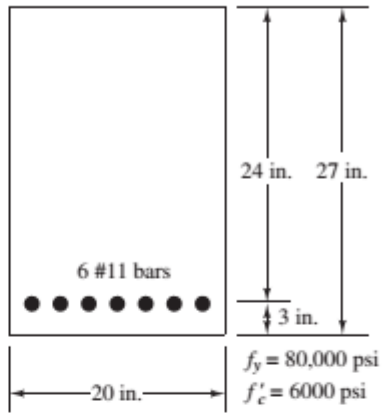
Problem 3.7 (*Ans.* $\phi M_n = 330.1$ ft-k)



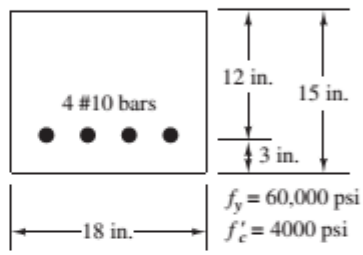
Problem 3.8



Problem 3.9 (*Ans.* $\epsilon_t = 0.00435$, $\phi = 0.827$, $\phi M_n = 1050.4$ ft-k)



Problem 3.10



Design of Rectangular Beams and One-Way Slabs

CHAPTER 4

4.1 Load Factors

Load factors are numbers, usually larger than 1.0, that are used to increase the estimated loads applied to structures. They are used for loads applied to all types of members, not just beams and slabs. The loads are increased to attempt to account for the uncertainties involved in estimating their magnitudes. How close can you estimate the largest wind or seismic loads that will be applied to the building that you are now occupying during its design life? How much uncertainty is present in your answer?

You should note that the load factors for dead loads are much smaller than the ones used for live and environmental loads. Obviously, the reason is that we can estimate the magnitudes of dead loads much more accurately than we can the magnitudes of those other loads. In this regard, you will notice that the magnitudes of loads that remain in place for long periods of time are much less variable than are those loads applied for brief periods, such as wind and snow.

ACI 318 Section 5.3 presents the load factors and combinations that are to be used for reinforced concrete design. The required strength, U, or the load-carrying ability of a particular reinforced concrete member, must at least equal the largest value obtained by substituting into ACI Equations 5.3.1a to 5.3.1g. The following equations conform to the requirements of the International Building Code (IBC)[1] as well as to the values required by ASCE/SEI 7-10.[2]

$$U = 1.4D \quad \text{(ACI Equation 5.3.1a)}$$

$$U = 1.2D + 1.6L + 0.5(L_r \text{ or } S \text{ or } R) \quad \text{(ACI Equation 5.3.1b)}$$

$$U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (L \text{ or } 0.5W) \quad \text{(ACI Equation 5.3.1c)}$$

$$U = 1.2D + 1.0W + L + 0.5(L_r \text{ or } S \text{ or } R) \quad \text{(ACI Equation 5.3.1d)}$$

$$U = 1.2D + 1.0E + L + 0.2S \quad \text{(ACI Equation 5.3.1e)}$$

$$U = 0.9D + 1.0W \quad \text{(ACI Equation 5.3.1f)}$$

$$U = 0.9D + 1.0E \quad \text{(ACI Equation 5.3.1g)}$$

In the preceding expressions, the following values are used:

U = the design or ultimate load the structure needs to be able to resist

D = dead load

L = live load

L_r = roof live load

S = snow load

R = rain load

W = wind load

E = seismic or earthquake load effects

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[1] International Code Council, 2012, *International Building Code*, Falls Church, Virginia 22041-3401.

[2] American Society of Civil Engineers, *Minimum Design Loads for Buildings and Other Structures*, ASCE 7-10 (Reston, VA: American Society of Civil Engineers), p. 7.

When impact effects need to be considered, they should be included with the live loads as per ACI Section 5.3.4. Such situations occur when those loads are quickly applied, as they are for parking garages, elevators, loading docks, cranes, and others. Other possible live loads included in Section 5.3.4 are concentrated live loads, vehicular loads, crane loads, loads on handrails, guardrails and vehicular barrier systems and vibration effects.

The load combinations presented in ACI Equations 5.3.1f and 5.3.1g contain a $0.9D$ value. This 0.9 factor accounts for cases where larger dead loads tend to reduce the effects of other loads. One obvious example of such a situation may occur in tall buildings that are subject to lateral wind and seismic forces where overturning may be a possibility. As a result, the dead loads are reduced by 10% to take into account situations where they may have been overestimated.

The reader must realize that the sizes of the load factors do not vary in relation to the seriousness of failure. You may think that larger load factors should be used for hospitals or high-rise buildings than for cattle barns, but such is not the case. The load factors were developed on the assumption that designers would consider the seriousness of possible failure in specifying the magnitude of their service loads. Furthermore, the ACI load factors are minimum values, and designers are perfectly free to use larger factors as they desire. The magnitude of wind loads and seismic loads, however, reflects the importance of the structure. For example, in ASCE-7,[3] a hospital must be designed for an earthquake load 50% larger than a comparable building with less serious consequences of failure.

For some special situations, ACI 318 permits reductions in the specified load factors. These situations are as follows:

(a) In ACI Equations 5.3.1c through 5.3.1e the factor used for live loads may be reduced to 0.5 except for garages, areas used for public assembly, and all areas where the live loads exceed 100 psf (Section 5.3.3).

(b) If the load W is based on service-level wind loads, replace $1.0W$ in ACI Equations 5.3.1d and 5.3.1f with $1.6W$. Also, replace $0.5W$ with $0.8W$ in ACI Equation 5.3.1c (Section 5.3.5).

(c) Self-restraining effects, T, in reinforced concrete structures include the effects of temperature, creep, shrinkage, and differential settlement. In some cases, the effects can be additive. For example, creep, shrinkage, and reduction in temperature all cause a reduction of concrete volume. Often such effects can be reduced or eliminated by proper use of control joints (Section 5.3.6).

(d) Fluid loads, F, resulting from the weight and pressure of fluids shall be included in ACI Equations 5.3.1a through 5.3.1e.

- If F acts alone or adds to the effects of D, it shall be included with a load factor of 1.4 in Equation 5.3.1a.
- If F adds to the primary load, it shall be included with a load factor of 1.2 in Equations (5.3.1b) through (5.3.1e).
- If the effect of F is permanent and *counteracts* the primary load, it shall be included with a load factor of 0.9 in Equation 5.3.1g.
- If the effect of F is not permanent but, when present, *counteracts* the primary load, F shall not be included in Equations 5.3.1a through 5.3.1g.

(e) Where soil loads, H, are present, they must be added to the load combinations in accordance with one of the following:

- where H acts alone or adds to the effects of other loads, it shall be included with a load factor of 1.6 in Equations 5.3.1a through 5.3.1e;

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[3] American Society of Civil Engineers, *Minimum Design Loads for Buildings and Other Structures*, ASCE 7-10 (Reston, VA: American Society of Civil Engineers), p. 5.

- where the effect of H is permanent and counteracts the effects of other loads, it shall be included with a load factor of 0.9 in Equations 5.3.1f and 5.3.1g;
- where the effect of H is not permanent but, when present, counteracts the effects of other loads, H shall not be included in Equations 5.3.1a through 5.3.1g.

Example 4.1 presents the calculation of factored loads for a reinforced concrete column using the ACI load combinations. The largest value obtained is referred to as the critical or governing load combination and is the value to be used in design. *Notice that the values of the wind and seismic loads can be different depending on the direction of those forces, and it may be possible for the sign of those loads to be different (i.e., compression or tension).* This is the situation assumed to exist in the column of this example. These rather tedious calculations can be easily handled with the Excel spreadsheet entitled Load Combinations on this book's website: www.wiley.com/college/mccormac.

Example 4.1

The compression gravity axial loads for a building column have been estimated with the following results: $D = 150$ k; live load from roof, $L_r = 60$ k; and live load from floors, $L = 300$ k. Compression wind, $W = 70$ k; tensile wind, $W = 60$ k; seismic compression load, E = 50 k; and tensile seismic load, E = 40 k. Determine the critical design load using the ACI load combinations.

SOLUTION

(5.3.1a) $U = 1.4D = (1.4)\ (150\text{ k}) = 210\text{ k}$

(5.3.1b) $U = 1.2D + 1.6L + 0.5(L_r \text{ or } S \text{ or } R) = (1.2)\ (150\text{ k}) + (1.6)\ (300\text{ k}) + (0.5)\ (60\text{ k}) = 690\text{ k}$

(5.3.1c)(1) $U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (L \text{ or } 0.5W) = (1.2)\ (150\text{ k}) + (1.6)\ (60\text{ k}) + (300\text{ k}) = 576\text{ k}$

(2) $U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (L \text{ or } 0.5W) = (1.2)\ (150\text{ k}) + (1.6)\ (60\text{ k}) + (0.5)\ (70\text{ k}) = 311\text{ k}$

(3) $U = 1.2D + 1.6(L_r \text{ or } S \text{ or } R) + (L \text{ or } 0.5W) = (1.2)\ (150\text{ k}) + (1.6)\ (60\text{ k}) + (0.5)\ (-60\text{ k}) = 246\text{ k}$

(5.3.1d)(1) $U = 1.2D + 1.0W + L + 0.5(L_r \text{ or } S \text{ or } R) = (1.2)\ (150\text{ k}) + (1.0)\ (70\text{ k}) + (300\text{ k}) + 0.5(60\text{ k}) = 580\text{ k}$

(2) $U = 1.2D + 1.0W + L + 0.5(L_r \text{ or } S \text{ or } R) = (1.2)\ (150\text{ k}) + (1.0)\ (-60\text{ k}) + (300\text{ k}) + 0.5(60\text{ k}) = 450\text{ k}$

(5.3.1e)(1) $U = 1.2D + 1.0E + L + 0.2S = (1.2)\ (150\text{ k}) + (1.0)\ (50\text{ k}) + (300\text{ k}) + (0.2)\ (0\text{ k}) = 530\text{ k}$

(2) $U = 1.2D + 1.0E + L + 0.2S = (1.2)\ (150\text{ k}) + (1.0)\ (-40\text{ k}) + (300\text{ k}) + (0.2)\ (0\text{ k}) = 440\text{ k}$

(5.3.1f)(1) $U = 0.9D + 1.0W = (0.9)\ (150\text{ k}) + (1.0)\ (70\text{ k}) = 205\text{ k}$

(2) $U = 0.9D + 1.0W = (0.9)\ (150\text{ k}) + (1.0)\ (-60\text{ k}) = 75\text{ k}$

(5.3.1g)(1) $U = 0.9D + 1.0E = (0.9)\ (150\text{ k}) + (1.0)\ (50\text{ k}) = 185\text{ k}$

(2) $U = 0.9D + 1.0E = (0.9)\ (150\text{ k}) + (1.0)\ (-40\text{ k}) = 95\text{ k}$

Answer: Largest value = 690 k from load case 5.3.1b.

For most of the example problems presented in this textbook, in the interest of reducing the number of computations, only dead and live loads are specified. As a result, the only load factor combination usually applied herein is the one presented by ACI Equation 5.3.1b. Occasionally, when the dead load is quite large compared to the live load, it is also necessary to consider Equation 5.3.1a.

4.2 Design of Rectangular Beams

Before the design of an actual beam is attempted, several miscellaneous topics need to be discussed. These include the following:

1. *Beam proportions.* Unless architectural or other requirements dictate the proportions of reinforced concrete beams, the most economical beam sections are usually obtained for shorter beams

(up to 20 ft or 25 ft in length), when the ratio of d to b is in the range of $1\frac{1}{2}$ to 2. For longer spans, better economy is usually obtained if deep, narrow sections are used. The depths may be as large as three or four times the widths. However, today's reinforced concrete designer is often confronted with the need to keep members rather shallow to reduce floor heights. As a result, wider and shallower beams are used more frequently than in the past. You will notice that the overall beam dimensions are selected to whole inches. This is done for simplicity in constructing forms or for the rental of forms, which are usually available in 1-in. or 2-in. increments. Furthermore, beam widths are often selected in multiples of 2 in. or 3 in.

2. *Deflections*. Considerable space is devoted in Chapter 6 to the topic of deflections in reinforced concrete members subjected to bending. However, the ACI Code in Tables 7.3.1.1 and 9.3.1.1 provide minimum thicknesses of beams and one-way slabs for which such deflection calculations are not required. These values are combined to form a single table, Table 4.1. The purpose of such limitations is to prevent deflections of such magnitudes as would interfere with the use of or cause injury to the structure. If deflections are computed for members of lesser thicknesses than those listed in the table and are found to be satisfactory, it is not necessary to abide by the thickness rules. For simply supported slabs, normal-weight concrete, and Grade 60 steel, the minimum depth given when deflections are not computed equals $\ell/20$, where ℓ is the span length of the slab. *For concrete of other weights and for steel of different yield strengths, the minimum depths required by the ACI Code are somewhat revised, as indicated in the footnotes to Table 4.1.* The ACI does not specify changes in the table for concretes weighing between 120 lb/ft and 145 lb/ft because substitution into the correction expression given yields correction factors almost equal to 1.0.

The minimum thicknesses provided apply only to members that are not supporting or attached to partitions or other construction likely to be damaged by large deflections.

3. *Estimated beam weight*. The weight of the beam to be selected must be included in the calculation of the bending moment to be resisted, because the beam must support itself as well as the external loads. The weight estimates for the beams selected in this text are generally very close because the authors were able to perform a little preliminary paperwork before making their estimates. You are not expected to be able to glance at a problem and give an exact estimate of the weight of the beam required. Following the same procedures as did the authors, however, you can do a little figuring on the side and make a very reasonable estimate. For instance, you could calculate the moment due to the external loads

TABLE 4.1 Minimum Thickness of Non-prestressed Beams or One-Way Slabs Unless Deflections Are Computed[1,2]

	Minimum Thickness, h			
	Simply supported	One end continuous	Both ends continuous	Cantilever
Member	Members not supporting or attached to partitions or other construction likely to be damaged by large deflections			
Solid one-way slabs	$\ell/20$	$\ell/24$	$\ell/28$	$\ell/10$
Beams or ribbed one-way slabs	$\ell/16$	$\ell/18.5$	$\ell/21$	$\ell/8$

[1] Span length, ℓ, is in inches.

[2] Values given shall be used directly for members with normal-weight concrete and Grade 60 reinforcement. For other conditions, the values shall be modified as follows:

(a) For lightweight concrete having equilibrium density in the range 90 lb/ft^3 to 115 lb/ft^3, the values shall be multiplied by $(1.65 - 0.005w_c)$ but not less than 1.09, where w_c is the unit weight in lb/ft^3.

(b) For f_y other than 60,000 psi, the values shall be multiplied by $(0.4 + f_y/100{,}000)$.

Source: Reproduced with permission from the American Concrete Institute.

only, select a beam size, and calculate its weight. From this beam size, you should be able to make a very good estimate of the weight of the final beam section.

Another practical method for estimating beam sizes is to assume a minimum overall depth, h, equal to the minimum depth specified by Table 4.1 if deflections are not to be calculated. The ACI minimum depth for the beam in question may be determined by referring to Table 4.1. Then the beam width can be roughly estimated equal to about two-thirds of the assumed value of h and the weight of this estimated beam calculated = bh/144 times the concrete weight per cubic foot. Because concrete weighs approximately 150 pcf (if the weight of steel is included), a "quick-and-dirty" calculation of self-weight is simply $b \times h$ because the concrete weight approximately cancels the 144 conversion factor.

After the design moment, M_u, is determined for all of the loads, including the estimated beam weight, the section is selected. If the dimensions of this section are significantly different from those initially assumed, it will be necessary to recalculate the weight and M_u and repeat the beam selection. At this point you may very logically ask, "What's a significant change?" Well, you must realize that we are not interested academically in how close our estimated weight is to the final weight, but rather we are extremely interested in how close our calculated M_u is to the actual M_u. In other words, our estimated weight may be considerably in error, but if it doesn't affect M_u by more than say 1% or $1\frac{1}{2}$%, forget it.

In Example 4.2, beam proportions are estimated as just described, and the dimensions so selected are taken as the final ones. As a result, you can see that it is not necessary to check the beam weight and recalculate M_u and repeat the design.

In Example 4.3, a beam is designed for which the total value of M_u (including the beam weight) has been provided, as well as a suggested reinforcement ratio.

Finally, with Example 4.4, the authors have selected a beam whose weight is unknown. Without a doubt, many students initially have a little difficulty understanding how to make reasonable member weight estimates for cases such as this one. To show how easily, quickly, and accurately this may be done for beams, this example is included.

We dreamed up a beam weight estimated out of the blue equal to 400 lb/ft. (We could just as easily and successfully have made it 10 lb/ft or 1000 lb/ft.) With this value, a beam section was selected and its weight calculated to equal 619 lb/ft. With this value, a very good weight estimate was then made. The new section obviously would be a little larger than the first one. So we estimated the weight a little above the 619 lb/ft value, recalculated the moment, selected a new section, and determined its weight. The results were very satisfactory.

4. *Selection of bars*. After the required reinforcing area is calculated, Appendix A, Table A.4 is used to select bars that provide the necessary area. For the usual situations, bars of sizes #11 and smaller are practical. It is usually convenient to use bars of one size only in a beam, although occasionally two sizes will be used. Bars for compression steel and stirrups are usually a different size, however. Otherwise the ironworkers may become confused.

5. *Cover*. The reinforcing steel for concrete members must be protected from the surrounding environment; that is, fire and corrosion protection need to be provided. To do this, the reinforcing steel is located at certain minimum distances from the surface of the concrete so that a protective layer of concrete, called *cover*, is provided. In addition, the cover improves the bond between the concrete and the steel. In Section 20.6.1.3 of the ACI 318 Code, specified cover is given for reinforcing bars under different conditions. Values are given for reinforced concrete beams, columns, and slabs; for cast-in-place members; for precast members; for prestressed members; for members exposed to earth and weather; for members not so exposed; and so on. The concrete for members that are to be exposed to deicing salts, brackish water, seawater, or spray from these sources must be especially proportioned to satisfy the exposure requirements of Section 20.6.1.4 of the code. These requirements pertain to air entrainment, water–cement ratios, cement types, concrete strength, and so on.

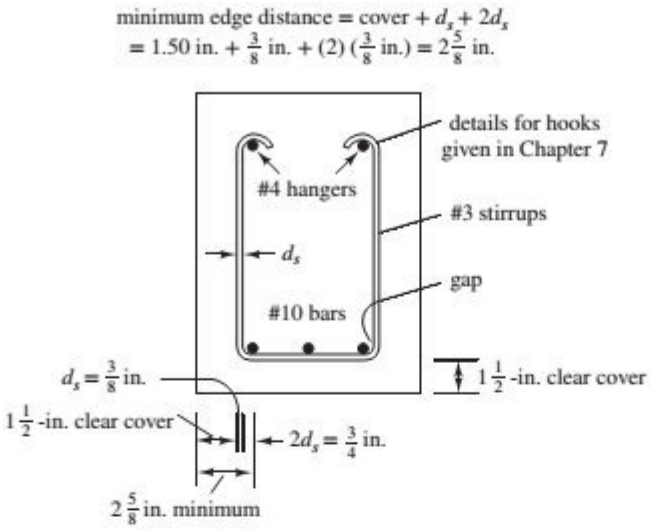

minimum edge distance = cover + d_s + $2d_s$
= 1.50 in. + $\frac{3}{8}$ in. + (2) ($\frac{3}{8}$ in.) = $2\frac{5}{8}$ in.
details for hooks given in Chapter 7
#4 hangers
#3 stirrups
d_s
gap
#10 bars
$1\frac{1}{2}$-in. clear cover
$d_s = \frac{3}{8}$ in.
$1\frac{1}{2}$-in. clear cover
$2d_s = \frac{3}{4}$ in.
$2\frac{5}{8}$ in. minimum

FIGURE 4.1 Determining minimum edge distance.

The beams designed in Examples 4.2 to 4.4 are assumed to be located inside a building and thus protected from the weather. For this case, the code requires a minimum cover of $1\frac{1}{2}$ in. of concrete outside of any reinforcement.

In Chapter 8, you will learn that vertical stirrups are used in most beams for shear reinforcing. A sketch of a stirrup is shown in the beam of Figure 4.1. The minimum stirrup diameter (d_s) that the code permits us to use is $\frac{3}{8}$ in. when the longitudinal bars are #10 or smaller; for #11 and larger bars, the minimum stirrup diameter is $\frac{1}{2}$ in. The minimum inside radius of the 90° stirrup bent around the outside longitudinal bars is two times the stirrup diameter ($2d_s$). As a result, when the longitudinal bars are #14 or smaller, there will be a gap between the bars and the stirrups, as shown in the figure. This is based on the assumption that each outside longitudinal bar is centered over the horizontal point of tangency of the stirrup corner bend. For #18 bars, however, the half-bar diameter is larger than $2d_s$ and controls.

For the beam of Figure 4.1 it is assumed that 1.50-in. clear cover, #3 stirrups, and #10 longitudinal bars are used. The minimum horizontal distance from the center of the outside longitudinal bars to the edge of the concrete can be determined as follows:

$$\text{Minimum edge distance} = \text{cover} + d_s + 2d_s = 1.50 \text{ in.} + \frac{3}{8} \text{ in.} + (2)\left(\frac{3}{8} \text{ in.}\right) = 2\frac{5}{8} \text{ in.}$$

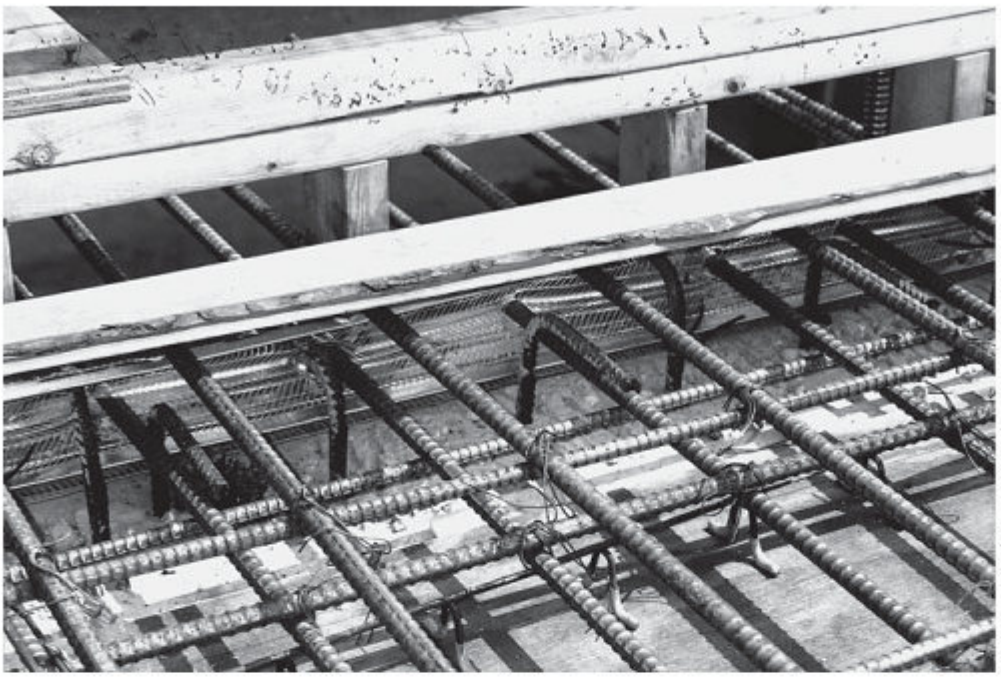
The minimum cover required for concrete cast against earth, as in a footing, is 3 in., and for concrete not cast against the earth but later exposed to it, as by backfill, is 2 in. Precast and prestressed concrete or other concrete cast under plant control conditions requires less cover, as described in Section 20.6.1.3.3 of the ACI Code.

Note that the two #4 bars called *hangers* are placed in the compression side of this beam. Their purpose is to provide support for the stirrups and to hold the stirrups in position.

If concrete members are exposed to very harsh surroundings, such as deicing salts, smoke, or acid vapors, the cover should be increased above these minimums.

6. *Minimum spacing of bars.* The ACI Code Section 25.2.1 states that the clear distance between parallel bars cannot be less than 1 in.[4] or less than the nominal bar diameter. If the bars are placed in more than one layer, those in the upper layers are required to be placed directly over the ones in the lower layers, and the clear distance between the layers must be not less than 1 in.

[4] 25 mm in SI.



Courtesy of Alabama Metal Industries Corporation.

Reinforcing bars. Note the supporting metal chairs.

A major purpose of these requirements is to enable the concrete to pass between the bars. The ACI Code further relates the spacing of the bars to the maximum aggregate sizes for the same purpose. In the ACI Code Section 25.2.1, maximum permissible aggregate sizes are limited to three-fourths of the minimum clear spacing between bars.

A reinforcing bar must extend an appreciable length in both directions from its point of highest stress in order to develop its stress by bonding to the concrete. The shortest length in which a bar's stress can be increased from 0 to f_y is called its *development length*.

If the distance from the end of a bar to a point where it theoretically has a stress equal to f_y is less than its required development length, the bar may very well pull loose from the concrete. Development lengths are discussed in detail in Chapter 7. There you will learn that required development lengths for reinforcing bars vary appreciably with their spacings and their cover. *As a result, it is sometimes wise to use greater cover and larger bar spacings than the specified minimum values in order to reduce development lengths.*

When selecting the actual bar spacing, the designer will comply with the preceding code requirements and, in addition, will give spacings and other dimensions in inches and fractions, not in decimals. The workers in the field are accustomed to working with fractions and would be confused by a spacing of bars such as 3 at 1.45 in. The designer should always strive for simple spacings, because such dimensions will lead to better economy.

Each time a beam is designed, it is necessary to select the spacing and arrangement of the bars. To simplify these calculations, Appendix A, Table A.5 is given. Corresponding information is provided in SI units in Appendix B, Table B.5. These tables show the minimum beam widths required for different numbers of bars. The values given are based on the assumptions that $\frac{3}{8}$-in. stirrups and $1\frac{1}{2}$-in. cover are required except for #18 bars, where the stirrup diameter is $\frac{1}{2}$ in. If three #10 bars are required, it can be seen from the table that a minimum beam width of 10.4 in. (say 11 in.) is required.

This value can be checked as follows, noting that $2d_s$ is the radius of bend of the bar, and the minimum clear spacing between bars in this case is d_b:

$$\begin{aligned}\text{Minimum beam width} &= \text{cover} + d_s + 2d_s + \frac{d_b}{2} + d_b + d_b + d_b + \frac{d_b}{2} + 2d_s + d_s + \text{cover} \\ &= 1.50 \text{ in.} + \frac{3}{8} \text{ in.} + (2)\left(\frac{3}{8} \text{ in.}\right) + \frac{1.27 \text{ in.}}{2} + (3)\,(1.27 \text{ in.}) + \frac{1.27 \text{ in.}}{2} \\ &\quad + (2)\left(\frac{3}{8} \text{ in.}\right) + \frac{3}{8} \text{ in.} + 1.50 \text{ in.} \\ &= 10.33 \text{ in. rounded to } 10.4 \text{ in.}\end{aligned}$$

4.3 Beam Design Examples

Example 4.2 illustrates the design of a simple span rectangular beam. For this introductory example, approximate dimensions are assumed for the beam cross section. The depth, h, is assumed to equal about one-tenth of the beam span, while its width, b, is assumed to equal about $\frac{1}{2}h$. Next the reinforcement ratio needed is determined with the equation derived in Section 3.4, and reinforcing bars are selected to satisfy that reinforcement ratio. Finally, ϕM_n is calculated for the final design.

Example 4.2

Design a rectangular beam for a 22-ft simple span if a dead load of 1 k/ft (not including the beam weight) and a live load of 2 k/ft are to be supported. Use $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

Estimating Beam Dimensions and Weight

$$\text{Assume } h = (0.10)(22 \text{ ft}) = 2.2 \text{ ft} \quad \underline{\text{Say 27 in.}}\ (d = 24.5 \text{ in.})$$

$$\text{Assume } b = \frac{1}{2}h = \frac{27 \text{ in.}}{2} \quad \underline{\text{Say 14 in.}}$$

$$\text{Beam wt} = \frac{(14 \text{ in.})(27 \text{ in.})}{144 \text{ in}^2/\text{ft}^2}\,(150 \text{ lb/ft}^3) = 394 \text{ lb/ft} = 0.394 \text{ k/ft (klf)}$$

Computing w_u and M_u

$$w_u = (1.2)\,(1 \text{ klf} + 0.394 \text{ klf}) + (1.6)\,(2 \text{ klf}) = 4.873 \text{ klf}$$

$$M_u = \frac{w_u L^2}{8} = \frac{(4.873 \text{ klf})(22 \text{ ft})^2}{8} = 294.8 \text{ ft-k}$$

Assuming $\phi = 0.90$ and computing ρ with the following expression, which was derived in Section 3.4

$$\rho = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right)$$

$$R_n = \frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(294{,}800 \text{ ft-lb})}{(0.90)\,(14 \text{ in.})(24.5 \text{ in.})^2} = 467.7 \text{ psi}$$

$$\rho = \frac{(0.85)\,(4000 \text{ psi})}{60{,}000 \text{ psi}}\left[1 - \sqrt{1 - \frac{(2)\,(467.7 \text{ psi})}{(0.85)\,(4000 \text{ psi})}}\right] = 0.00842$$

Selecting Reinforcing Steel

$$A_s = \rho bd = (0.00842)(14 \text{ in.})(24.5 \text{ in.}) = 2.89 \text{ in.}^2$$

Use 3 #9 bars ($A_s = 3.00$ in.^{2})

Appendix A.5 indicates a minimum beam width of 9.8 in. for interior exposure for three #9 bars. If five #7 bars had been selected, a minimum width of 12.8 in. would be required. Either choice would be acceptable because the beam width of 14 in. exceeds either requirement. If we had selected a beam width of 12 in. earlier in the design process, we might have been limited to the larger #9 bars because of this minimum beam width requirement.

Checking Solution

$$\rho = \frac{A_s}{bd} = \frac{3.00 \text{ in.}^2}{(14 \text{ in.})\,(24.5 \text{ in.})} = 0.00875 > \rho_{min} = 0.0033$$

$< \rho_{max} = 0.0181$ (ρ values from Appendix A, Table A.7.) $\therefore$ Section is ductile and $\phi = 0.90$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(3.00 \text{ in.}^2)\,(60 \text{ ksi})}{(0.85)\,(4 \text{ ksi})\,(14 \text{ in.})} = 3.78 \text{ in.}$$

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = (0.90)\,(3.00 \text{ in.}^2)\,(60 \text{ ksi}) \left(24.5 \text{ in.} - \frac{3.78 \text{ in.}}{2}\right)$$

$$= 3662 \text{ in-k} = 305.2 \text{ ft-k} > 294.8 \text{ ft-k} \quad \underline{\underline{\text{OK}}}$$

Final Section (Figure 4.2)

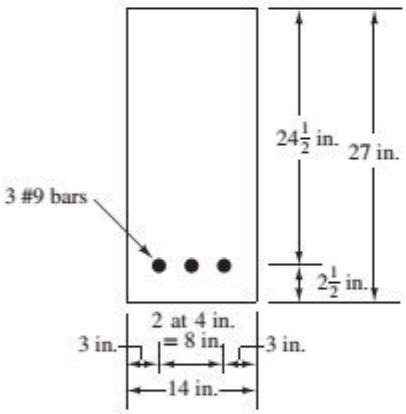


FIGURE 4.2 Final beam cross section for Example 4.2.

Use of Graphs and Tables

In Section 3.4, the following equation was derived:

$$M_u = \phi A_s f_y d \left(1 - \frac{1}{1.7} \frac{\rho f_y}{f'_c}\right)$$

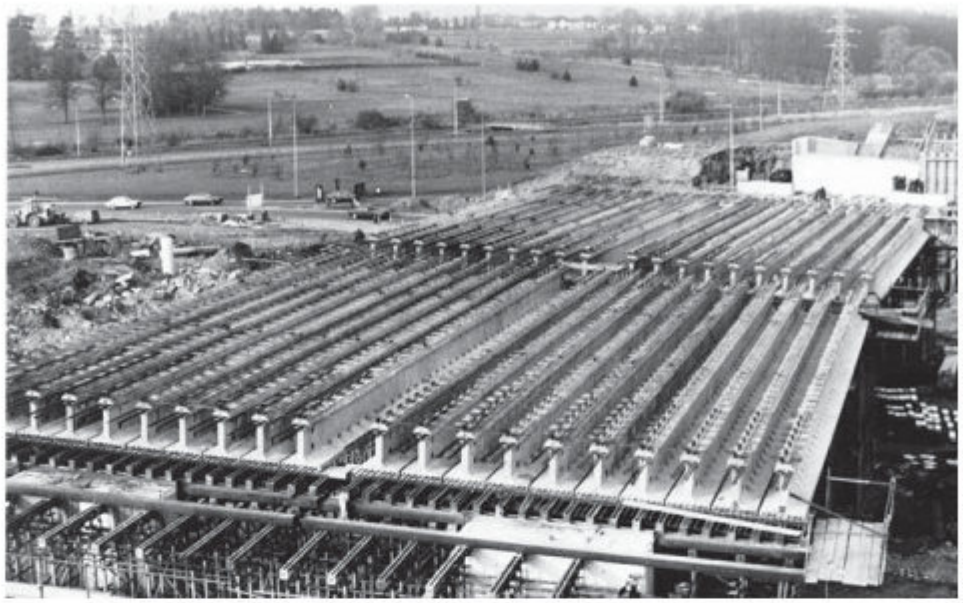
If A_s in this equation is replaced with ρbd, the resulting expression can be solved for $M_u/\phi bd^2$.

$$M_u = \phi \rho bd\, f_y d \left(1 - \frac{1}{1.7} \frac{\rho f_y}{f'_c}\right)$$

and dividing both sides of the equation by ϕbd^2,

$$\frac{M_u}{\phi bd^2} = \rho f_y \left(1 - \frac{1}{1.7} \frac{\rho f_y}{f'_c}\right)$$

For a given reinforcement ratio, ρ, certain concrete, f'_c, and certain steel, f_y, the value of $M_u/\phi bd^2$ can be calculated and listed in tables, as is illustrated in Appendix A, Tables A.8 through A.13, or in



C&CA Collection, Courtesy of The Concrete Society.

Barnes Meadow Interchange, Northampton, England.

graphs (see Graph 1 of Appendix A). SI values are provided in Appendix B, Tables B.8 and B.9. It is much easier to accurately read the tables than the graphs (at least to the scale to which the graphs are shown in this text). For this reason, the tables are used for the examples here. The units for $M_u/\phi bd^2$ in both the tables and the graphs of Appendix A are pounds per square inch. In Appendix B, the units are MPa.

Once $M_u/\phi bd^2$ is determined for a particular beam, the value of M_u can be calculated by simply multiplying the value obtained from the tables for $M_u/\phi bd^2$ by ϕbd^2. The same tables and graphs can be used for either the design or analysis of beams.

The value of ρ, determined in Example 4.2 by substituting into that long and tedious equation, can be directly selected from Appendix A, Table A.13. We enter that table with the $M_u/\phi bd^2$ value previously calculated in the example, and we read a value of ρ between 0.0084 and 0.0085. Interpolation can be used to find the actual value of 0.00842, but such accuracy is not really necessary. It is conservative to use the higher value (0.0085) to calculate the steel area.

In Example 4.3, which follows, a value of ρ was specified in the problem statement, and the long equation was used to determine the required dimensions of the structure as represented by bd^2. Again, it is much easier to use the appropriate appendix table to determine this value. In nearly every other case in this textbook, the tables are used for design or analysis purposes.

Once the numerical value of bd^2 is determined, the authors take what seems to be reasonable values for b (in this case 12 in., 14 in., and 16 in.) and compute the required d for each width so that the required bd^2 is satisfied. Finally, a section is selected in which b is roughly one-half to two-thirds of d. (For long spans, d may be two and a half or three or more times b for economical reasons.)

Example 4.3

A beam is to be selected with $\rho \cong 0.0120$, $M_u = 600$ ft-k, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi.

SOLUTION

Assuming $\phi = 0.90$ and substituting into the following equation from Section 3.4:

$$\frac{M_u}{\phi bd^2} = \rho f_y \left(1 - \frac{1}{1.7}\frac{\rho f_y}{f'_c}\right)$$

$$\frac{(12\text{ in/ft})\,(600{,}000\text{ ft-lb})}{(0.9)\,(bd^2)} = (0.0120)\,(60{,}000\text{ psi})\left[1-\left(\frac{1}{1.7}\right)\frac{(0.0120)\,(60{,}000\text{ psi})}{4000\text{ psi}}\right]$$

$$bd^2 = 12{,}427\text{ in.}^3 \begin{cases} b \times d \\ 12\text{ in.} \times 32.18\text{ in.} \\ 14\text{ in.} \times 29.79\text{ in.} \\ 16\text{ in.} \times 27.87\text{ in.} \end{cases} \leftarrow \begin{cases} \text{This one seems} \\ \text{pretty reasonable} \\ \text{to the authors.} \end{cases}$$

Note: Alternatively, we could have used tables to help calculate bd^2. Upon entering Appendix A, Table A.13, we find $M_u/\phi bd^2 = 643.5$ psi when $\rho = 0.0120$.

$$\therefore bd^2 = \frac{(12\text{ in/ft})\,(600{,}000\text{ ft-lb})}{(0.90)\,(643.5\text{ psi})} = 12{,}432\text{ in.}^3 \quad \underline{\text{OK}}$$

$$\underline{\text{Try } 14\text{ in.} \times 33\text{ in.}\ (d = 30.00\text{ in.})}$$

$$A_s = \rho bd = (0.0120)(14\text{ in.})(30\text{ in.}) = 5.04\text{ in.}^2$$

$$\underline{\underline{\text{Use 4 \#10 } (A_s = 5.06\text{ in.}^2)}}$$

Note: Appendix A.5 indicates a minimum beam width of 12.9 in. for this bar selection. Since our width is 14 in., the bars will fit.

Checking Solution

$$\rho = \frac{A_s}{bd} = \frac{5.06\text{ in.}^2}{(14\text{ in.})\,(30\text{ in.})} = 0.01205 > \rho_{\min} = 0.0033$$

$$< \rho_{\max} = 0.0181 \text{ (from Appendix A, Table A.7)}$$

Note: A value of $\rho = 0.0206$ is permitted by the code, but the corresponding value of ϕ would be less than 0.9 (see Figure 3.5 and Table A.7). Since a value of ϕ of 0.9 was used in the above calculations, it is necessary to use a maximum value of $\rho = 0.0181$.

With $\rho = 0.01205$, $M_u/\phi bd^2$ by interpolation from Table A.13 equals 645.85.

$$\phi M_n = (645.85\text{ psi})\,(\phi bd^2) = (645.85\text{ psi})\,(0.9)\,(14\text{ in.})(30\text{ in.})^2$$
$$= 7{,}323{,}939\text{ in-lb} = 610.3\text{ ft-k} > 600\text{ ft-k}$$

Final Section (Figure 4.3)

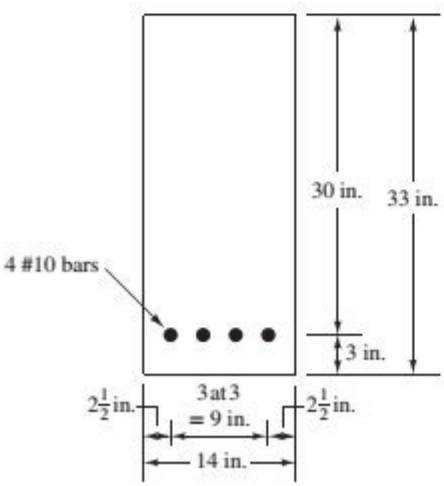


FIGURE 4.3 Final cross section for Example 4.3.

Through quite a few decades of reinforced concrete design experience, it has been found that if reinforcement ratios are kept fairly small, say roughly $0.18f'_c/f_y$ or perhaps $0.375\rho_b$, beam cross sections will be sufficiently large so that deflections will seldom be a problem. As the areas of steel required will be fairly small, there will be little problem fitting them into beams without crowding.

If these relatively small reinforcement ratios are used, there will be little difficulty in placing the bars and in getting the concrete between them. Of course, from the standpoint of deflection, higher reinforcement ratios, and thus smaller beams, can be used for short spans where deflections present no problem. Whatever reinforcement ratios are used, the resulting members will have to be carefully checked for deflections, particularly for long-span beams, cantilever beams, and shallow beams and slabs. Of course, such deflection checks are not required if the minimum depths specified in Table 4.1 of this chapter are met.

Another reason for using smaller reinforcement ratios is given in ACI Section 6.6.5, where a plastic redistribution of moments (a subject to be discussed in Chapter 14) is permitted in continuous members whose ϵ_t values are 0.0075 or greater. Such tensile strains will occur when smaller reinforcement ratios are used. For the several reasons mentioned here, structural designers believe that keeping reinforcement ratios fairly low will result in good economy.

Example 4.4

A rectangular beam is to be sized with $f_y = 60{,}000$ psi, $f'_c = 3000$ psi, and a reinforcement ratio ρ approximately equal to $0.18f'_c/f_y$. It is to have a 25-ft simple span and to support a dead load, in addition to its own weight, equal to 2 k/ft and a live load equal to 3 k/ft.

SOLUTION

Assume Beam wt = 400 lb/ft

$$w_u = (1.2)\,(2 \text{ klf} + 0.400 \text{ klf}) + (1.6)\,(3 \text{ klf}) = 7.68 \text{ klf (k/ft)}$$

$$M_u = \frac{(7.68 \text{ klf})(25 \text{ ft})^2}{8} = 600 \text{ ft-k}$$

$$\rho = \frac{(0.18)\,(3 \text{ ksi})}{60 \text{ ksi}} = 0.009$$

$$\frac{M_u}{\phi bd^2} = 482.6 \text{ psi (from Appendix A, Table A.12)}$$

$$bd^2 = \frac{M_u}{\phi(482.6 \text{ psi})} = \frac{(12 \text{ in/ft})\,(600{,}000 \text{ ft-lb})}{(0.9)\,(482.6 \text{ psi})}$$

Solving this expression for bd^2 and trying varying values of b and d.

$$bd^2 = 16{,}577 \text{ in.}^3 \quad \begin{matrix} b \times d \\ \left\{\begin{matrix} 16 \text{ in.} \times 32.19 \text{ in.} \\ 18 \text{ in.} \times 30.35 \text{ in.} \\ 20 \text{ in.} \times 28.79 \text{ in.} \end{matrix}\right. \end{matrix} \quad \leftarrow \text{ seems reasonable}$$

Try 18-in. × 33-in. Beam ($d = 30.50$ in.)

$$\text{Beam wt} = \frac{(18 \text{ in.})\,(33 \text{ in.})}{144 \text{ in}^2/\text{ft}^2}(150 \text{ lb/ft}^3) = 619 \text{ lb/ft}$$

$$> \text{the estimated } 400 \text{ lb/ft} \quad \underline{\text{No good}}$$

Assume Beam wt a Little Higher Than 619 lb / ft

$$\text{Estimated wt} = 650 \text{ lb/ft}$$

$$w_u = (1.2)\,(2 \text{ klf} + 0.650 \text{ klf}) + (1.6)\,(3 \text{ klf}) = 7.98 \text{ klf}$$

$$M_u = \frac{(7.98 \text{ klf})(25 \text{ ft})^2}{8} = 623.4 \text{ ft-k}$$

$$bd^2 = \frac{M_u}{\phi(482.6 \text{ psi})} = \frac{(12 \text{ in/ft})\,(623{,}400 \text{ ft-lb})}{(0.9)\,(482.6 \text{ psi})}$$

$$= 17{,}223 \text{ in.}^3 \begin{cases} 16 \text{ in.} \times 32.81 \text{ in.} \\ 18 \text{ in.} \times 30.93 \text{ in.} \quad \leftarrow \quad \text{seems reasonable} \\ 20 \text{ in.} \times 29.35 \text{ in.} \end{cases}$$

Try 18-in. × 34-in. Beam ($d = 31.00$ in.)

$$\text{Beam wt} = \frac{(18 \text{ in.})\,(34 \text{ in.})}{144 \text{ in}^2/\text{ft}^2}\,(150 \text{ lb/ft}^3) = 637.5 \text{ lb/ft} < 650 \text{ lb/ft} \quad \underline{\underline{\text{OK}}}$$

$$A_s = \rho bd = (0.009)\,(18 \text{ in.})\,(31 \text{ in.}) = 5.02 \text{ in.}^2$$

Try five #9 bars (minimum width is 14.3 in. from Appendix A, Table A.5) $\underline{\underline{\text{OK}}}$

Normally a bar selection should exceed the theoretical value of A_s. In this case, the area chosen was less than, but very close to, the theoretical area, and it will be checked to be sure it has enough capacity.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{5.00 \text{ in.}^2 (60 \text{ ksi})}{(0.85)\,(3 \text{ ksi})\,(18 \text{ in.})} = 6.54 \text{ in.}$$

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = 0.9(5.00 \text{ in.}^2)\,(60 \text{ ksi})\,\left(31 \text{ in.} - \frac{6.54 \text{ in.}}{2}\right)$$

$$= 7487.6 \text{ in-lb} = 623.9 \text{ ft-k} > M_u$$

The reason a beam with less reinforcing steel than calculated is acceptable is that a value of d exceeding the theoretical value was selected ($d = 31$ in. > 30.93 in.). Whenever the value of b and d selected results in a bd^2 that exceeds the calculated value based on the assumed ρ, the actual value of ρ will be lower than the assumed value.

If a value of $b = 18$ in. and $d = 30$ in. had been selected, the result would have been that the actual value of ρ would be greater than the assumed value of 0.009. Using the actual values of b and d to recalculate ρ

$$\frac{M_u}{\phi bd^2} = \frac{(12 \text{ in/ft})\,(623{,}400 \text{ ft-lb})}{(0.9)\,(18 \text{ in.})(30 \text{ in.})^2} = 513.1 \text{ psi}$$

From Appendix A, Table A.12, $\rho = 0.00965$, which exceeds the assumed value of 0.009. The required value of A_s will be larger than that required for $d = 31$ in.

$$A_s = \rho bd = (0.00965)\,(18 \text{ in.})\,(30 \text{ in.}) = 5.21 \text{ in.}^2 \qquad (\text{Use 7 \#8 bars, } A_s = 5.50 \text{ in.}^2)$$

Either design is acceptable. This kind of flexibility is sometimes perplexing to the student who simply wants to know the right answer. One of the best features of reinforced concrete is that there is so much flexibility in the choices that can be made.

4.4 Miscellaneous Beam Considerations

This section introduces two general limitations relating to beam design: lateral bracing and deep beams.

Lateral Support

It is unlikely that laterally unbraced reinforced concrete beams of any normal proportions will buckle laterally, even if they are deep and narrow, unless they are subject to appreciable lateral torsion. As a result, the ACI 318 Code (9.2.3.1a) states that lateral bracing for a beam is not required closer than 50 times the least width, b, of the compression flange or face. Should appreciable torsion be present, however, it must be considered in determining the maximum spacing for lateral support.

Skin Reinforcement for Deep Beams

Beams with web depths that exceed 3 ft have a tendency to develop excessively wide cracks in the upper parts of their tension zones. To reduce these cracks, it is necessary to add some additional longitudinal reinforcement in the zone of flexural tension near the vertical side faces of their webs, as shown in Figure 4.4. The ACI Code Section 9.7.2.3 states that additional skin reinforcement must be uniformly distributed along both side faces of members with $h > 36$ in. for distances equal to $h/2$ nearest the flexural reinforcing steel.

The spacing, s, between this skin reinforcement shall be as provided in ACI 24.3.2. These additional bars may be used in computing the bending strengths of members only if appropriate strains for their positions relative to neutral axes are used to determine bar stresses. The total area of the skin reinforcement in both side faces of the beam does not have to exceed one-half of the required bending tensile reinforcement in the beam. The ACI does not specify the actual area of skin reinforcement; it merely states that some additional reinforcement should be placed near the vertical faces of the tension zone to prevent cracking in the beam webs.

Some special requirements must be considered relating to shear in deep beams, as described in the ACI Code Section 9.9 and in Section 8.14 of this text.

For a beam designed in SI units with an effective depth > 1 m, additional skin reinforcement must be determined with the following expression, in which A_{sk} is the area of skin reinforcement per meter of height on each side of the beam:

Its maximum spacing may not exceed $d/6$ on 300 mm or $1000A_b/(d - 750)$.

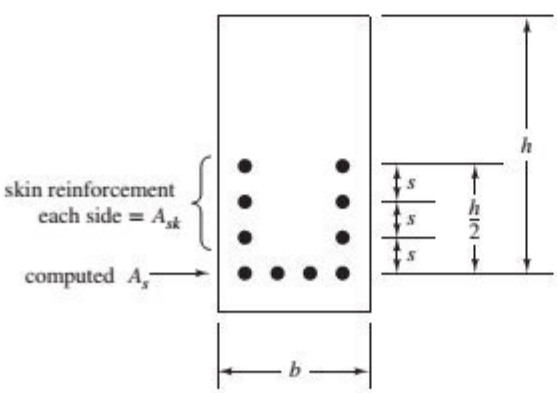


FIGURE 4.4 Skin reinforcement for deep beams with $h > 36$ in., as required by ACI 318 Code Section 9.7.2.3.

Other Items

The next four chapters of this book are devoted to several other important items relating to beams. These include different shaped beams, compression reinforcing steel, cracks, bar development lengths, and shear.

Further Notes on Beam Sizes

From the standpoints of economy and appearance, only a few different sizes of beams should be used in a particular floor system. Such a practice will save appreciable amounts of money by simplifying the formwork and at the same time will provide a floor system that has a more uniform and attractive appearance.

If a group of college students studying the subject of reinforced concrete were to design a floor system and then compare their work with a design of the same floor system made by an experienced structural designer, the odds are that the major difference between the two designs would be in the number of beam sizes. The practicing designer would probably use only a few different sizes, whereas the average student would probably use a larger number.

The designer would probably examine the building layout to decide where to place the beams and then would make the beam subject to the largest bending moment as small as practically possible (i.e., with fairly high reinforcement ratios). Then he or she would proportion as many as possible of the other similar beams with the same outside dimensions. The reinforcement ratios of these latter beams might vary quite a bit because of their different moments.

4.5 Determining Steel Area When Beam Dimensions Are Predetermined

Sometimes the external dimensions of a beam are predetermined by factors other than moments and shears. The depth of a member may have been selected on the basis of the minimum thickness requirements discussed in Section 4.2 for deflections. The size of a whole group of beams may have been selected to simplify the formwork, as discussed in Section 4.4. Finally, a specific size may have been chosen for architectural reasons. Next we briefly mention three methods for computing the reinforcing steel required. Example 4.5 illustrates the application of each of these methods.

Appendix Tables

The value of $M_u/\phi bd^2$ can be computed, and ρ can be selected from the tables. For most situations this is the quickest and most practical method. *The tables given in Appendices A and B of this text apply only to reinforced rectangular sections reinforced with tension steel only. Furthermore, we must remember to check ϕ values.*

Use of ρ Formula

The following equation was previously developed in Section 3.4 for rectangular sections.

$$\rho = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85f'_c}}\right)$$

Trial-and-Error (Iterative) Method

A value of a can be assumed, the value of A_s computed, the value of a determined for that value of A_s, another value of a calculated, and so on. Alternatively, a value of the lever arm from C to T (it's $d - a/2$ for rectangular sections) can be estimated and used in the trial-and-error procedure. *This method is a general one that will work for all cross sections with tensile reinforcement.* It is particularly useful for T beams, as will be illustrated in the next chapter.

Example 4.5

The dimensions of the beam shown in Figure 4.5 have been selected for architectural reasons. Determine the reinforcing steel area by each of the methods described in this section.

SOLUTION

Using Appendix Tables

$$\frac{M_u}{\phi bd^2} = \frac{(12 \text{ in/ft})(160{,}000 \text{ ft-lb})}{(0.9)(16 \text{ in.})(21 \text{ in.})^2} = 302.3 \text{ psi}$$

ρ (from Appendix A, Table A.12) = 0.00538 (by interpolation)

$$A_s = (0.00538)(16 \text{ in.})(21 \text{ in.}) = 1.81 \text{ in.}^2$$

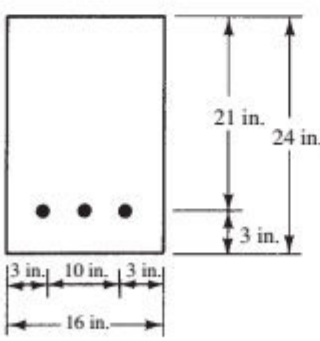
Use 6 #5 bars (1.84 in.^{2})

Using ρ Formula

$$R_n = \frac{M_u}{\phi bd^2} = 302.3 \text{ psi}$$

$$\rho = \frac{(0.85)(3000 \text{ psi})}{60{,}000 \text{ psi}} \left[1 - \sqrt{1 - \frac{(2)(302.3 \text{ psi})}{(0.85)(3000 \text{ psi})}}\right]$$

$= 0.00538$ (same as that obtained from Appendix A)



M_u = 160 ft-k
f_c = 3000 psi
f_y = 60,000 psi

FIGURE 4.5 Beam cross section for Example 4.5.

Trial-and-Error (Iterative) Method

Here it is necessary to estimate the value of a. The student probably has no idea of a reasonable value for this quantity, but the accuracy of the estimate is not a matter of importance. He or she can assume some value probably around $0.1d$ to $0.2d$ and then compute $d - a/2$ and A_s. With this value of A_s, a new value of a can be computed and the cycle repeated. After two or three cycles, a very good value of a will be obtained.

Assume $a = 2$ in.:

$$A_s = \frac{M_u}{\phi f_y \left(d - \frac{a}{2}\right)} = \frac{(12 \text{ in/ft})\,(160{,}000 \text{ ft-lb})}{(0.9)\,(60{,}000 \text{ psi})\left(21 \text{ in.} - \frac{2 \text{ in.}}{2}\right)} = 1.78 \text{ in.}^2$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(1.78 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.85)(3000 \text{ psi})(16 \text{ in.})} = 2.62 \text{ in.}$$

Assume $a = 2.6$ in.:

$$A_s = \frac{(12 \text{ in/ft})(160{,}000 \text{ ft-lb})}{(0.9)(60{,}000 \text{ psi})\left(21 \text{ in.} - \frac{2.62 \text{ in.}}{2}\right)} = 1.81 \text{ in.}^2$$

$$a = \frac{(1.81 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.85)(3000 \text{ psi})(16 \text{ in.})} = 2.66 \text{ in.} \quad \underline{\text{(close enough)}}$$

Based on this method, use a theoretical value of $A_s = 1.81$ in.^{2}

Note that all three methods result in the same value of $A_s = 1.81$ in.^{2}.

4.6 Bundled Bars

Sometimes when large amounts of steel reinforcing are required in a beam or column, it is very difficult to fit all the bars in the cross section. For such situations, groups of parallel bars may be bundled together. Up to four bars can be bundled, provided they are enclosed by stirrups or ties. The ACI Code Section 25.6.1.3 states that bars larger than #11 shall not be bundled in beams or girders. This is primarily because of crack control problems, a subject discussed in Chapter 6 of this text. That is, if the ACI crack control provisions are to be met, bars larger than #11 cannot practically be used.

Typical configurations for two-, three-, and four-bar bundles are shown in Figure 4.6. When bundles of more than one bar deep vertically are used in the plane of bending, they may not practically be hooked or bent as a unit. If end hooks are required, it is preferable to stagger the hooks of the individual bars within the bundle.

Although the ACI permits the use of bundled bars, their use in the tension areas of beams may very well be counterproductive because of the other applicable code restrictions that are brought into play as a result of their use.

FIGURE 4.6 Bundled-bar arrangements.

When spacing limitations and cover requirements are based on bar sizes, the bundled bars may be treated as a single bar for computation purposes; the diameter of the fictitious bar is to be calculated from the total equivalent area of the group. When individual bars in a bundle are cut off within the span of beams or girders, they should terminate at different points. The ACI Code Section 25.6.1.4 requires that there be a stagger of at least 40 bar diameters.

4.7 One-Way Slabs

Reinforced concrete slabs are large flat plates that are supported by reinforced concrete beams, walls, or columns; by masonry walls; by structural steel beams or columns; or by the ground. If they are supported on two opposite sides only, they are referred to as *one-way slabs* because the bending is in one direction only—that is, perpendicular to the supported edges. Should the slab be supported by beams on all four edges, it is referred to as a *two-way slab* because the bending is in both directions. Actually, if a rectangular slab is supported on all four sides, but the long side is two or more times as long as the short side, the slab will, for all practical purposes, act as a one-way slab, with bending primarily occurring in the short direction. Such slabs are designed as one-way slabs. You can easily verify these bending moment ideas by supporting a sheet of paper on two opposite sides or on four sides with the support situation described. This section is concerned with one-way slabs; two-way slabs are considered in Chapters 16 and 17. It should be realized that a large percentage of reinforced concrete slabs fall into the one-way slab class.

A one-way slab is assumed to be a rectangular beam with a large ratio of width to depth. Normally, a 12-in.-wide piece of such a slab is designed as a beam (see Figure 4.7), the slab being assumed to consist of a series of such beams side by side. The method of analysis is somewhat conservative because of the lateral restraint provided by the adjacent parts of the slab. Normally, a beam will tend to expand laterally somewhat as it bends, but this tendency to expand by each of the 12-in. strips is resisted by the adjacent 12-in.-wide strips, which tend to expand also. In other words, Poisson's ratio is assumed to be zero. Actually, the lateral expansion tendency results in a very slight stiffening of the beam strips, which is neglected in the design procedure used here.

The 12-in.-wide beam is quite convenient when thinking of the load calculations because loads are normally specified as so many pounds per square foot, and thus the load carried per foot of length of the 12-in.-wide beam is the load supported per square foot by the slab. The load supported by the one-way slab, including its own weight, is transferred to the members supporting the edges of the slab. Obviously, the reinforcing steel for flexure is placed perpendicular to these supports—that is, parallel

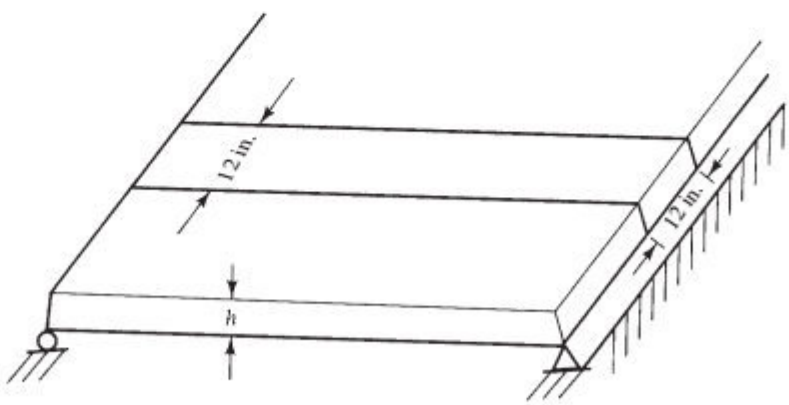


FIGURE 4.7 A 12-in. strip in a simply supported one-way slab.

to the long direction of the 12-in.-wide beams. This flexural reinforcing steel may not be spaced farther on center than three times the slab thickness, or 18 in., according to the ACI Code Section 7.7.2.3. Of course, there will be some reinforcing steel placed in the other direction to resist shrinkage and temperature stresses.

The thickness required for a particular one-way slab depends on the bending, the deflection, and shear requirements. As described in Section 4.2, the ACI Code Section 7.3.1.1 provides certain span/depth limitations for concrete flexural members where deflections are not calculated.

Because of the quantities of concrete involved in floor slabs, their depths are rounded off to closer values than are used for beam depths. Slab thicknesses are usually rounded off to the nearest $\frac{1}{4}$ in. on the high side for slabs of 6 in. or less in thickness and to the nearest $\frac{1}{2}$ in. on the high side for slabs thicker than 6 in.

As concrete hardens, it shrinks. In addition, temperature changes occur that cause expansion and contraction of the concrete. When cooling occurs, the shrinkage effect and the shortening due to cooling add together. The ACI Code Section 7.7.6 states that shrinkage and temperature reinforcement must be provided in a direction perpendicular to the main reinforcement for one-way slabs. (For two-way slabs, reinforcement is provided in both directions for bending.) The code states that for Grade 40 or 50 deformed bars, the minimum area of this steel is 0.002 times the gross cross-sectional area of the slab. Note that the gross cross-sectional area is bh (where h is the slab thickness). The ACI Code Section 24.4.3.3 states that shrinkage and temperature reinforcement may not be spaced farther apart than five times the slab thickness, or 18 in. When Grade 60 deformed bars or welded wire fabric is used, the minimum area is $0.0018bh$. For slabs with $f_y > 60{,}000$ psi, the minimum value is $(0.0018 \times 60{,}000)/f_y \geq 0.0014$.

In SI units, the minimum area of reinforcing steel are 0.002 for Grades 300 and 350 steels and 0.0018 for Grade 420 steel. When $f_y > 420$ MPa, the minimum area equals $(0.0018 \times 420)/f_y$. These reinforcement ratios are then multiplied by the gross area of concrete, bh, to obtain minimum reinforcing steel areas. The reinforcing steel may not be spaced farther apart than five times the slab thickness, or 500 mm.

Should structural walls or large columns provide appreciable resistance to shrinkage and temperature movements, it may very well be necessary to increase the minimum amounts listed.

Shrinkage and temperature steel serves as mat steel in that it is tied perpendicular to the main flexural reinforcing steel and holds it firmly in place as a mat. This steel also helps to distribute concentrated loads transversely in the slab.

Areas of steel are often determined for 1-ft widths of reinforced concrete slabs, footings, and walls. A table of areas of bars in slabs such as Appendix A, Table A.6 is very useful in such cases for selecting the specific bars to be used. A brief explanation of the preparation of this table is provided here.

For a 1-ft width of concrete, the total steel area obviously equals the total or average number of bars in a 1-ft width times the cross-sectional area of one bar. This can be expressed as (12 in./bar spacing c. to c.)(area of 1 bar). Some examples follow, and the values obtained can be checked in the table. Understanding these calculations enables one to expand the table as desired.

1. #9 bars, 6-in. o.c. total area in 1-ft width $= \left(\frac{12}{6}\right)(1.00) = 2.00 \text{ in.}^2$
2. #9 bars, 5-in. o.c. total area in 1-ft width $= \left(\frac{12}{5}\right)(1.00) = 2.40 \text{ in.}^2$

Example 4.6 illustrates the design of a one-way slab. It will be noted that the ACI Code Section 20.6.1 cover requirement for reinforcement in cast-in-place slabs (#11 and smaller bars) is $\frac{3}{4}$ in. clear, unless corrosion or fire protection requirements are more severe.

Example 4.6

Design a one-way slab for the inside of a building using the span, loads, and other data given in Figure 4.8. Normal-weight aggregate concrete is specified with a density of 145 pcf.

SOLUTION

Minimum Total Slab Thickness *h* if Deflections Are Not Computed (See Table 4.1)

$$h = \frac{\ell}{20} = \frac{(12 \text{ in/ft})\,(10 \text{ ft})}{20} = 6 \text{ in.}$$

Assume 6-in. slab (with d = approximately 6 in. $-\frac{3}{4}$ in. Cover $-\frac{1}{4}$ in. [for estimated half-diameter of bar size] = 5.0 in.). The moment is calculated, and then the amount of steel required is determined. If this value seems unreasonable, a different thickness is tried.

Design a 12-in.-wide strip of the slab. Thus, $b = 12$ in., and the load on the slab in units of lb/ft^2 becomes lb/ft. Usually 5 pcf is added to account for the weight of reinforcement, so 150 pcf is used in calculating the weight of a normal-weight concrete member.

$$DL = \text{slab wt} = \left(\frac{6 \text{ in.}}{12 \text{ in/ft}}\right)(150 \text{ pcf}) = 75 \text{ psf}$$

$$LL = 200 \text{ psf}$$

$$w_u = (1.2)\,(75 \text{ psf}) + (1.6)\,(200 \text{ psf}) = 410 \text{ psf}$$

$$M_u = \frac{(0.410 \text{ klf})\,(10 \text{ ft})^2}{8} = 5.125 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in/ft})\,(5125 \text{ ft-lb})}{(0.9)\,(12 \text{ in.})\,(5.00 \text{ in.})^2} = 227.8 \text{ psi}$$

$$\rho = 0.00393 \text{ (from Appendix A, Table A.13)}$$

$$> \rho_{min} = 0.0033$$

$$A_s = \rho b d = (0.00393)\,(12 \text{ in.})\,(5.0 \text{ in.}) = 0.236 \text{ in}^2\text{/ft}$$

Use #4 at 10 in. from Table A.6 ($A_s = 0.24$ in^2/ft)

Spacing < maximum of 18 in. as per ACI Code Section 7.6.5

Transverse Direction—Shrinkage and Temperature Steel

$$A_s = 0.0018bd = (0.0018)\,(12 \text{ in.})\,(6 \text{ in.}) = 0.1296 \text{ in}^2\text{/ft}$$

Use #3 at 10 in. (0.13 in^2/ft) as selected from Table A.6

Spacing < maximum of 18 in. as per ACI Code Section 7.12.2.2 OK

The #4 bars are placed below the #3 bars in this case. The #4 bars are the primary flexural reinforcing steel, and the value of d is based on this assumption. The #3 bars are for temperature and shrinkage control, and their depth within the slab is not as critical.

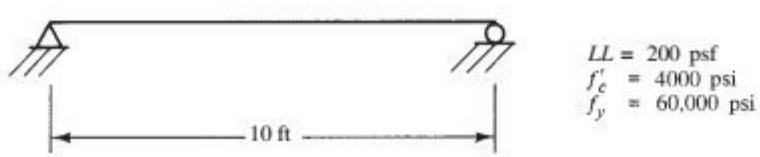


FIGURE 4.8 Given information for Example 4.6.

The designers of reinforced concrete structures must be very careful to comply with building code requirements for fire resistance. If the applicable code requires a certain fire resistance rating for floor systems, that requirement may very well cause the designer to use thicker slabs than might otherwise be required to meet the ACI strength design requirements. *In other words, the designer of a building should study carefully the fire resistance provisions of the governing building code before proceeding with the design*. Section 4.11 of ACI 318-14 includes such a requirement.

4.8 Cantilever Beams and Continuous Beams

Cantilever beams supporting gravity loads are subject to negative moments throughout their lengths. As a result, their reinforcement is placed in their top or tensile sides, as shown in Figures 4.9 and 4.10(a). The reader will note that for such members the maximum moments occur at the faces of the fixed supports. As a result, the largest amounts of reinforcing steel are required at those points. You should also note that the bars cannot be stopped at the support faces. They must be extended or anchored in the concrete beyond the support face. We call this as *development length* and it does not have to be straight as shown in the figure, because the bars may be hooked at 90° or 180°. Development lengths and hooked bars will be discussed in depth in Chapter 7.

Up to this point, only statically determinate members have been considered. The very common situation, however, is for beams and slabs to be continuous over several supports, as shown in Figure 4.10. Because reinforcing steel is needed on the tensile sides of the beams, we will place it in the bottoms when we have positive moments and in the tops when we have negative moments. There are several ways in which the reinforcing bars can be arranged to resist the positive and negative moments in continuous members. One possible arrangement is shown in Figure 4.10(a). These members, including bar arrangements, will be discussed in detail in Chapter 14.

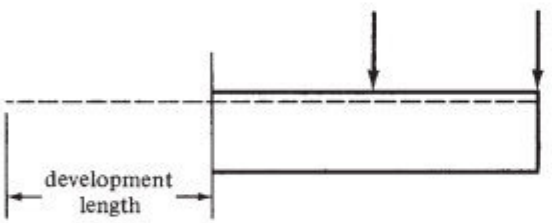


FIGURE 4.9 Cantilever beam development length.

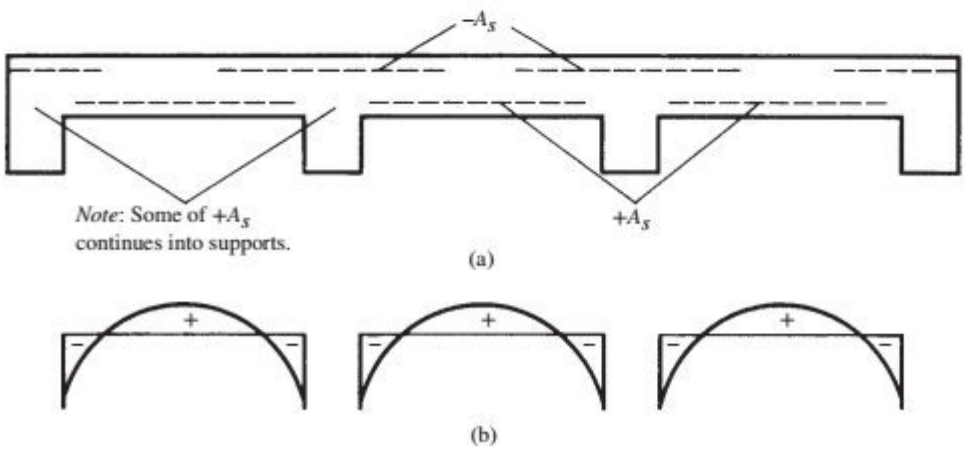


FIGURE 4.10 Continuous slab showing theoretical placement of bars for given moment diagram.

Chris Condon/US PGA TOUR/Getty Images, Inc.

Workers pour the first concrete of the new clubhouse during the TPC Sawgrass renovation (May 10, 2006).

4.9 SI Example

Example 4.7 illustrates the design of a beam using SI units.

Example 4.7

Design a rectangular beam for a 10-m simple span to support a dead load of 20 kN/m (not including beam weight) and a live load of 30 kN/m. Use $\rho = 0.5\rho_b$, $f'_c = 28$ MPa, and $f_y = 420$ MPa, and concrete weight is 23.5 kN/m^3. Do not use the ACI thickness limitation.

SOLUTION

Assume that the beam weight is 10 kN/m and $\phi = 0.90$.

$$w_u = (1.2)\,(30\text{ kN/m}) + (1.6)\,(30\text{ kN/m}) = 84\text{ kN/m}$$

$$M_u = \frac{(84\text{ kN/m})(10\text{ m})^2}{8} = 1050\text{ kN}\cdot\text{m}$$

$$\rho = \left(\frac{1}{2}\right)(0.0283) = 0.01415 \text{ (from Appendix B, Table B.7)}$$

$$M_u = \phi\rho f_y bd^2\left(1 - \frac{1}{1.7}\rho\frac{f_y}{f'_c}\right)$$

$$(10^6)\,(1050\text{ kN}\cdot\text{m}) = (0.9)\,(0.01415)\,(420\text{ MPa})\,(bd^2)\left[1 - \left(\frac{1}{1.7}\right)(0.01415)\left(\frac{420\text{ MPa}}{28\text{ MPa}}\right)\right]$$

$$bd^2 = 2.2432 \times 10^8\text{ mm}^3 \begin{cases} 400\text{ mm} \times 749\text{ mm} \\ 450\text{ mm} \times 706\text{ mm} \\ 500\text{ mm} \times 670\text{ mm} \leftarrow \text{ select} \end{cases}$$

Use 500-mm × 800-mm Section ($d = 680$ mm)

$$\text{Beam wt} = \frac{(500\text{ mm})\,(800\text{ mm})}{10^6\text{ mm}^2/\text{m}^2}\,(23.5\text{ kN/m}^3) = 9.4\text{ kN/m}$$

$$< 10\text{ kN/m assumed}$$

$$A_s = (0.01415)\,(500\text{ mm})\,(680\text{ mm}) = 4811\text{ mm}^2 \quad \underline{\text{OK}}$$

Use six #32 bars in two rows (4914 mm^2). One row could be used here.

A value of $\phi = 0.9$ was assumed earlier. This requires that ϵ_t be ≥ 0.005.

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(4914\text{ mm}^2)\,(420\text{ MPa})}{(0.85)\,(28\text{ MPa})\,(500\text{ mm})} = 173\text{ mm}$$

$$c = \frac{a}{\beta_1} = \frac{173\text{ mm}}{0.85} = 204\text{ mm}$$

$$\epsilon_t = \frac{680\text{ mm} - 204\text{ mm}}{204\text{ mm}}\,(0.003) = 0.0070 > 0.005 \qquad \underline{\therefore \phi = 0.90}$$

Note: Can more easily be checked with ρ values. Table B.7 indicates that a value of $\rho < 0.0181$ is needed to ensure $\phi = 0.9$. Our value of ρ satisfies that requirement.

$$b_{\min} = 267\text{ mm (from Appendix B, Table B.5 for three bars in a layer)}$$

$$< 500\text{ mm} \quad \underline{\text{OK}}$$

The final section is shown in Figure 4.11.

Note: This problem can be solved more quickly by making use of the Appendix tables. In Table B.9 with $f_y = 420$ MPa, $f'_c = 28$ MPa, and $\rho = 0.01415$.

$$\frac{M_u}{\phi b d^2} = 5.201\text{ MPa} \quad \text{(by interpolation)}$$

$$bd^2 = \frac{M_u}{\phi(5.201\text{ MPa})} = \frac{(1050\text{ kN} \cdot \text{m})\,(10)^3}{(0.9)\,(5.201\text{ MPa})} = 2.2432 \times 10^8\text{ mm}^3$$

After this step, proceed as shown above, when bd^2 was found using equations.

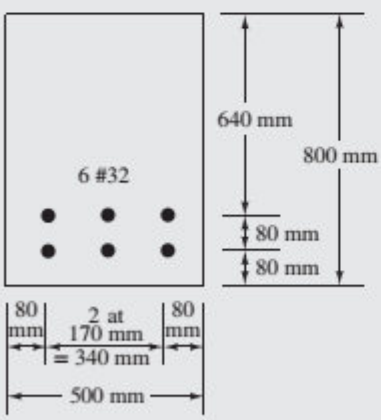


FIGURE 4.11 Beam cross section for Example 4.7.

4.10 Computer Example

Example 4.8

Repeat Example 4.4 using the Excel spreadsheet for Chapter 4.

SOLUTION

Use the worksheet called Beam Design. Enter material properties (f'_c, f_y) and M_u (can be taken from the bottom part of the spreadsheet or just entered if you already know it). Input $\rho = 0.009$ (given in the example). The two tables with headings b and d give some choices for b and d based on the ρ value you picked. Larger assumed values of ρ result in smaller values of b and d and vice versa. Select $b = 18$ in. and $d = 31$ in. (many other choices are also correct). Add 2.5 in. or more to d to get h, and enter that value (used only to find beam weight below). The spreadsheet recalculates ρ and A_s from actual values of b and d chosen, so note that ρ is not the same as originally assumed (0.00895 instead of 0.009). This results in a slightly smaller calculated steel area than in Example 4.4. You can also enter the number of bars and size to get a value for A_s. This value must exceed the theoretical value or an error message will appear. You should check to see if this bar selection will fit within the width selected.

At the bottom of the spreadsheet, the design moment M_u can be obtained if the beam is simply supported and uniformly loaded with only dead and live loads. The beam self-weight is calculated based on the input values for b and h (Cells D23 and D25). You may have to iterate a few times before these values all agree. In this example, the dead load is 2 klf plus self-weight. The input value for w_D is 2.0 + 0.65 plf, with the second term being taken from the spreadsheet. In working this problem the first time, you probably would not have these dimensions for b and h, hence the self-weight would not be correct. Iteration as done in Example 4.4 is also required with the spreadsheet, although it is much faster.

Design of singly reinforced rectangular beams

$f'_c =$	3	ksi
$f_y =$	60	ksi
$\beta_1 =$	0.85	
$M_u =$	623.4	ft-k

***Instructions*: Enter values only in shaded cells. Other values are calculated from those input values.**

Assume ρ = 0.009

$$bd^2 = \frac{M_u}{\phi f_y \rho \dfrac{1-\rho f_y}{1.7 f'_c}} = 17{,}215 \text{ in.}^3$$

$R = d/b$

R	b	d	b	d
1	25.82	25.82	14	35.07
1.2	22.86	27.44	15	33.88
1.4	20.63	28.88	16	32.80
1.5	19.70	29.56	17	31.82
1.6	18.87	30.20	18	30.93
1.7	18.13	30.82	19	30.10
1.8	17.45	31.41	20	29.34
1.9	16.83	31.98	21	28.63
2	16.27	32.53	22	27.97

Select b and d

$b =$ 18 in.
$d =$ 31 in.
$h =$ 34 in.
$R_n =$ 480.5

These tables give some choices for b and d that you may round up to enter here.

$$\rho = 0.85 f'_c / f_y \left\{1 - \left[1 - 2R_n/(.85f'_c)\right]^{0.5}\right\} = 0.00895$$

$A_s = \rho bd = 4.99$ in.^{2} <-- theoretical steel area

select bars: No. of bars 5 — Bar size # 9 — $A_s =$ 5.00 in.^{2}

Calculation of M_u for simply supported beam with D and L uniformly distributed loads

$w_D =$	**2.65**	**klf**
$w_L =$	**3**	**klf**
span =	**25**	**ft**
$w_u =$	**7.980**	**klf**
$M_u =$	**623.4**	**ft-k**
$\gamma_c =$	**145**	**pcf**
self-wt =	**0.6375**	**klf**

PROBLEMS

Problem 4.1 The estimated service or working axial loads and bending moments for a particular column are as follows: $P_D = 100$ k, $P_L = 60$ k, $M_D = 30$ ft-k, and $M_L = 20$ ft-k. Compute the axial load and moment values that must be used in the design. (*Ans.* $P_u = 216$ k; $M_u = 68$ ft-k)

Problem 4.2 Determine the required design strength for a column for which $P_D = 120$ k, $P_L = 40$ k, and wind $P_W = 60$ k compression or 80 k tension.

Problem 4.3 A reinforced concrete slab must support a dead working floor load of 90 psf, which includes the weight of the concrete slab and a live working load of 40 psf. Determine the factored uniform load for which the slab must be designed. (*Ans.* $w_u = 172$ psf)

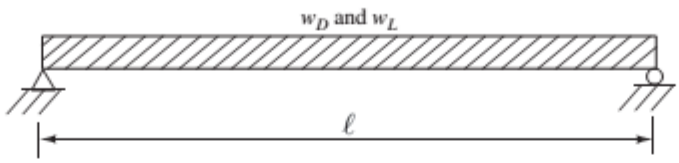
Problem 4.4 Using the Chapter 4 spreadsheet, Load Combination worksheet, repeat the following problems:

(a) Problem 4.1

(b) Problem 4.2

(c) Problem 4.3

For Problems 4.5 to 4.9, design rectangular sections for the beams, loads, and values given. Beam weights are not included in the given loads. Show sketches of beam cross sections, including bar sizes, arrangement, and spacing. Assume concrete weighs 150 lb/ft^3. Use $h = d + 2.5$ in.

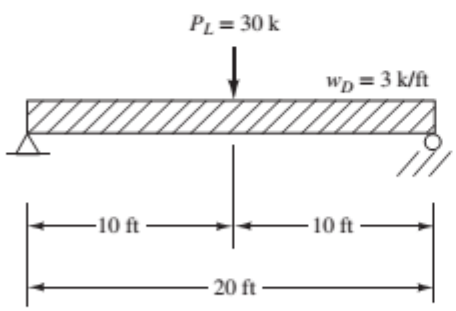


Problem No.	f_y (psi)	f'_c (psi)	Span ℓ (ft)	w_D not incl. beam wt (k/ft)	w_L (k/ft)	ρ*
4.5	60,000	4000	24	2	2.8	$\epsilon_t = 0.0075$
4.6	60,000	4000	30	2	2	$0.18f'_c/f_y$
4.7	50,000	3000	28	3	3	$\frac{1}{2}\rho_b$
4.8	60,000	4000	32	2	1.8	$\frac{1}{2}\rho_b$
4.9	60,000	3000	25	1.8	1.5	$\epsilon_t = 0.005$

*See Appendix A, Table A.7 for ρ values that correspond to the ϵ_t values listed.
One ans. Problem 4.5: 18 in. × 26 in. with 6 #9 bars.
One ans. Problem 4.7: 22 in. × 33 in. with 6 #11 bars.
One ans. Problem 4.9: 16 in. × 25 in. with 5 #9 bars.

For Problems 4.10 to 4.22, design rectangular sections for the beams, loads, and ρ values shown. Beam weights are not included in the loads shown. Show sketches of cross sections, including bar sizes, arrangement, and spacing. Assume concrete weighs 150 lb/ft^3, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi, unless given otherwise.

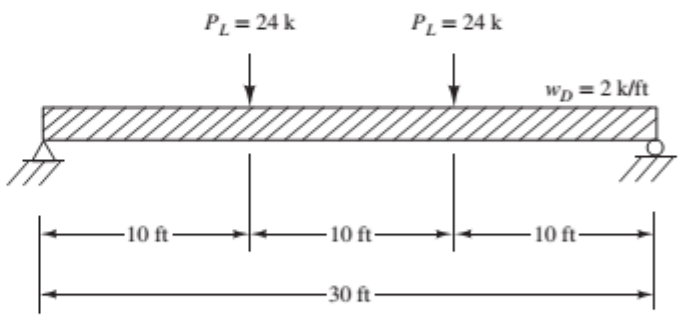
Problem 4.10



Use $\rho = \frac{1}{2}\rho_b$

Problem 4.11 Repeat Problem 4.10 if $w_D = 2$ k/ft and if $P_L = 10$ k. (*One ans.* 12 in. × 21 in. with 4 #8 bars)

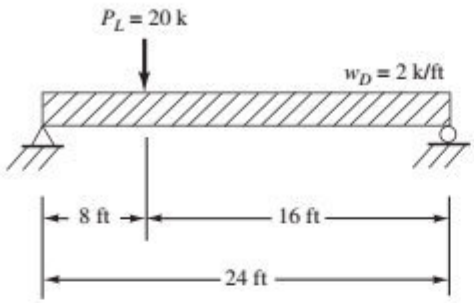
Problem 4.12



Use $\rho = \dfrac{0.18f'_c}{f_y}$

Problem 4.13 Repeat Problem 4.12 if $w_D = 1.5$ k/ft and $P_L = 30$ k. (*One ans.* 16 in. × 34 in. with 4 #11 bars)

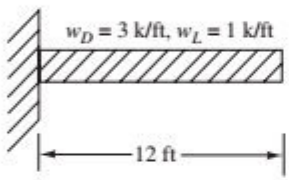
Problem 4.14



Use $\rho = \frac{1}{2}\rho_b$

Problem 4.15 Repare Problem 4.14 if $w_D = 3$ k/ft, $P_L = 40$ k, $f'_c = 3000$ psi, and $\rho = 0.5\rho_b$. (*One ans.* 16 in. × 33 in. with 5 #10 bars)

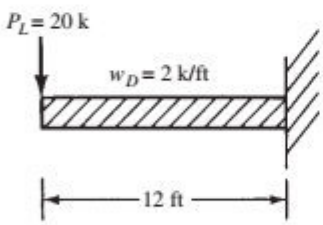
Problem 4.16



Use $\rho = \dfrac{0.18 f'_c}{f_y}$

Problem 4.17 Repeat Problem 4.16 if the beam span = 14 ft. (*One ans.* 20 in. × 27 in. with 6 #9 bars in top)

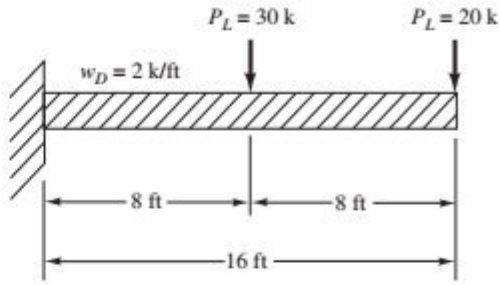
Problem 4.18



Use $\rho = \frac{1}{2}\rho_b$

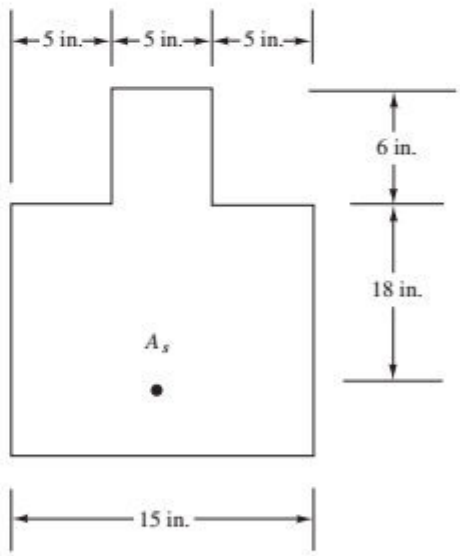
Problem 4.19 Repeat Problem 4.18 if $P_L = 10$ k, $\ell = 16$ ft. (*One ans.* 14 in. × 27 in. with 4 #10 bars in top)

Problem 4.20



Use $\rho = \frac{1}{2}\rho_{max}$

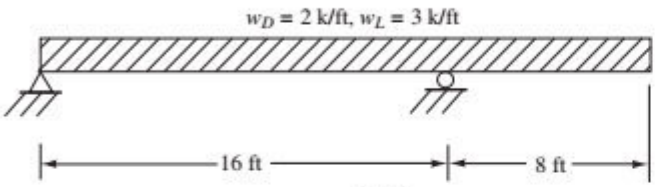
Problem 4.21 Select reinforcing bars for the beam shown if $M_u = 150$ ft-k, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi. (*Hint*: Assume that the distance from the c.g. of the tensile steel to the c.g. of the compression block equals 0.9 times the effective depth, d, of the beam.) After a steel area is computed, check the assumed distance and revise the steel area if necessary. Is $\epsilon_t \geq 0.005$? (*Ans.* $A_s = 1.57$ in.^{2}, $\epsilon_t = 0.00804 > 0.005$)



Problem 4.22 Repeat Problem 4.21 for $M_u = 250$ ft-k.

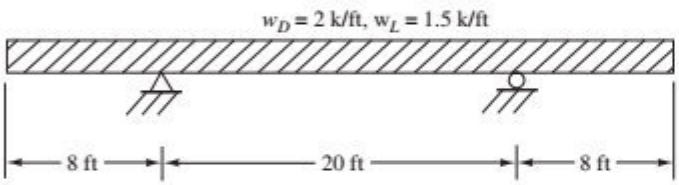
For Problems 4.23 and 4.24, design rectangular sections for the beams and loads shown. Beam weights are not included in the given loads. $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. Live loads are to be placed where they will cause the most severe conditions at the sections being considered. Select beam size for the largest moment (positive or negative), and then select the steel required for maximum positive and negative moment. Finally, sketch the beam and show approximate bar locations.

Problem 4.23 (*One ans.* 12 in. × 23 in. with 3 #9 bars negative reinforcement and 4 #7 bars positive reinforcement)



Use $\rho = \dfrac{0.18f'_c}{f_y}$

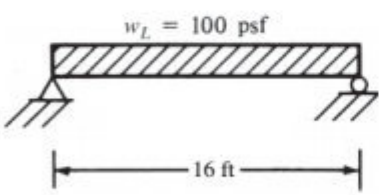
Problem 4.24



Use $\rho = 0.5\rho_b$

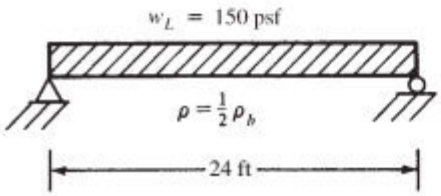
For Problems 4.25 and 4.26, design interior one-way slabs for the situations shown. Concrete weight = 150 lb/ft^3, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi. Do not use the ACI Code's minimum thickness for deflections (Table 4.1). Reinforcement ratios are given in the figures. The only dead load is the weight of the slab.

Problem 4.25 (*One ans.* 5 in. slab with #6 at 10 in. main reinf.)



Use $\rho = \dfrac{0.18f'_c}{f_y}$

Problem 4.26

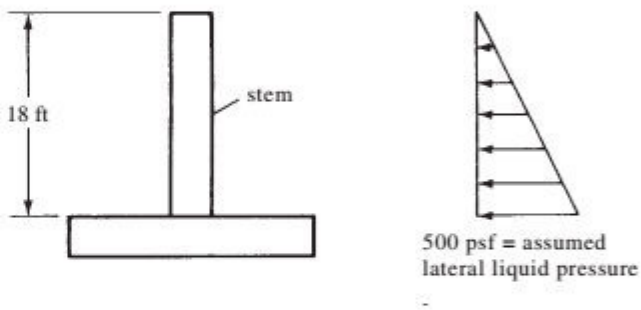


Problem 4.27 Repeat Problem 4.26 using the ACI Code's minimum thickness requirement for cases where deflections are not computed (Table 4.1). Do not use the ρ given in Problem 4.25. (*Ans.* 14.5-in. slab with #6 at 9 in. main reinf.)

Problem 4.28 Using $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, and ρ corresponding to $\epsilon_t = 0.005$, determine the depth required for a simple beam to support itself for a 200-ft simple span.

Problem 4.29 Determine the depth required for a beam to support itself only for a 100-ft span. Neglect concrete cover in self-weight calculations. Given $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and $\rho \cong 0.5\rho_b$. (*Ans.* $d = 32.5$ in.)

Problem 4.30 Determine the stem thickness for maximum moment for the retaining wall shown in the accompanying illustration. Also, determine the steel area required at the bottom and mid-depth of the stem if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. Assume that #8 bars are to be used and that the stem thickness is constant for the 18-ft height. Also, assume that the clear cover required is 2 in. and $\rho = 0.5\rho_b$.



Problem 4.31

(a) Design a 24-in.-wide precast concrete slab to support a 60-psf live load for a simple span of 15 ft. Assume the minimum concrete cover required to be $\frac{5}{8}$ in. Use welded wire fabric for reinforcing steel. $f_y = 60{,}000$ psi, $f'_c = 3000$ psi, and $\rho = 0.18f'_c/f_y$. (*Ans.* 4-in. slab with 4 × 8 D12/D6)

(b) Can a 300-lb football tackle walk across the center of the span when the other live load is not present? Assume 100% impact. (*Ans.* yes)

Problem 4.32 Prepare a flow chart for the design of tensilely reinforced rectangular beams.

Problem 4.33 Using the Chapter 4 spreadsheets, solve the following problems.

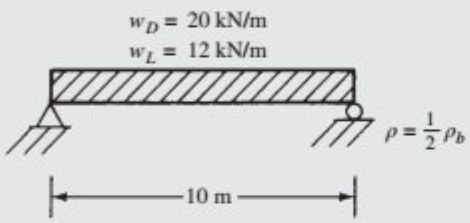
(a) Problem 4.6. (*Ans.* 16 in. × 33 in. with 5 #10 bars)

(b) Problem 4.18. (*Ans.* 18 in. × 27 in. with 5 #10 bars)

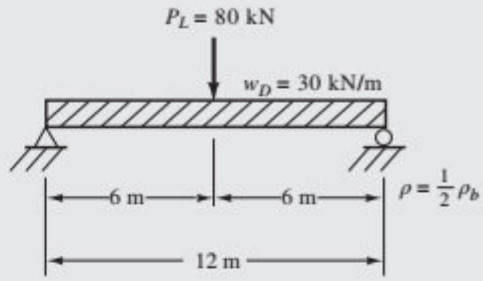
Problems in SI Units

For Problems 4.34 to 4.39, design rectangular sections for the beams, loads, and ρ values shown. Beam weights are not included in the loads given. Show sketches of cross sections including bar sizes, arrangements, and spacing. Assume concrete weighs 23.5 kN/m^3, $f_y = 420$ MPa, and $f'_c = 28$ MPa.

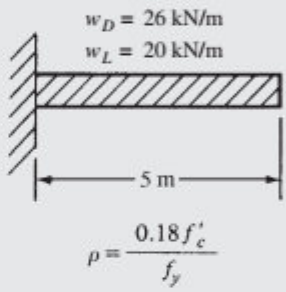
Problem 4.34



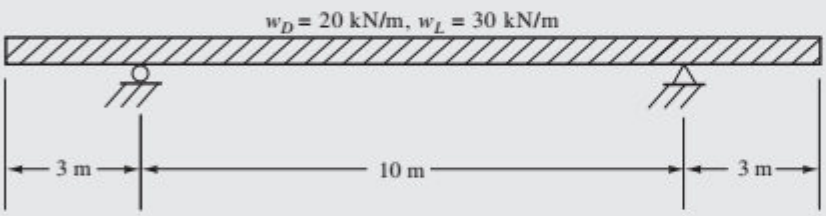
Problem 4.35 (*One ans.* 500 mm × 830 mm with 5 #36 bars)



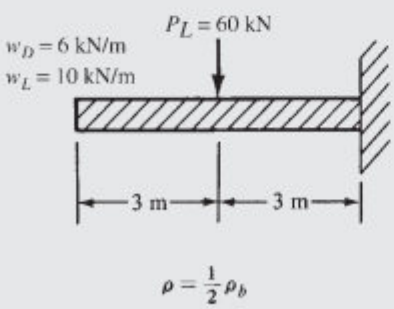
Problem 4.36



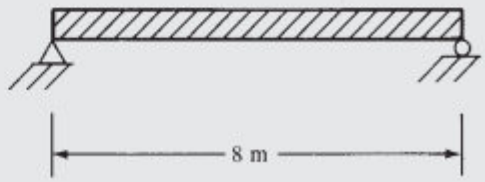
Problem 4.37 Place live loads to cause maximum positive and negative moments. $\rho = 0.18 f'_c / f_y$. (*One ans.* 400 mm × 830 mm with 5 #32 bars positive reinf.)



Problem 4.38



Problem 4.39 Design the one-way slab shown in the accompanying figure to support a live load of 10 kN/m^2. Do not use the ACI thickness limitation for deflections. Assume concrete weighs 23.5 kN/m^3, $f'_c = 28$ MPa, and $f_y = 420$ MPa. Use $\rho = \rho_{max}$ corresponding to $\epsilon_t = 0.005$. (*One ans.* 220-mm slab with #29 bars at 200-mm main steel)



CHAPTER 5 Analysis and Design of T Beams and Doubly Reinforced Beams

5.1 T Beams

Reinforced concrete floor systems normally consist of slabs and beams that are placed monolithically. As a result, the two parts act together to resist loads. In effect, the beams have extra widths at their tops, called *flanges*, and the resulting T-shaped beams are called *T beams*. The part of a T beam below the slab is referred to as the *web* or *stem*. (The beams may be L shaped if the stem is at the end of a slab.) The stirrups (described in Chapter 8) in the webs extend up into the slabs, as perhaps do bent-up bars, with the result that they further make the beams and slabs act together.

There is a problem involved in estimating how much of the slab acts as part of the beam. Should the flanges of a T beam be rather stocky and compact in cross section, bending stresses will be fairly uniformly distributed across the compression zone. If, however, the flanges are wide and thin, bending stresses will vary quite a bit across the flange due to shear deformations. The farther a particular part of the slab or flange is away from the stem, the smaller will be its bending stress.

Instead of considering a varying stress distribution across the full width of the flange, the ACI Code 6.3.2 calls for a smaller width with an assumed uniform stress distribution for design purposes. The objective is to have the same total compression force in the reduced width that actually occurs in the full width with its varying stresses.

The hatched area in Figure 5.1 shows the effective size of a T beam. For T beams with flanges on both sides of the web, the code states that the effective flange width may not exceed one-fourth of the beam span, and the overhanging width on each side may not exceed eight times the slab thickness or one-half the clear distance to the next web. An isolated T beam must have a flange thickness no less than one-half the web width, and its effective flange width may not be larger than four times the web width (ACI 6.3.2.2). If there is a flange on only one side of the web, the width of the overhanging flange cannot exceed one-twelfth the span, $6h_f$, or half the clear distance to the next web (ACI 6.3.2.1).

The analysis of T beams is quite similar to the analysis of rectangular beams in that the specifications relating to the strains in the reinforcement are identical. To repeat briefly, it is desirable to have ϵ_t values ≥ 0.005, and they may not be less than 0.004 unless the member is subjected to an axial load $\geq 0.10f'_cA_g$. You will learn that ϵ_t values are almost always much larger than 0.005 in T beams because of their very large compression flanges. For such members, the values of c are normally very small, and calculated ϵ_t values are very large.

The neutral axis (N.A.) for T beams can fall either in the flange or in the stem, depending on the proportions of the slabs and stems. If it falls in the flange, as it almost always does for positive moments, the rectangular beam formulas apply, as can be seen in Figure 5.2(a). The concrete below the neutral axis is assumed to be cracked, and its shape has no effect on the flexure calculations (other than weight). The section above the neutral axis is rectangular. If the neutral axis is below the flange, however, as shown for the beam of Figure 5.2(b), the compression concrete above the neutral axis no longer consists of a single rectangle, and thus the normal rectangular beam expressions do not apply.

If the neutral axis is assumed to fall within the flange (Figure 5.2a), the value of a can be computed as it was for rectangular beams:

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{\rho f_y d}{0.85 f'_c}$$

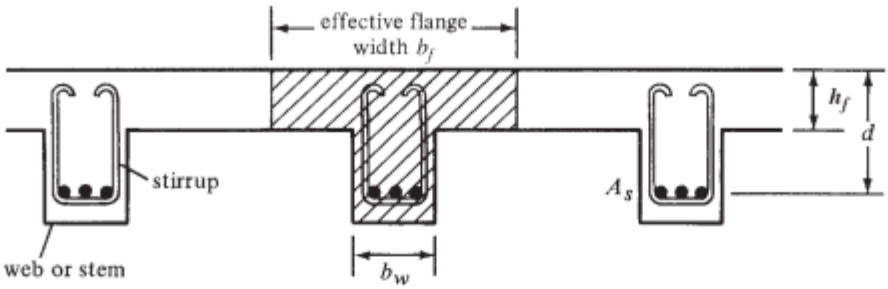


FIGURE 5.1 Effective width of T beams.

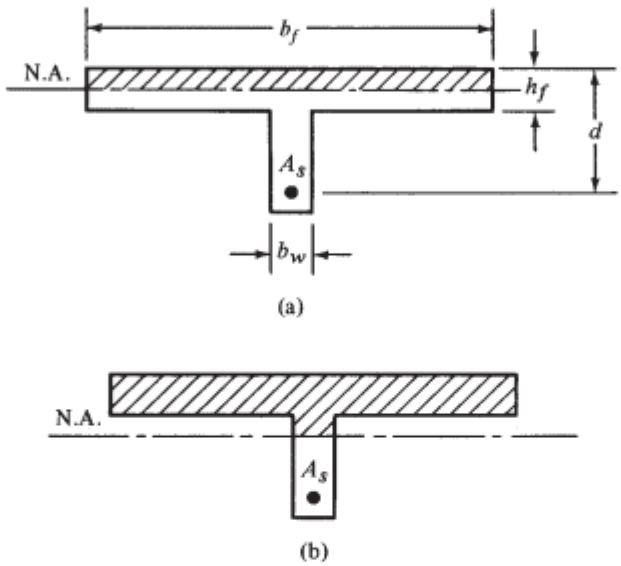


FIGURE 5.2 Neutral axis locations.

The distance to the neutral axis, c, equals a/β_1. If the computed value of a is equal to or less than the flange thickness, the section for all practical purposes can be assumed to be rectangular, even though the computed value of c is actually greater than the flange thickness.

A beam does not really have to look like a T beam to be one. This fact is shown by the beam cross sections shown in Figure 5.3. For these cases the compression concrete is T shaped, and the shape or size of the concrete on the tension side, which is assumed to be cracked, has no effect on the theoretical resisting moments. It is true, however, that the shapes, sizes, and weights of the tensile concrete do affect the deflections that occur (as is described in Chapter 6), and their dead weights affect the magnitudes of the moments to be resisted.

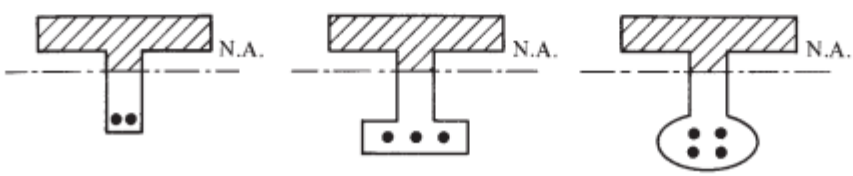


FIGURE 5.3 Various cross sections of T beams.

C&CA Collection, courtesy of The Concrete Society.

Natural History Museum, Kensington, London, England.

5.2 Analysis of T Beams

The calculation of the design strengths of T beams is illustrated in Examples 5.1 and 5.2. In the first of these problems, the neutral axis falls in the flange, whereas for the second, it is in the web. The procedure used for both examples involves the following steps:

1. Check $A_{s\,\min}$ as per ACI Section 9.6.1.2 using b_w as the web width.
2. Compute $T = A_s f_y$.
3. Determine the area of the concrete in compression (A_c) stressed to $0.85f'_c$.

$$C = T = 0.85f'_c A_c$$
$$A_c = \frac{T}{0.85f'_c}$$

4. Calculate a, c, and ϵ_t.
5. Calculate ϕM_n.

For Example 5.1, where the neutral axis falls in the flange, it would be logical to apply the normal rectangular equations of Section 3.4 of this book, but the authors have used the couple method as a background for the solution of Example 5.2, where the neutral axis falls in the web. This same procedure can be used for determining the design strengths of reinforced concrete beams of any shape (T, ⌜, ⊓, triangular, circular, etc.) as long as they are reinforced only with tension steel.

Example 5.1

Determine the design strength of the T beam shown in Figure 5.4, with $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. The beam has a 30-ft span and is cast integrally with a floor slab that is 4 in. thick. The clear distance between webs is 50 in.

SOLUTION

Check Effective Flange Width

$$b_f \le 16h_f + b_w = 16\ (4\text{ in.}) + 10\text{ in.} = 74\text{ in.}$$

$$b_f \le \text{average clear distance to adjacent webs} + b_w = 50\text{ in.} + 10\text{ in.} = 60\text{ in.} \leftarrow$$

$$b_f \le \frac{\text{span}}{4} = \frac{30\text{ ft}}{4} = 7.5\text{ ft} = 90\text{ in.}$$

Checking $A_{s\ min}$

$$A_{s\ min} = \frac{3\sqrt{f'_c}}{f_y} b_w d = \frac{(3\sqrt{4000}\text{ psi})}{60{,}000\text{ psi}} (10\text{ in.})\ (24\text{ in.}) = 0.76\text{ in.}^2$$

$$\text{nor less than } \frac{200 b_w d}{f_y} = \frac{(200)\ (10\text{ in.})\ (24\text{ in.})}{60{,}000\text{ psi}} = 0.80\text{ in.}^2 \leftarrow$$

$$< A_s = 6.00\text{ in.}^2 \qquad \underline{\text{OK}}$$

Computing T

$$T = A_s f_y = (6.00\text{ in.}^2)\ (60\text{ ksi}) = 360\text{ k}$$

Determining A_c

$$A_c = \frac{T}{0.85 f'_c} = \frac{360\text{ k}}{(0.85)\ (4\text{ ksi})} = 105.88\text{ in.}^2$$

$< \text{flange area} = (60\text{ in.})\ (4\text{ in.}) = 240\text{ in.}^2$ $\qquad \therefore$ Compression stress block, a, is in flange

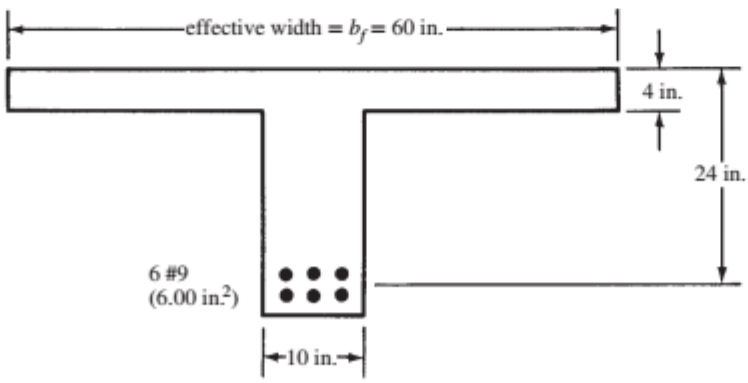


FIGURE 5.4 Beam cross section for Example 5.1.

Calculating a, c, and ϵ_t

$$a = \frac{105.88 \text{ in.}^2}{60 \text{ in.}} = 1.76 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{1.76 \text{ in.}}{0.85} = 2.07 \text{ in.}$$

$$\epsilon_t = \left(\frac{d-c}{c}\right)(0.003) = \left(\frac{24 \text{ in.} - 2.07 \text{ in.}}{2.07 \text{ in.}}\right)(0.003)$$

$$= 0.0318 > 0.005 \qquad \therefore \text{ Section is ductile and } \phi = 0.90$$

Calculating ϕM_n

Obviously, the stress block is entirely within the flange, and the rectangular formulas apply. However, using the couple method as follows:

$$\text{Lever arm} = z = d - \frac{a}{2} = 24 \text{ in.} - \frac{1.76 \text{ in.}}{2} = 23.12 \text{ in.}$$

$$\phi M_n = \phi T z = (0.90)\ (360 \text{ k})\ (23.12 \text{ in.})$$

$$= 7490.9 \text{ in.-k} = \underline{\underline{624.2 \text{ ft-k}}}$$

Example 5.2

Compute the design strength for the T beam shown in Figure 5.5, in which $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

Checking $A_{s\ \text{min}}$

$$A_{s\ \text{min}} = \frac{3\sqrt{4000} \text{ psi}}{60{,}000 \text{ psi}}(14 \text{ in.})\ (30 \text{ in.}) = 1.33 \text{ in.}^2$$

$$\text{nor less than } \frac{(200)\ (14 \text{ in.})\ (30 \text{ in.})}{60{,}000 \text{ psi}} = 1.40 \text{ in.}^2 \leftarrow$$

$$< A_s = 10.12 \text{ in.}^2 \quad \underline{\underline{\text{OK}}}$$

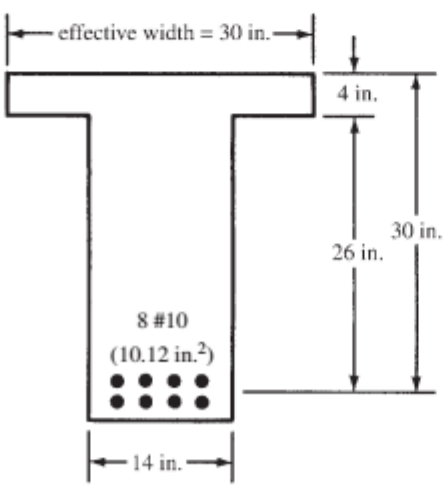


FIGURE 5.5 Beam cross section for Example 5.2.

Computing T

$$T = A_s f_y = (10.12 \text{ in.}^2)\ (60 \text{ ksi}) = 607.2 \text{ k}$$

Determining A_c and Its Center of Gravity

$$A_c = \frac{T}{0.85 f'_c} = \frac{607.2 \text{ k}}{(0.85)\ (4 \text{ ksi})} = 178.59 \text{ in.}^2$$

$$> \text{flange area} = (30 \text{ in.})\ (4 \text{ in.}) = 120 \text{ in.}^2$$

Obviously, the stress block must extend below the flange to provide the necessary compression area, $178.6 \text{ in.}^2 - 120 \text{ in.}^2 = 58.6 \text{ in.}^2$, *as shown in Figure 5.6. In order for the area in the web to be* 58.6 in.^2, *the depth of the stress block must be* $58.6 \text{ in.}^2/14 \text{ in.} = 4.19 \text{ in.}$ *below the flange.*

Computing the Distance $\bar{y}$ from the Top of the Flange to the Center of Gravity of A_c

$$\bar{y} = \frac{(120 \text{ in.}^2)\ (2 \text{ in.}) + (58.6 \text{ in.}^2)\left(4 \text{ in.} + \dfrac{4.19 \text{ in.}}{2}\right)}{178.6 \text{ in.}^2} = 3.34 \text{ in.}$$

The Lever Arm Distance from T to $C = 30.00 \text{ in.} - 3.34 \text{ in.} = 26.66 \text{ in.} = z$

Calculating a, c, and ϵ_t

$$a = 4 \text{ in.} + 4.19 \text{ in.} = 8.19 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{8.19 \text{ in.}}{0.85} = 9.64 \text{ in.}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \left(\frac{30 \text{ in.} - 9.64 \text{ in.}}{9.64 \text{ in.}}\right)(0.003) = 0.00634$$

$$> 0.005 \qquad \therefore \text{ Section is ductile and } \phi = 0.90$$

Calculating ϕM_n

$$\phi M_n = \phi T z = (0.90)\ (607.2 \text{ k})\ (26.66 \text{ in.}) = 14{,}569 \text{ in.-k}$$

$$= 1214 \text{ ft-k}$$

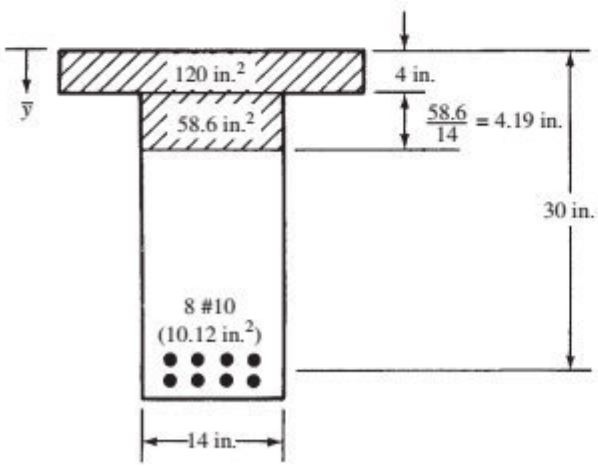


FIGURE 5.6 Area of concrete in compression.

5.3 Another Method for Analyzing T Beams

The preceding section presented a very important method of analyzing reinforced concrete beams. It is a general method that is applicable to tensilely reinforced beams of any cross section, including T beams. T beams are so very common, however, that many designers prefer another method that is specifically designed for T beams.

First, the value of a is determined as previously described in this chapter. Should it be less than the flange thickness, h_f, we will have a rectangular beam and the rectangular beam formulas will apply. Should it be greater than the flange thickness, h_f (as was the case for Example 5.2), the special method to be described here will be very useful.

The beam is divided into a set of rectangular parts consisting of the overhanging parts of the flange and the compression part of the web (see Figure 5.7).

The total compression, C_w, in the web rectangle, and the total compression in the overhanging flange, C_f, are computed:

$$C_w = 0.85f'_c ab_w$$
$$C_f = 0.85f'_c(b_f - b_w)\,(h_f)$$

Then the nominal moment, M_n, is determined by multiplying C_w and C_f by their respective lever arms from their centroids to the centroid of the steel:

$$M_n = C_w\left(d - \frac{a}{2}\right) + C_f\left(d - \frac{h_f}{2}\right)$$

This procedure is illustrated in Example 5.3. Although it seems to offer little advantage in computing M_n, we will learn that it does simplify the design of T beams when $a > h_f$ because it permits a direct solution of an otherwise trial-and-error problem.

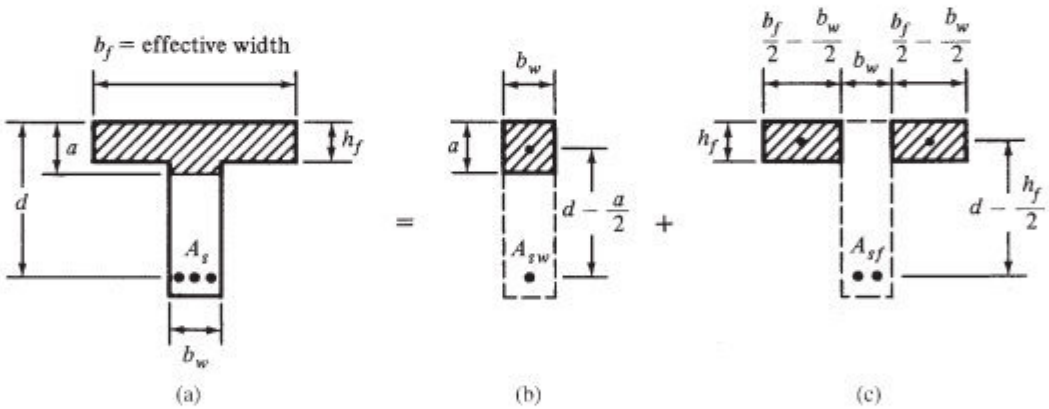


FIGURE 5.7 Separation of T beam into rectangular parts.

Example 5.3

Repeat Example 5.2 using the value of a (8.19 in.) previously obtained and the alternate formulas just developed. Reference is made to Figure 5.8, the dimensions of which were taken from Figure 5.5.

SOLUTION

(Noting that $a > h_f$)

Computing C_w and C_f

$$C_w = (0.85)\,(4\text{ ksi})\,(8.19\text{ in.})\,(14\text{ in.}) = 389.8\text{ k}$$
$$C_f = (0.85)\,(4\text{ ksi})\,(30\text{ in.} - 14\text{ in.})\,(4\text{ in.}) = 217.6\text{ k}$$

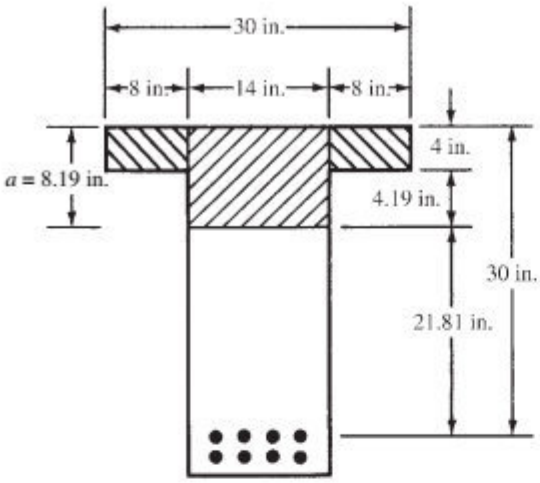


FIGURE 5.8 Concrete compression areas for Example 5.3.

Computing c and ϵ_t

$$c = \frac{a}{\beta_1} = \frac{8.19 \text{ in.}}{0.85} = 9.64 \text{ in.}$$

$$\epsilon_t = \left(\frac{d-c}{c}\right)(0.003) = \left(\frac{30 \text{ in.} - 9.64 \text{ in.}}{9.64 \text{ in.}}\right)(0.003) = 0.00634$$

$$> 0.005 \qquad \therefore \text{ Section is ductile and } \phi = 0.90$$

Calculating M_n and ϕM_n

$$M_n = C_w\left(d - \frac{a}{2}\right) + C_f\left(d - \frac{h_f}{2}\right)$$

$$= (389.8 \text{ k})\left(30 \text{ in.} - \frac{8.19 \text{ in.}}{2}\right) + (217.6 \text{ k})\left(30 \text{ in.} - \frac{4 \text{ in.}}{2}\right)$$

$$= 16{,}190 \text{ in.-k} = 1349 \text{ ft-k}$$

$$\phi M_n = (0.90)(1349 \text{ ft-k}) = \underline{\underline{1214 \text{ ft-k}}}$$

5.4 Design of T Beams

For the design of T beams, the flange has normally already been selected in the slab design, as it is for the slab. The size of the web is normally not selected on the basis of moment requirements but probably is given an area based on shear requirements; that is, a sufficient area is used so as to provide a certain minimum shear capacity, as will be described in Chapter 8. It is also possible that the width of the web may be selected on the basis of the width estimated to be needed to put in the reinforcing bars. Sizes may also be preselected, as previously described in Section 4.5, to simplify formwork for architectural requirements or for deflection reasons. For the examples that follow (Examples 5.4 and 5.5), the values of h_f, d, and b_w are given.

The flanges of most T beams are usually so large that the neutral axis probably falls within the flange, and thus the rectangular beam formulas apply. Should the neutral axis fall within the web, a trial-and-error process is often used for the design. In this process, a lever arm from the center of gravity of the compression block to the center of gravity of the steel is estimated to equal the larger of $0.9d$ or $d - (h_f/2)$, and from this value, called z, a trial steel area is calculated ($A_s = M_n/f_y z$). Then

by the process used in Example 5.2, the value of the estimated lever arm is checked. If there is much difference, the estimated value of z is revised and a new A_s is determined. This process is continued until the change in A_s is quite small. T beams are designed in Examples 5.4 and 5.5 by this process.

Often a T beam is part of a continuous beam that spans over interior supports, such as columns. The bending moment over the support is negative, so the flange is in tension. Also, the magnitude of the negative moment is usually larger than that of the positive moment near midspan. This situation will control the design of the T beam because the depth and web width will be determined for this case. When the beam is designed for positive moment at midspan, the width and depth are already known. See Section 5.5 for other details on T beams with negative moments.

Example 5.6 presents a more direct approach for the case where $a > h_f$. This is the case where the beam is assumed to be divided into its rectangular parts.

Example 5.4

Design a T beam for the floor system shown in Figure 5.9, for which b_w and d are given. $M_D = 80$ ft-k, $M_L = 100$ ft-k, $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and simple span = 20 ft.

SOLUTION

Effective Flange Width

(a) $\frac{1}{4}$ ft $\times$ 20 ft = 5 ft = 60 in.
(b) 12 in. + (2) (8) (4 in.) = 76 in.
(c) 10 ft = 120 in.

Therefore $b_f = 60$ in.

Computing Moments Assuming $\phi = 0.90$

$$M_u = (1.2)\,(80 \text{ ft-k}) + (1.6)\,(100 \text{ ft-k}) = 256 \text{ ft-k}$$

$$M_n = \frac{M_u}{\phi} = \frac{256 \text{ ft-k}}{0.90} = 284.4 \text{ ft-k}$$

Assuming a Lever Arm z Equal to the Larger of 0.9d or $d - (h_f/2)$

$$z = (0.9)\,(18 \text{ in.}) = \underline{\underline{16.20 \text{ in.}}}$$

$$z = 18 \text{ in.} - \frac{4 \text{ in.}}{2} = \underline{\underline{16.00 \text{ in.}}}$$

Trial Steel Area

$$A_s f_y z = M_n$$

$$A_s = \frac{(12 \text{ in./ft})\,(284.4 \text{ ft-k})}{(60 \text{ ksi})\,(16.20 \text{ in.})} = 3.51 \text{ in.}^2$$

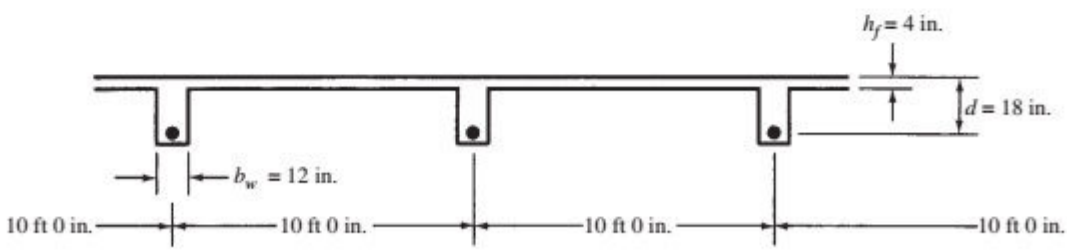


FIGURE 5.9 Cross section of T-beam floor system for Example 5.4.

Computing Values of *a* and *z*

$$0.85f'_c A_c = A_s f_y$$

$$(0.85)\ (4\ \text{ksi})\ (A_c\ \text{in.}^2) = (3.51\ \text{in.}^2)\ (60\ \text{ksi})$$

$$A_c = 61.9\ \text{in.}^2 < (4\ \text{in.})\ (60\ \text{in.}) = 240\ \text{in.}^2 \qquad \therefore \text{N.A. in flange}$$

$$a = \frac{61.9\ \text{in.}^2}{60\ \text{in.}} = 1.03\ \text{in.}$$

$$z = d - \frac{a}{2} = 18\ \text{in.} - \frac{1.03\ \text{in.}}{2} = 17.48\ \text{in.}$$

Calculating A_s with This Revised *z*

$$A_s = \frac{(12\ \text{in./ft})\ (284.4\ \text{ft-k})}{(60\ \text{ksi})\ (17.48\ \text{in.})} = 3.25\ \text{in.}^2$$

Computing Values of *a* and *z*

$$A_c = \frac{(3.25\ \text{in.}^2)\ (60\ \text{ksi})}{(0.85)\ (4\ \text{ksi})} = 57.4\ \text{in.}^2$$

$$a = \frac{57.4\ \text{in.}^2}{60\ \text{in.}} = 0.96\ \text{in.}$$

$$z = 18\ \text{in.} - \frac{0.96\ \text{in.}}{2} = 17.52\ \text{in.}$$

Calculating A_s with This Revised *z*

$$A_s = \frac{(12\ \text{in./ft})\ (284.4\ \text{ft-k})}{(60\ \text{ksi})\ (17.52\ \text{in.})} = 3.25\ \text{in.}^2 \qquad \text{OK, close enough to previous value}$$

Checking Minimum Reinforcing

$$A_{s\ \min} = \frac{3\sqrt{f'_c}}{f_y} b_w d = \frac{3\sqrt{4000}\ \text{psi}}{60{,}000\ \text{psi}} (12\ \text{in.})\ (18\ \text{in.}) = 0.68\ \text{in.}^2$$

but not less than

$$A_{s\ \min} = \frac{200 b_w d}{f_y} = \frac{(200)\ (12\ \text{in.})\ (18\ \text{in.})}{60{,}000\ \text{psi}} = 0.72\ \text{in.}^2 < 3.25\ \text{in.}^2 \qquad \text{OK}$$

or $\rho_{\min}$ (from Appendix A, Table A.7) = 0.0033

$$A_{s\ \min} = (0.0033)\ (12\ \text{in.})\ (18\ \text{in.}) = 0.71\ \text{in.}^2 < 3.25\ \text{in.}^2 \qquad \text{OK}$$

Computing *c*, ϵ_t, and ϕ

$$c = \frac{a}{\beta_1} = \frac{0.96\ \text{in.}}{0.85} = 1.13\ \text{in.}$$

$$\epsilon_t = \left(\frac{d-c}{c}\right)\ (0.003) = \left(\frac{18\ \text{in.} - 1.13\ \text{in.}}{1.13\ \text{in.}}\right)\ (0.003)$$

$$= 0.045 > 0.005 \qquad \therefore \phi = 0.90 \text{ as assumed}$$

$$A_{s\ \text{reqd}} = 3.25\ \text{in.}^2$$

Example 5.5

Design a T beam for the floor system shown in Figure 5.10, for which b_w and d are given. $M_D = 200$ ft-k, $M_L = 425$ ft-k, $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, and simple span = 18 ft.

SOLUTION

Effective Flange Width

(a) $\frac{1}{4}$ ft $\times$ 18 ft = 4 ft 6 in. = <u>54 in.</u>
(b) 15 in. + (2) (8) (3 in.) = 63 in.
(c) 6 ft = 72 in.

Therefore $b_f = 54$ in.

Moments Assuming $\phi = 0.90$

$$M_u = (1.2)\ (200 \text{ ft-k}) + (1.6)\ (425 \text{ ft-k}) = 920 \text{ ft-k}$$

$$M_n = \frac{M_u}{0.90} = \frac{920 \text{ ft-k}}{0.90} = 1022 \text{ ft-k}$$

Assuming a Lever Arm z

(Note that the compression area in the slab is very wide, and thus its required depth is very small.)

$$z = (0.90)\ (24 \text{ in.}) = 21.6 \text{ in.}$$

$$z = 24 \text{ in.} - \frac{3 \text{ in.}}{2} = \underline{\underline{22.5 \text{ in.}}}$$

Trial Steel Area

$$A_s = \frac{(12 \text{ in./ft})\ (1022 \text{ ft-k})}{(60 \text{ ksi})\ (22.5 \text{ in.})} = 9.08 \text{ in.}^2$$

Checking Values of a and z

$$A_c = \frac{(60 \text{ ksi})\ (9.08 \text{ in.}^2)}{(0.85)\ (3 \text{ ksi})} = 213.6 \text{ in.}^2$$

The stress block extends down into the web, as shown in Figure 5.11.

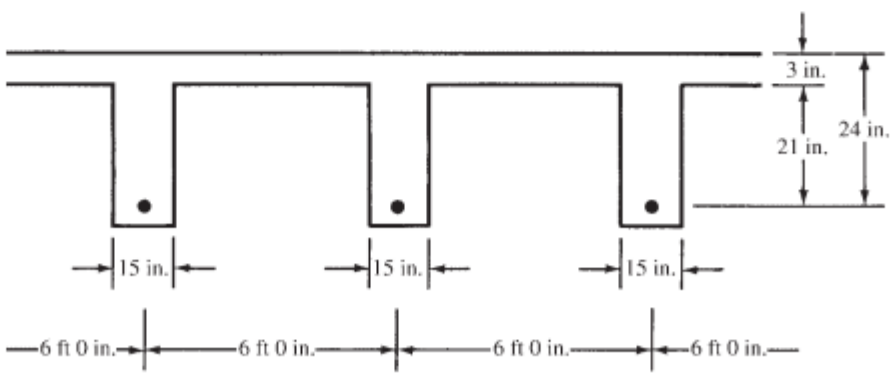


FIGURE 5.10 Cross section for T-beam floor system of Example 5.5.

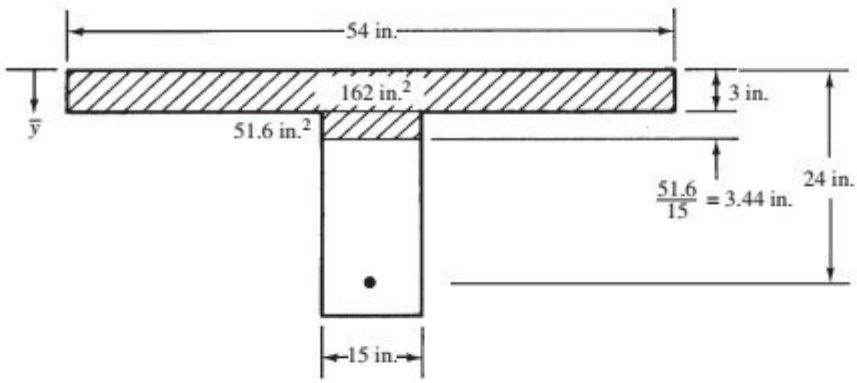


FIGURE 5.11 Concrete compression area for Example 5.5.

Computing the Distance $\bar{y}$ from the Top of the Flange to the Center of Gravity of A_c

$$\bar{y} = \frac{(162 \text{ in.}^2)(1.5 \text{ in.}) + (51.6 \text{ in.}^2)\left(3 \text{ in.} + \frac{3.44 \text{ in.}}{2}\right)}{213.6 \text{ in.}^2} = 2.28 \text{ in.}$$

$$z = 24 \text{ in.} - 2.28 \text{ in.} = 21.72 \text{ in.}$$

$$A_s = \frac{(12 \text{ in./ft})(1022 \text{ ft-k})}{(60 \text{ ksi})(21.72 \text{ in.})} = 9.41 \text{ in.}^2$$

The steel area required (9.41 in.^{2}) could be refined a little by repeating the design, but space is not used to do this. (If this is done, $A_s = 9.51$ in.^{2}.)

Checking Minimum Reinforcing

$$\rho_{min} \text{ (from Appendix A, Table A.7)} = 0.00333$$

or

$$A_{s\,min} = (0.00333)(15 \text{ in.})(24 \text{ in.}) = 1.20 \text{ in.}^2 < 9.51 \text{ in.}^2 \qquad \underline{\underline{\text{OK}}}$$

Checking Values of ϵ_t and ϕ

$$a = 3 \text{ in.} + 3.44 \text{ in.} = 6.44 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{6.44 \text{ in.}}{0.85} = 7.58 \text{ in.}$$

$$\epsilon_t = \left(\frac{d-c}{c}\right)(0.003) = \left(\frac{24 \text{ in.} - 7.58 \text{ in.}}{7.58 \text{ in.}}\right)(0.003)$$

$$= 0.00650 > 0.005 \qquad \therefore \phi = 0.90 \text{ as assumed}$$

If the calculations for ϵ_t and ϕ are repeated using the more refined values of $A_s = 9.51$ in.^{2}, then $a = 7.12$ in., $\epsilon_t = 0.0056$, and $\phi = 0.90$.

Our procedure for designing T beams has been to assume a value of z, compute a trial steel area of A_s, determine a for that steel area assuming a rectangular section, and so on. Should $a > h_f$, we will have a real T beam. A trial-and-error process was used for such a beam in Example 5.5. It is easily possible, however, to determine A_s directly using the method described in Section 5.3, where the member was broken down into its rectangular components. For this discussion, reference is made to Figure 5.7.

The compression force provided by the overhanging flange rectangles must be balanced by the tensile force in part of the tensile steel, A_{sf}, while the compression force in the web is balanced by the tensile force in the remaining tensile steel, A_{sw}.

For the overhanging flange, we have

$$0.85f'_c(b_f - b_w)\,(h_f) = A_{sf} f_y$$

from which the required area of steel, A_{sf}, equals

$$A_{sf} = \frac{0.85f'_c(b_f - b_w)h_f}{f_y}$$

The design strength of these overhanging flanges is

$$M_{uf} = \phi A_{sf} f_y \left(d - \frac{h_f}{2}\right)$$

The remaining moment to be resisted by the web of the T beam and the steel required to balance that value are determined next.

$$M_{uw} = M_u - M_{uf}$$

The steel required to balance the moment in the rectangular web is obtained by the usual rectangular beam expression. The value $M_{uw}/\phi b_w d^2$ is computed, and ρ_w is determined from the appropriate Appendix table or using the expression for ρ_w previously given in Section 3.4 of this book. Think of ρ_w as the reinforcement ratio for the beam shown in Figure 5.7(b). Then,

$$A_{sw} = \rho_w b_w d$$
$$A_s = A_{sf} + A_{sw}$$

Example 5.6

Rework Example 5.5 using the rectangular component method just described.

SOLUTION

First assume $a \leq h_f$ (which is very often the case). Then the design would proceed like that of a rectangular beam with a width equal to the effective width of the T-beam flange.

$$\frac{M_u}{\phi b d^2} = \frac{920 \text{ ft-k } (12{,}000 \text{ in.-lb/ft-k})}{(0.9)\,(54 \text{ in.})\,(24 \text{ in.})^2} = 394.4 \text{ psi}$$

$\rho = 0.0072$ (from Appendix A, Table A.12)

$$a = \frac{\rho f_y d}{0.85 f'_c} = \frac{0.0072\,(60 \text{ ksi})\,(24 \text{ in.})}{(0.85)\,(3 \text{ ksi})} = 4.06 \text{ in.} > h_f = 3 \text{ in.}$$

The beam acts like a T beam, not a rectangular beam, and the values for ρ and a above are not correct. If the value of a had been $\leq h_f$, the value of A_s would have been simply $\rho b_f d = 0.0072$ (54 in.) (24 in.) = 9.33 in.^{2}. Now break the beam up into two parts (Figure 5.7) and design it as a T beam.

Assuming $\phi = 0.90$

$$A_{sf} = \frac{(0.85)\,(3 \text{ ksi})\,(54 \text{ in.} - 15 \text{ in.})\,(3 \text{ in.})}{60 \text{ ksi}} = 4.97 \text{ in.}^2$$

$$M_{uf} = (0.9)\,(4.97 \text{ in.}^2)\,(60 \text{ ksi})\left(24 \text{ in.} - \frac{3}{2} \text{ in.}\right) = 6039 \text{ in.-k} = 503 \text{ ft-k}$$

$$M_{uw} = 920 \text{ ft-k} - 503 \text{ ft-k} = 417 \text{ ft-k}$$

Designing a Rectangular Beam with $b_w = 15$ in. and $d = 24$ in. to Resist 417 ft-k

$$\frac{M_{uw}}{\phi b_w d^2} = \frac{(12 \text{ in./ft})\,(417 \text{ ft-k})\,(1000 \text{ lb/k})}{(0.9)\,(15 \text{ in.})\,(24 \text{ in.})^2} = 643.5 \text{ psi}$$

$$\rho_w = 0.0126 \text{ (from Appendix A, Table A.12)}$$

$$A_{sw} = (0.0126)\,(15 \text{ in.})\,(24 \text{ in.}) = 4.54 \text{ in.}^2$$

$$A_s = 4.97 \text{ in.}^2 + 4.54 \text{ in.}^2 = \underline{9.51 \text{ in.}^2}$$

5.5 Design of T Beams for Negative Moments

When T beams are resisting negative moments, their flanges will be in tension and the bottom of their stems will be in compression, as shown in Figure 5.12. Obviously, for such situations, the rectangular beam design formulas will be used. Section 24.3.4 of the ACI Code requires that part of the flexural steel in the top of the beam in the negative-moment region be distributed over the effective width of the flange or over a width equal to one-tenth of the beam span, whichever is smaller. Should the effective width be greater than one-tenth of the span length, the code requires that some additional longitudinal steel be placed in the outer portions of the flange. The intention of this part of the code is to minimize the size of the flexural cracks that will occur in the top surface of the flange perpendicular to the stem of a T beam subject to negative moments.

In Section 3.8, it was stated that if a rectangular section had a very small amount of tensile reinforcing, its design-resisting moment, ϕM_n, might very well be less than its cracking moment. If this were the case, the beam might fail without warning when the first crack occurred. The same situation applies to T beams with a very small amount of tensile reinforcing.

When the flange of a T beam is in tension, the amount of tensile reinforcing needed to make its ultimate resisting moment equal to its cracking moment is about twice that of a rectangular section or that of a T section with its flange in compression. As a result, ACI Section 9.6.1 states that the minimum amount of reinforcing steel required equals the larger of the two values that follow:

$$A_{s\,\min} = \frac{3\sqrt{f'_c}}{f_y} b_w d \qquad \text{(ACI Equation 9.6.1.2a)}$$

or

$$A_{s\,\min} = \frac{200 b_w d}{f_y} \qquad \text{(ACI Equation 9.6.1.2b)}$$

For statically determinate members with their flanges in tension, b_w in the above expression is to be replaced with either $2b_w$ or the width of the flange, b_f, whichever is smaller.

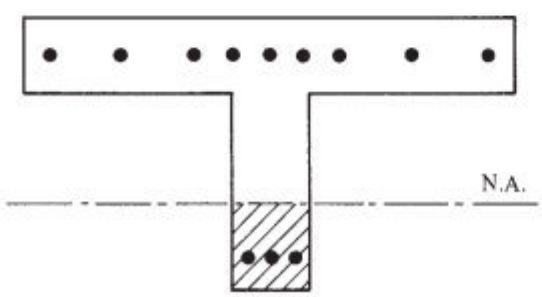


FIGURE 5.12 T beam with flange in tension and bottom (hatched) in compression (a rectangular beam).

Courtesy of EFCO Corp.

Reinforced concrete building in Calgary, Canada.

Courtesy of EFCO Corp.

Increased seating capacity at Bryant-Denny Stadium home of the Alabama Crimson Tide, Tuscaloosa, Alabama.

5.6 L-Shaped Beams

The author assumes for this discussion that L beams (i.e., edge T beams with a flange on one side only) are not free to bend laterally. Thus they will bend about their horizontal axes and will be handled as symmetrical sections, exactly as with T beams.

For L beams, the effective width of the overhanging flange may not be larger than one-twelfth the span length of the beam, six times the slab thickness, or one-half the clear distance to the next web (ACI 6.3.2.1).

If an L beam is assumed to be free to deflect both vertically and horizontally, it will be necessary to analyze it as an unsymmetrical section with bending about both the horizontal and vertical axes. An excellent reference on this topic is given in a book by MacGregor.[1]

5.7 Compression Steel

The steel that is occasionally used on the compression sides of beams is called *compression steel*, and beams with both tensile and compressive steel are referred to as *doubly reinforced beams*. Compression steel is not normally required in sections designed by the strength method because use of the full compressive strength of the concrete decidedly decreases the need for such reinforcement, as compared to designs made with the working-stress design method.

Occasionally, however, space or aesthetic requirements limit beams to such small sizes that compression steel is needed in addition to tensile steel. To increase the moment capacity of a beam beyond that of a tensilely reinforced beam with the maximum percentage of steel [when ($\epsilon_t = 0.005$)], it is necessary to introduce another resisting couple in the beam. This is done by adding steel in both the compression and tensile sides of the beam. Compression steel increases not only the resisting moments of concrete sections but also the amount of curvature that a member can take before flexural failure. This means that the ductility of such sections will be appreciably increased. Although expensive, compression steel makes beams tough and ductile, enabling them to withstand large moments, deformations, and stress reversals that might occur during earthquakes. As a result, many building codes for earthquake zones require that certain minimum amounts of compression steel be included in flexural members.

Compression steel is very effective in reducing long-term deflections due to shrinkage and plastic flow. In this regard you should note the effect of compression steel on the long-term deflection expression in Section 24.2.4.1 of the code (to be discussed in Chapter 6 of this text). Continuous compression bars are also helpful for positioning stirrups (by tying them to the compression bars) and keeping them in place during concrete placement and vibration.

Tests of doubly reinforced concrete beams have shown that even if the compression concrete crushes, the beam may very well not collapse if the compression steel is enclosed by stirrups. Once the compression concrete reaches its crushing strain, the concrete cover spalls or splits off the bars, much as in columns (see Chapter 9). If the compression bars are confined by closely spaced stirrups, the bars will not buckle until additional moment is applied. This additional moment cannot be considered in practice because beams are not practically useful after part of their concrete breaks off. (Would you like to use a building after some parts of the concrete beams have fallen on the floor?)

Section 9.7.6.4.1 of the ACI Code states that compression steel in beams must be enclosed by ties or stirrups or by welded wire fabric of equivalent area. In Section 9.7.6.4.2, the code states that the ties must be at least #3 in size for longitudinal bars #10 and smaller and at least #4 for larger longitudinal bars and bundled longitudinal bars. The ties may not be spaced farther apart than 16 bar diameters, 48 tie diameters, or the least dimension of the beam cross section (Code 9.7.6.4.3).

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[1] Wight, J. K. and MacGregor, J. G., 2011, *Reinforced Concrete Mechanics and Design*, 6th ed. (Upper Saddle River, NJ: Pearson Prentice Hall), pp. 165–168.

For doubly reinforced beams, an initial assumption is made that the compression steel yields as well as the tensile steel. (The tensile steel is always assumed to yield because of the ductile requirements of the ACI Code.) If the strain at the extreme fiber of the compression concrete is assumed to equal 0.00300 and the compression steel, A_s', is located two-thirds of the distance from the neutral axis to the extreme concrete fiber, then the strain in the compression steel equals $\frac{2}{3} \times 0.003 = 0.002$. If this is greater than the strain in the steel at yield, say $50{,}000/(29 \times 10^6) = 0.00172$ for 50,000-psi steel, the steel has yielded. It should be noted that actually the creep and shrinkage occurring in the compression concrete help the compression steel to yield.

Sometimes the neutral axis is quite close to the compression steel. As a matter of fact, in some beams with low steel percentages, the neutral axis may be right at the compression steel. For such cases, the addition of compression steel adds little, if any, moment capacity to the beam. It can, however, make the beam more ductile.

When compression steel is used, the nominal resisting moment of the beam is assumed to consist of two parts: the part due to the resistance of the compression concrete and the balancing tensile reinforcement, and the part due to the nominal moment capacity of the compression steel and the balancing amount of the additional tensile steel. This situation is illustrated in Figure 5.13. In the expressions developed here, the effect of the concrete in compression, which is replaced by the Compression steel, A_s', is neglected. This omission will cause us to overestimate M_n by a very small and negligible amount (less than 1%). The first of the two resisting moments is illustrated in Figure 5.13(b).

$$M_{n1} = A_{s1} f_y \left(d - \frac{a}{2}\right)$$

The second resisting moment is that produced by the additional tensile and Compression steel (A_{s2} and A_s'), which is presented in Figure 5.13(c).

$$M_{n2} = A_s' f_y (d - d')$$

Up to this point it has been assumed that the compression steel has reached its yield stress. If such is the case, the values of A_{s2} and A_s' will be equal because the addition to T of $A_{s2} f_y$ must be equal to the addition to C of $A_s' f_y$ for equilibrium. If the compression steel has not yielded, A_s' must be larger than A_{s2}, as will be described later in this section.

Combining the two values, we obtain

$$M_n = A_{s1} f_y \left(d - \frac{a}{2}\right) + A_{s2} f_y (d - d')$$

$$\phi M_n = \phi \left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A_{s2} f_y (d - d')\right]$$

The addition of compression steel only on the compression side of a beam will have little effect on the nominal resisting moment of the section. The lever arm, z, of the internal couple is not affected very

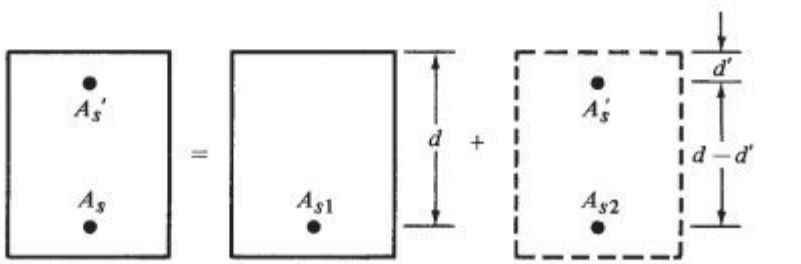


FIGURE 5.13 Doubly reinforced beam broken into parts.

much by the presence of the compression steel, and the value of T will remain the same. Thus, the value $M_n = Tz$ will change very little. To increase the nominal resisting moment of a section, it is necessary to add reinforcement on both the tension and the compression sides of the beam, thus providing another resisting moment couple.

Examples 5.7 and 5.8 illustrate the calculations involved in determining the design strengths of doubly reinforced sections. In each of these problems, the strain, f'_s, in the compression steel is checked to determine whether it has yielded. With the strain obtained, the compression steel stress, f'_s, is determined, and the value of A_{s2} is computed with the following equation:

$$A_{s2} f_y = A'_s f'_s$$

In addition, it is necessary to compute the strain in the tensile steel, ϵ_t, because if it is less than 0.005, the value of the bending, ϕ, will have to be computed, inasmuch as it will be less than its usual 0.90 value. The beam may not be used in the unlikely event that ϵ_t is less than 0.004.

To determine the value of these strains, an equilibrium equation is written, which when solved will yield the value of c and thus the location of the neutral axis. To write this equation, the nominal tensile strength of the beam is equated to its nominal compressive strength. Only one unknown appears in the equation, and that is c.

Initially the stress in the compression steel is assumed to be at yield ($f'_s = f_y$). From Figure 5.14, summing forces horizontally in the force diagram and substituting $\beta_1 c$ for a leads to the following:

$$A_s f_y = 0.85 f'_c \beta_1 cb + A'_s f_y$$

$$c = \frac{(A_s - A'_s) f_y}{0.85 f'_c \beta_1 b}$$

Referring to the strain diagram of Figure 5.14, from similar triangles it follows:

$$\epsilon'_s = \frac{c - d'}{c}(0.003)$$

If the strain in the compression steel $\epsilon'_s > \epsilon_y = f_y/E_s$, the assumption is valid and f'_s is at yield, f_y. If $\epsilon'_s < \epsilon_y$, the compression steel is not yielding, and the value of c calculated above is not correct. A new equilibrium equation must be written that assumes $f'_s < f_y$.

$$A_s f_y = 0.85 f'_c \beta_1 cb + A'_s \left(\frac{c - d'}{c}\right)(0.003) E_s$$

where $E_s = 29{,}000{,}000$ psi $= 29{,}000$ ksi.

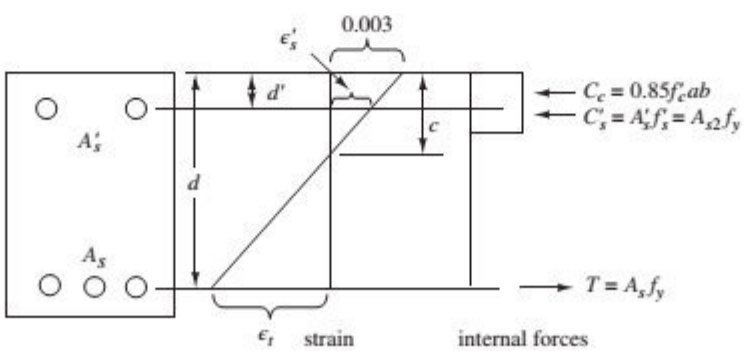


FIGURE 5.14 Internal strains and forces for doubly reinforced rectangular beam.

The value of c determined enables us to compute the strains in both the compression and tensile steels and thus their stresses. Even though the writing and solving of this equation are not too tedious, use of the Excel spreadsheet for beams with compression steel makes short work of the whole business.

Examples 5.7 and 5.8 illustrate the computation of the design moment strength of doubly reinforced beams. In the first of these examples, the compression steel yields, whereas in the second, it does not.

Example 5.7

Determine the design moment capacity of the beam shown in Figure 5.15 for which $f_y = 60{,}000$ psi and $f'_c = 3000$ psi.

SOLUTION

Writing the Equilibrium Equation Assuming $f'_s = f_y$

$$A_s f_y = 0.85 f'_c b \beta_1 c + A'_s f_y$$

$$(6.25 \text{ in.}^2)\,(60 \text{ ksi}) = (0.85)\,(3 \text{ ksi})\,(14 \text{ in.})\,(0.85c) + (2.00 \text{ in.}^2)\,(60 \text{ ksi})$$

$$c = \frac{(6.25 \text{ in.}^2 - 2.00 \text{ in.}^2)\,(60 \text{ ksi})}{(0.85)\,(3 \text{ ksi})\,(0.85)\,(14 \text{ in.})} = 8.40 \text{ in.}$$

$$a = \beta_1 c = (0.85)\,(8.40 \text{ in.}) = 7.14 \text{ in.}$$

Computing Strains in Compression Steel to Verify the Assumption That It Is Yielding

$$\epsilon'_s = \frac{c - d'}{c}(0.003) = \frac{8.40 \text{ in.} - 2.5 \text{ in.}}{8.40 \text{ in.}}(0.003) = 0.00211$$

$$\epsilon_y = \frac{f_y}{E_s} = \frac{60{,}000 \text{ psi}}{29{,}000{,}000 \text{ psi}} = 0.00207 < \epsilon'_s \qquad \therefore f'_s = f_y \text{ as assumed.}$$

Note: Example 5.8 shows what to do if this assumption is not correct.

$$A_{s2} = \frac{A'_s f'_s}{f_y} = \frac{(2.00 \text{ in.}^2)\,(60{,}000 \text{ psi})}{60{,}000 \text{ psi}} = 2.00 \text{ in.}^2$$

$$A_{s1} = A_s - A_{s2} = 6.25 \text{ in.}^2 - 2.00 \text{ in.}^2 = 4.25 \text{ in.}^2$$

$$\epsilon_t = \frac{d - c}{c}0.003 = \frac{24 \text{ in.} - 8.40 \text{ in.}}{8.40 \text{ in.}}(0.003) = 0.00557 > 0.005 \qquad \therefore \phi = 0.9$$

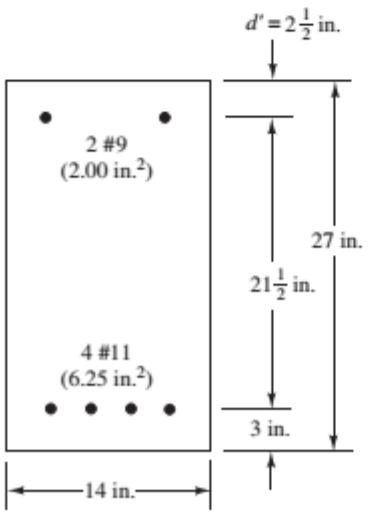


FIGURE 5.15 Beam cross section for Example 5.7.

Then the design moment strength is calculated as follows:

$$\phi M_n = \phi\left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A_s' f_s' (d - d')\right]$$

$$= 0.9\left[(4.25 \text{ in.}^2)(60 \text{ ksi})\left(24 \text{ in.} - \frac{7.14 \text{ in.}}{2}\right) + (2.00 \text{ in.}^2)(60 \text{ ksi})(24 \text{ in.} - 2.5 \text{ in.})\right]$$

$$= 7010 \text{ in.-k} = 584.2 \text{ ft-k}$$

Example 5.8

Compute the design moment strength of the section shown in Figure 5.16 if $f_y = 60{,}000$ psi and $f_c' = 4000$ psi.

SOLUTION

Writing the Equilibrium Equation Assuming $f_s' = f_y$

$$A_s f_y = 0.85 f_c' b \beta_1 c + A_s' f_y$$

$$(5.06 \text{ in.}^2)(60 \text{ ksi}) = (0.85)(4 \text{ ksi})(14 \text{ in.})(0.85c) + (1.20 \text{ in.}^2)(60 \text{ ksi})$$

$$c = \frac{(5.06 \text{ in.}^2 - 1.20 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(4 \text{ ksi})(0.85)} = 5.72 \text{ in.}$$

$$a = \beta_1 c = (0.85)(5.72 \text{ in.}) = 4.86 \text{ in.}$$

Computing Strains in Compression Steel to Verify the Assumption that It Is Yielding

$$\epsilon_s' = \frac{c - d'}{c}(0.003) = \frac{5.72 \text{ in.} - 2.5 \text{ in.}}{5.72 \text{ in.}}(0.003) = 0.00169$$

$$\epsilon_y = \frac{f_y}{E_s} = \frac{60{,}000 \text{ psi}}{29{,}000{,}000 \text{ psi}} = 0.00207 > \epsilon_s' \qquad \therefore f_s' \neq f_y \text{ as assumed.}$$

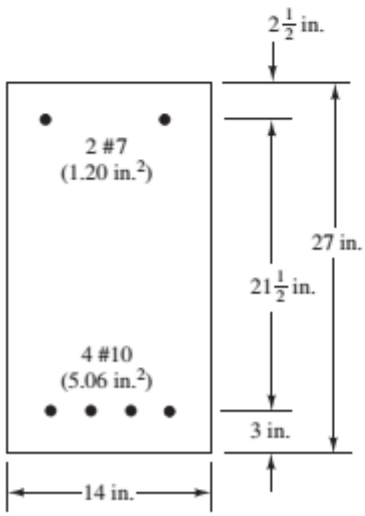


FIGURE 5.16 Beam cross section for Example 5.8.

Because the assumption is not valid, we have to use the equilibrium equation that is based on f'_s not yielding.

$$A_s f_y = 0.85 f'_c \beta_1 cb + A'_s \left(\frac{c - d'}{c}\right) (0.003) E_s$$

$$(5.06 \text{ in.}^2)(60 \text{ ksi}) = (0.85)(4 \text{ ksi})(0.85c)(14 \text{ in.}) + (1.20 \text{ in.}^2)\left(\frac{c - 2.5 \text{ in.}}{c}\right)(0.003)(29{,}000 \text{ ksi})$$

Solving the Quadratic Equation for $c = 6.00$ in. and $a = \beta_1 c = 5.10$ in.

Compute strains, stresses, and steel areas

$$\epsilon'_s = \left(\frac{c - d'}{c}\right)(0.003) = \frac{6.00 \text{ in.} - 2.5 \text{ in.}}{6.00 \text{ in.}}(0.003) = 0.00175 < \epsilon_y$$

$$f'_s = \epsilon'_s E_s = (0.00175)(29{,}000 \text{ ksi}) = 50.75 \text{ ksi}$$

$$A_{s2} = \frac{A'_s f'_s}{f_y} = \frac{(1.20 \text{ in.}^2)(50{,}750 \text{ psi})}{60{,}000 \text{ psi}} = 1.015 \text{ in.}^2$$

$$A_{s1} = A_s - A_{s2} = 5.06 \text{ in.}^2 - 1.015 \text{ in.}^2 = 4.045 \text{ in.}^2$$

$$\epsilon_t = \left(\frac{d - c}{c}\right)(0.003) = \frac{24 \text{ in.} - 6.00 \text{ in.}}{6.00 \text{ in.}}(0.003) = 0.0090 > 0.005 \qquad \therefore \phi = 0.9$$

Then the design moment strength is calculated as follows:

$$\phi M_n = \phi\left[A_{s1} f_y \left(d - \frac{a}{2}\right) + A'_s f'_s (d - d')\right]$$

$$= 0.9\left[(4.045 \text{ in.}^2)(60 \text{ ksi})\left(24 \text{ in.} - \frac{5.10 \text{ in.}}{2}\right) + (1.20 \text{ in.}^2)(50.75 \text{ ksi})(24 \text{ in.} - 2.5 \text{ in.})\right]$$

$$= 5863 \text{ in.-k} = 488.6 \text{ ft-k}$$

5.8 Design of Doubly Reinforced Beams

Sufficient tensile steel can be placed in most beams so that compression steel is not needed. But if it is needed, the design is usually quite straightforward. Examples 5.9 and 5.10 illustrate the design of doubly reinforced beams. The solutions follow the theory used for analyzing doubly reinforced sections.

Example 5.9

Design a rectangular beam for $M_D = 325$ ft-k and $M_L = 400$ ft-k if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. The maximum permissible beam dimensions are shown in Figure 5.17.

$$M_u = (1.2)(325 \text{ ft-k}) + (1.6)(400 \text{ ft-k}) = 1030 \text{ ft-k}$$

SOLUTION

Assuming $\phi = 0.90$

$$M_n = \frac{M_u}{\phi} = \frac{1030 \text{ ft-k}}{0.90} = 1144.4 \text{ ft-k}$$

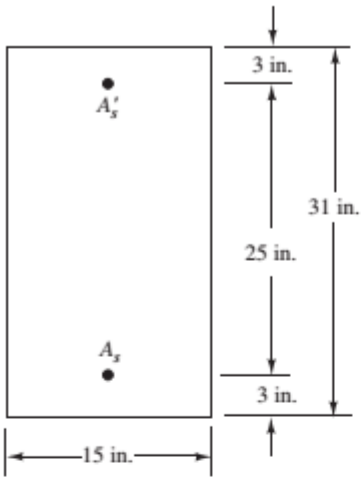


FIGURE 5.17 Beam cross section for Example 5.9.

Assuming Maximum Possible Tensile Steel with No Compression Steel and Computing Beam's Nominal Moment Strength

ρ_{max} (from Appendix A, Table A.7) = 0.0181

$A_{s1} = (0.0181)\ (15 \text{ in.})\ (28 \text{ in.}) = 7.60 \text{ in.}^2$

For $\rho = 0.0181$, $\dfrac{M_u}{\phi b d^2}$ (from Table A.13) = 912.0 psi

$M_{u1} = (912.0 \text{ psi})\ (0.9)\ (15 \text{ in.})\ (28 \text{ in.})^2 = 9{,}652{,}608 \text{ in.-lb}$

= 804.4 ft-k, which is less than the design moment. Therefore compression steel is required.

$$M_{n1} = \frac{804.4}{0.90} = 893.8 \text{ ft-k}$$

$M_{n2} = M_n - M_{n1} = 1144.4 \text{ ft-k} - 893.8 \text{ ft-k} = 250.6 \text{ ft-k}$

Checking to See Whether Compression Steel Has Yielded

$$a = \frac{(7.60 \text{ in.}^2)\ (60 \text{ ksi})}{(0.85)\ (4 \text{ ksi})\ (15 \text{ in.})} = 8.94 \text{ in.}$$

$$c = \frac{8.94 \text{ in.}}{0.85} = 10.52 \text{ in.}$$

$$\epsilon'_s = \left(\frac{10.52 \text{ in.} - 3 \text{ in.}}{10.52 \text{ in.}}\right)\ (0.00300) = 0.00214 > 0.00207$$

Therefore, compression steel has yielded.

$$\text{Theoretical } A'_s \text{ required} = \frac{M_{n2}}{(f_y)\ (d - d')} = \frac{(12 \text{ in./ft})\ (250.6 \text{ ft-k})}{(60 \text{ ksi})\ (28 \text{ in.} - 3 \text{ in.})} = 2.00 \text{ in.}^2$$

$$A'_s f'_s = A_{s2} f_y$$

$$A_{s2} = \frac{A'_s f'_s}{f_y} = \frac{(2.00 \text{ in.}^2)\ (60 \text{ ksi})}{60 \text{ ksi}} = 2.00 \text{ in.}^2$$

Try 2 #9 (2.00 in.^{2})

$$A_s = A_{s1} + A_{s2}$$

$$A_s = 7.60 \text{ in.}^2 + 2.00 \text{ in.}^2 = \underline{9.60 \text{ in.}^2}$$

Try 8 #10 (10.12 in.^{2})

If we had been able to select bars with exactly the same areas as calculated here, ϵ_t would have remained equal to 0.005 as originally assumed and $\phi = 0.90$, but such was not the case.

From the equation for c in Section 5.7, c is found to equal 11.24 in. and $a = \beta_1 c = 9.55$ in. using actual, not theoretical, bar areas for A_s and A'_s.

$$\epsilon'_s = \left(\frac{11.24 \text{ in.} - 3 \text{ in.}}{11.24 \text{ in.}}\right)(0.003) = 0.00220 > 0.00207 \text{ compression steel yields}$$

$$\epsilon_t = \left(\frac{28 \text{ in.} - 11.24 \text{ in.}}{11.24 \text{ in.}}\right)(0.003) = 0.00447 < 0.005$$

$$\phi = 0.65 + (0.00447 - 0.002)\left(\frac{250}{3}\right) = 0.855$$

$$\phi M_n = 0.855\left[(10.12 \text{ in.}^2 - 2.00 \text{ in.}^2)(60 \text{ ksi})\left(28 \text{ in.} - \frac{9.55 \text{ in.}}{2}\right) + (2.00 \text{ in.}^2)(60 \text{ ksi})(25 \text{ in.})\right]$$

$$= 12{,}241 \text{ in.-lb} = 1020 \text{ ft-k} < 1030 \text{ ft-k} \quad \underline{\text{No good}}$$

The beam does not have sufficient capacity because of the variable ϕ factor. This can be avoided if you are careful in picking bars. Note that the actual value of A'_s is exactly the same as the theoretical value. The actual value of A_s, however, is higher than the theoretical value by $10.12 - 9.6 = 0.52$ in.^{2}. If a new bar selection for A'_s is made whereby the actual value of A'_s exceeds the theoretical value by about this much (0.52 in.^{2}), the design will be adequate. Select three #8 bars ($A'_s = 2.36$ in.^{2}) and repeat the previous steps. Note that the actual steel areas are used below, not the theoretical ones. As a result, the values of c, a, ϵ'_s, and f'_s must be recalculated.

Assuming $f'_s = f_y$

$$c = \frac{(A_s - A'_s)f_y}{0.85 f'_c b \beta_1} = \frac{(10.12 \text{ in.}^2 - 2.36 \text{ in.}^2)(60 \text{ ksi})}{(0.85)(4 \text{ ksi})(15 \text{ in.})(0.85)} = 10.74 \text{ in.}$$

$$\epsilon'_s = \left(\frac{c - d'}{c}\right)(0.003) = \frac{10.74 \text{ in.} - 3 \text{ in.}}{10.74 \text{ in.}}(0.003) = 0.00216 > \epsilon_y \qquad \therefore \text{ Assumption is valid.}$$

$$\epsilon_t = \left(\frac{d - c}{c}\right)(0.003) = \frac{28 \text{ in.} - 10.74 \text{ in.}}{10.74 \text{ in.}}(0.003) = 0.00482 < 0.005 \qquad \therefore \phi \neq 0.9$$

$$\phi = 0.65 + (\epsilon_t - 0.002)\left(\frac{250}{3}\right) = 0.88$$

$$A_{s2} = \frac{A'_s f'_s}{f_y} = \frac{(2.36 \text{ in.}^2)(60 \text{ ksi})}{60 \text{ ksi}} = 2.36 \text{ in.}^2$$

$$A_{s1} = A_s - A_{s2} = 10.12 \text{ in.}^2 - 2.36 \text{ in.}^2 = 7.76 \text{ in.}^2$$

$$M_{n1} = A_{s1} f_y \left(d - \frac{a}{2}\right) = (7.76 \text{ in.}^2)(60 \text{ ksi})\left[28 \text{ in.} - \frac{(0.85)(10.74 \text{ in.})}{2}\right] = 10{,}912 \text{ in.-k} = 909.3 \text{ ft-k}$$

$$M_{n2} = A_{s2} f_y (d - d') = (2.36 \text{ in.}^2)(60 \text{ ksi})(28 \text{ in.} - 3 \text{ in.}) = 3540 \text{ in.-k} = 295 \text{ ft-k}$$

$$M_n = M_{n1} + M_{n2} = 909.3 \text{ ft-k} + 295 \text{ ft-k} = 1204.3 \text{ ft-k}$$

$$\phi M_n = (0.88)(1204.3 \text{ ft-k}) = 1059.9 \text{ ft-k} > M_u \quad \underline{\text{OK}}$$

Note that eight #10 bars will not fit in a single layer in this beam. If they were placed in two layers, the centroid would have to be more than 3 in. from the bottom of the section. It would be necessary to increase the beam depth, h, in order to provide for two layers or to use bundled bars (Section 7.4).

Example 5.10

A beam is limited to the dimensions $b = 15$ in., $d = 20$ in., and $d' = 4$ in. If $M_D = 170$ ft-k, $M_L = 225$ ft-k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi, select the reinforcement required.

SOLUTION

$$M_u = (1.2)\,(170 \text{ ft-k}) + (1.6)\,(225 \text{ ft-k}) = 564 \text{ ft-k}$$

Assuming $\phi = 0.90$

$$M_n = \frac{564 \text{ ft-k}}{0.90} = 626.7 \text{ ft-k}$$

$$\text{Max } A_{s1} = (0.0181)\,(15 \text{ in.})\,(20 \text{ in.}) = 5.43 \text{ in.}^2$$

$$\text{For } \rho = 0.0181, \quad \frac{M_u}{\phi b d^2} = 912.0 \text{ psi (from Appendix A, Table A.13)}$$

$$M_{u1} = (912 \text{ psi})\,(0.90)\,(15 \text{ in.})\,(20 \text{ in.})^2 = 4{,}924{,}800 \text{ in.-lb} = 410.4 \text{ ft-k}$$

$$M_{n1} = \frac{410.4 \text{ ft-k}}{0.90} = 456.0 \text{ ft-k}$$

$$M_{n2} = 626.7 \text{ ft-k} - 456.0 \text{ ft-k} = 170.7 \text{ ft-k}$$

Checking to See if Compression Steel Has Yielded

$$a = \frac{A_{s1} f_y}{0.85 f'_c b} = \frac{(5.43 \text{ in.}^2)\,(60 \text{ ksi})}{(0.85)\,(4 \text{ ksi})\,(15 \text{ in.})} = 6.39 \text{ in.}$$

$$c = \frac{6.39 \text{ in.}}{0.85} = 7.52 \text{ in.}$$

$$\epsilon'_s = \left(\frac{7.52 \text{ in.} - 4.00 \text{ in.}}{7.52 \text{ in.}}\right)(0.003) = 0.00140 < \frac{60 \text{ ksi}}{29{,}000 \text{ ksi}} = 0.00207$$

$$\therefore f'_s = (0.00140)\,(29{,}000 \text{ ksi}) = 40.6 \text{ ksi}$$

$$\text{Theoretical } A'_{s\,\text{reqd}} = \frac{M_{n2}}{f'_s (d - d')}$$

$$= \frac{(12 \text{ in./ft})\,(170.7 \text{ ft-k})}{(40.6 \text{ ksi})\,(20 \text{ in.} - 4 \text{ in.})} = \underline{3.15 \text{ in.}^2} \qquad \underline{\text{Try 4 \#8 } (3.14 \text{ in.}^2)}$$

$$A'_s f'_s = A_{s2} f_y$$

$$A_{s2} = \frac{(3.14 \text{ in.}^2)\,(40.6 \text{ ksi})}{60 \text{ ksi}} = 2.12 \text{ in.}^2$$

$$A_s = A_{s1} + A_{s2} = 5.43 \text{ in.}^2 + 2.12 \text{ in.}^2 = \underline{7.55 \text{ in.}^2} \qquad \underline{\text{Try 6 \#10 } (7.59 \text{ in.}^2)}$$

Subsequent checks using actual steel areas reveal $\epsilon_t = 0.00495$, $\phi = 0.896$, and $\phi M_n = 459.4$ ft-k, which is less than M_u by about 0.1%.

5.9 SI Examples

Examples 5.11 and 5.12 illustrate the analysis of a T beam and the design of a doubly reinforced beam using SI units.

Example 5.11

Determine the design strength of the T beam shown in Figure 5.18 if $f_y = 420$ MPa, $f'_c = 35$ MPa, and $E_s = 200{,}000$ MPa.

SOLUTION

Computing T and A_c

$$T = (3060 \text{ mm}^2)\,(420 \text{ MPa}) = 1{,}285{,}200 \text{ N}$$

$$A_c = \frac{T}{0.85f'_c} = \frac{1{,}285{,}200 \text{ N}}{(0.85)\,(35 \text{ MPa})} = 43{,}200 \text{ mm}^2$$

$$\rho = \frac{A_s}{b_w d} = \frac{3060 \text{ mm}^2}{(300 \text{ mm})\,(550 \text{ mm})} = 0.0185 < \rho_{max}$$

$$= 0.0216 \text{ (from Appendix B, Table B.7)} \quad \underline{\underline{\text{OK}}}$$

$$\rho_{min} = \frac{\sqrt{f'_c}}{4f_y} = \frac{\sqrt{35} \text{ MPa}}{(4)\,(420 \text{ MPa})} = 0.00352 < 0.0185 \quad \underline{\underline{\text{OK}}}$$

or

$$\frac{1.4}{f_y} = \frac{1.4}{420 \text{ MPa}} = 0.00333 < 0.0185 \quad \underline{\underline{\text{OK}}}$$

Calculating Design Strength

$$a = \frac{43{,}200 \text{ mm}^2}{1200 \text{ mm}} = 36 \text{ mm} < h_f = 100 \text{ mm}$$

$\therefore$ stress block is entirely within flange

$$z = d - \frac{a}{2} = 550 \text{ mm} - \frac{36 \text{ mm}}{2} = 532 \text{ mm}$$

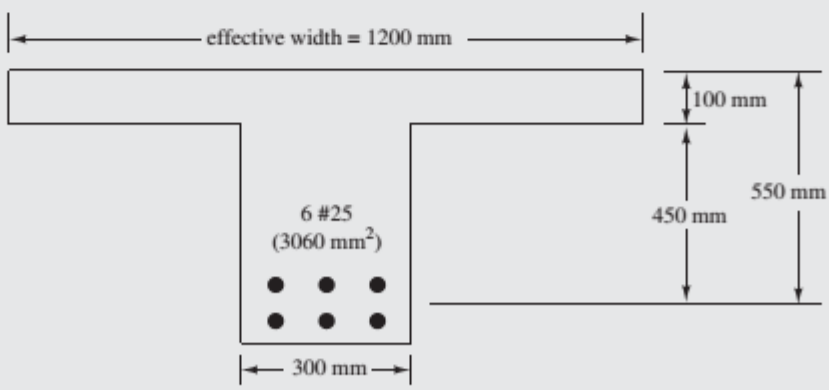


FIGURE 5.18 Beam cross section for Example 5.11.

$$\phi M_n = \phi T z$$
$$= (0.9)\ (1{,}285{,}200\ \text{N})\ (532\ \text{mm})$$
$$= 6.153 \times 10^8\ \text{N} \cdot \text{mm} = \underline{615.3\ \text{kN} \cdot \text{m}}$$

Example 5.12

If $M_u = 1225\ \text{kN} \cdot \text{m}$, determine the steel area required for the section shown in Figure 5.19. Should compression steel be required, assume that it will be placed 70 mm from the compression face. $f'_c = 21\ \text{MPa}$, $f_y = 420\ \text{MPa}$, and $E_s = 200{,}000\ \text{MPa}$.

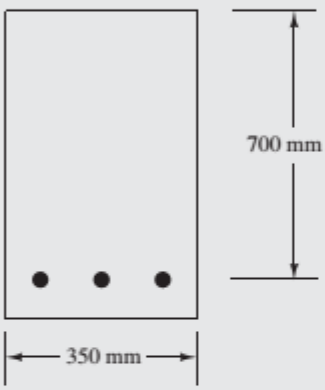


FIGURE 5.19 Beam cross section for Example 5.12.

SOLUTION

$$M_n = \frac{1225\ \text{kN} \cdot \text{m}}{0.9} = 1361\ \text{kN} \cdot \text{m}$$

ρ_{max} if singly reinforced = 0.0135 (from Appendix B, Table B.7)

$$A_{s1} = (0.0135)\ (350\ \text{mm})\ (700\ \text{mm}) = 3307\ \text{mm}^2$$

$$\frac{M_{u1}}{\phi b d^2} = 4.769\ \text{MPa (from Table B.8)}$$

$$M_{u1} = (\phi b d^2)\ (4.769\ \text{MPa}) = \frac{(4.769\ \text{MPa})\ (0.9)\ (350\ \text{mm})\ (700\ \text{mm})^2}{10^6} = 736.1\ \text{kN} \cdot \text{m}$$

$$M_{n1} = \frac{736.1\ \text{kN} \cdot \text{m}}{0.9} = 818\ \text{kN} \cdot \text{m}$$

$< M_n$ of 1361 kN · m $\quad\quad$ $\therefore$ Double reinforcement required

$$M_{n2} = M_n - M_{n1} = 1361\ \text{kN} \cdot \text{m} - 818\ \text{kN} \cdot \text{m} = 543\ \text{kN} \cdot \text{m}$$

Checking to See if Compression Steel Yields

$$a = \frac{(3307\ \text{mm}^2)\,(420\ \text{MPa})}{(0.85)\,(21\ \text{MPa})\,(350\ \text{mm})} = 222.32\ \text{mm}$$

$$c = \frac{222.32\ \text{mm}}{0.85} = 261.55\ \text{mm}$$

$$\epsilon_s' = \left(\frac{261.55\ \text{mm} - 70\ \text{mm}}{261.55\ \text{mm}}\right)(0.003) = 0.00220$$

$$> \frac{420\ \text{MPa}}{200{,}000\ \text{MPa}} = 0.00210 \qquad \therefore \text{Compression steel yields.}$$

$$A'_{s\,\text{reqd}} = \frac{M_{n2}}{f_y(d - d')} = \frac{543\ \text{kN} \cdot \text{m} \times 10^6}{(700\ \text{mm} - 70\ \text{mm})\,(420\ \text{MPa})} = \underline{2052\ \text{mm}^2}$$

Use 3 #32 bars (2457 mm^2)

$$A_s = A_{s1} + A_{s2} = 3307\ \text{mm}^2 + 2052\ \text{mm}^2 = \underline{5359\ \text{mm}^2}$$

Use 6 #36 bars (A_s = 6036 mm^2)

5.10 Computer Examples

Example 5.13

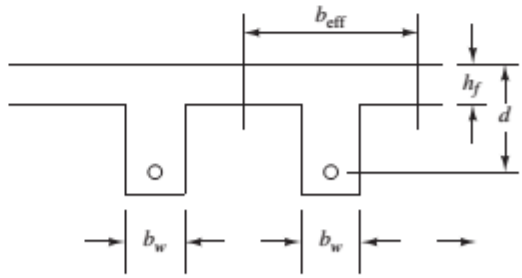
Repeat Example 5.3 using the Excel spreadsheet.

SOLUTION

Open the Excel spreadsheet for T beams and select the Analysis worksheet tab at the bottom. Input only cells C3 through C9 highlighted in yellow (only in the Excel spreadsheets, not in the printed example).

T-Beam Analysis

f'_c =	4000	psi
f_y =	60,000	psi
b_{eff} =	30	in.
b_w =	14	in.
d =	30	in.
h_f =	4	in.
A_s =	10.12	in.^{2}
β_1 =	0.85	



$$A_{s\,\text{min}} = \frac{3\sqrt{f'_c}}{f_y} b_w d = 1.33\ \text{in.}^2$$

$$A_{s\,\text{min}} = \frac{200}{f_y} b_w d = 1.40\ \text{in.}^2 \qquad A_{s\,\text{min}} = 1.40\ \text{in.}^2 \quad \text{Minimum steel is OK.}$$

If $a \le h_f$: $a > h_f$, so this analysis is not valid.

$$a = \frac{A_s f_y}{0.85 f'_c b} = 5.95 \text{ in.}$$

$M_n = A_s f_y (d - a/2) =$ – in.-lb = See solution below ft-k

$c = a/\beta_1 = 7.00346$ in.

$\epsilon_t = \frac{d-c}{c}(0.003) = 0.009851$ –

$\phi = 0.9$

$\phi M_n =$ – in.-lb = See solution below ft-k

If $a > h_f$: $a > h_f$, so this analysis is valid—acts like a T beam.

$$A_{sf} = \frac{0.85 f'_c (b - b_w) h_f}{f_y} = 3.627 \text{ in.}^2$$

$$A_{sw} = A_s - A_{sf} = 6.493 \text{ in.}^2$$

$$a = \frac{A_{sw} f_y}{0.85 f'_c b} = 8.185 \text{ in.}$$

$$M_n = A_{sf} f_y \left(d - \frac{h_f}{2}\right) + A_{sw} f_y \left(d - \frac{a}{2}\right) = 16{,}186{,}387 \text{ in.-lb} = 1348.9 \text{ ft-k}$$

$c = a/\beta_1 = 9.629263$ in.

$\epsilon_t = \frac{d-c}{c}(0.003) = 0.006347$ –

$\phi = 0.9$

$\phi M_n = 14{,}567{,}748$ in.-lb = 1214.0 ft-k

Example 5.14

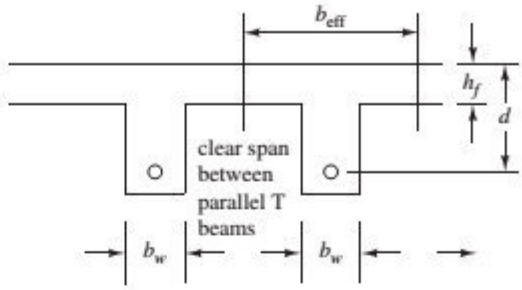
Repeat Example 5.6 using the Excel spreadsheet.

SOLUTION

Open the Excel spreadsheet for T beams and select the T-Beam Design worksheet tab at the bottom. Input only cells C3 through C9 highlighted in yellow.

T-Beam Design

$f'_c =$	3000	psi
$f_y =$	60,000	psi
$b_{eff} =$	54	in.
$b_w =$	15	in.
$d =$	24	in.
$h_f =$	3	in.
$M_u =$	920	ft-k
$\beta_1 =$	0.85	
$M_{tu} =$	697.1	ft-k



If $a \leq h_f$: $a > h_f$, so this analysis is not valid.

$$R_n = \frac{M_u}{\phi b d^2} = 394.38 \text{ in.}$$

$$\rho = \frac{0.85 f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_n}{0.85 f'_c}}\right) = 0.007179$$

$$a = \frac{\rho f_y d}{0.85 f'_c} = 4.054201 \text{ in.}$$

$$c = a/\beta_1 = 4.769648 \text{ in.}$$

$$\epsilon_t = \frac{d-c}{c}(0.003) = 0.012095 \qquad A_{s\,min} = \frac{3\sqrt{f'_c}}{f_y} b_w d = 0.985901 \text{ in.}^2$$

$$\phi = 0.9 \qquad - \qquad -$$

$$A_s = \rho b d = 9.304391 \text{ in.}^2 \qquad A_{s\,min} = \frac{200}{f_y} b_w d = 1.2 \text{ in.}^2$$

$$-$$

If $a > h_f$: $a > h_f$, so this analysis is valid—acts like a T beam.

$$A_{sf} = \frac{0.85 f'_c\,(b - b_w)\,h_f}{f_y} = 4.97 \text{ in.}^2$$

$$M_{uf} = A_{sf} f_y \left(d - \frac{h_f}{2}\right) = 503.5 \text{ ft-k}$$

$$M_{uw} = M_{uf} - M_u = 416.5 \text{ ft-k}$$

$$R_{nw} = \frac{M_{uw}}{\phi b d^2} = 642.8$$

$$\rho_w = \frac{0.85f'_c}{f_y}\left(1 - \sqrt{1 - \frac{2R_{nw}}{0.85f'_c}}\right) = 0.012573$$

$$a = \frac{\rho_w f_y d}{0.85f'_c} = 7.100128 \text{ in.}$$

$$c = a/\beta_1 = 8.353092 \text{ in.}$$

$$\epsilon_t = \frac{d-c}{c}(0.003) = 0.00562 \quad -$$

$$\phi = 0.9 \quad -$$

$$A_{sw} = \rho_\omega b_w d = 4.53 \text{ in.}^2 \qquad A_{s\,\min} = \frac{3\sqrt{f'_c}}{f_y} b_w d = 0.985901 \text{ in.}^2 \quad -$$

$$\boxed{A_s = A_{sw} + A_{sf} = 9.50 \text{ in.}^2} \qquad A_{s\,\min} = \frac{200}{f_y} b_w d = 1.2 \text{ in.}^2$$

Note: Solution is based on $\phi = 0.9$.

Example 5.15

Repeat Example 5.7 using the Excel spreadsheet.

SOLUTION

Open the Excel spreadsheet for Beams with Compression Steel and select the Analysis worksheet tab at the bottom. Input only cells C3 through C9 highlighted in yellow. Other values are calculated from those input values. See comment on cell E22 for Goal Seek instructions.

Analysis of Doubly Reinforced Beams by ACI 318-14
$A_s, A'_s, b, d, M_u, f'_c, f_y$ are known or specified.

$b =$	14	in.
$d =$	24	in.
$d' =$	2.5	in.
$A'_s =$	2.00	in.^{2}
$A_s =$	6.25	in.^{2}
$f_c =$	3000	psi
$f_y =$	60,000	psi
$\beta_1 =$	0.85	

Determine the location of the neutral axis, c.

$c =$	8.40	in.	
$\epsilon_s' =$	0.00211		
$a =$	7.14	in.	
$f_s' =$	60.00	psi	Compression steel is at yield
$0.85 f_c' \beta_1 cb + A_s' f_s' =$	$A_s f_y$		
$0.85 f_c' \beta_1 cb =$	255		
$A_s' f_s' =$	120	kips	
$A_s f_y =$	375	kips	
$0.85 f_c' \beta_1 cb + A_s' f' - A_s f_y =$	0	OK	
$A_{s2} = A_s' f_s' / f_y =$	2.000	$in.^2$	
$A_{s1} = A_s - A_{s2} =$	4.250	$in.^2$	

You must use "Goal Seek" to set this cell = 0 by changing c in cell D14. Go to "Data" at the top of the screen, then do "What If Analysis" to find Goal Seek.

$\epsilon_t = \frac{d-c}{c}(0.003) =$	0.005568	OK	tensile strain exceeds 0.004	
$\phi =$	0.900	–		
$M_{n1} = (A_s - f_y - A_s' f_s')(d - a/2) =$	5209.29	in.-lb =	434.11	kip-ft
$\phi M_{n1} =$	4688.36	in.-lb =	390.70	kip-ft
$M_{n2} =$	2580.00	in.-lb =	215.00	kip-ft
$\phi M_{n2} =$	2322.00	in.-lb =	193.50	kip-ft
$M_n =$	7789.29	in.-lb =	649.11	kip-ft
$\phi M_n =$	7010.36	in.-lb =	584.20	kip-ft

Example 5.16

Repeat Example 5.9 using the Excel spreadsheet.

SOLUTION

Open the Excel spreadsheet for Beams with Compression Steel and select the ACI 318-14 Case I worksheet tab at the bottom. Input only cells C3 through C9 highlighted in yellow. Other values are calculated from those input values.

Design of Doubly Reinforced Beams by ACI 318-14 When Both A_s and A_s' Are Unknown

Case I: A_s and A_s' are unknown; b, d, M_u, f_c', f_y are known or specified.

$M_u =$	1030.00	ft-k
$b =$	15	in.
$d =$	28	in.
$d' =$	3	in.
$f_c' =$	4000	psi
$f_y =$	60,000	psi
$\beta_1 =$	0.85	

1. Determine the maximum ultimate moment permitted by the code for the beam if it were singly reinforced (using the maximum value of ρ associated with $\phi = 0.9$).

$$\rho = 0.375(0.85\beta_1 f'_c/f_y) = 0.018063$$

$$M_{max} = \phi\rho bd^2 f_y\left(1 - \frac{\rho f_y}{1.7 f'_m}\right) = \boxed{9{,}642{,}313.4}\ \text{in.-lb}$$

$$= \boxed{803.53}\ \text{ft-k}$$

2. If $M_{max} \geq M_u$, compression steel is not needed. Design as singly reinforced beam. If $M_{max} < M_u$, continue to step 3.
3. The most economical design uses $M_{u1} = M_{max}$, which corresponds to ρ_1 = the maximum value of ρ associated with $\phi = 0.9$.

$$\rho_1 = \boxed{0.018063}$$

$$A_{s1} = \rho_1 bd = \boxed{7.586}\ \text{in.}^2$$

$$M_{u1} = \phi A_{s1} f_y(d - a/2) = \boxed{9{,}642{,}313.41}\ \text{in.-lb}$$

$$= \boxed{803.53}\ \text{ft-k}$$

$$a = \frac{A_s f_y}{0.85 f'_c b} = \boxed{8.925}\ \text{in.} \qquad c = \frac{\alpha}{\beta_1} = 10.500\ \text{in.}$$

$$\epsilon_t = \frac{d - c}{c}(0.003) = \boxed{0.00500}$$

$$\phi = 0.65 + (\epsilon_t - 0.002)(250/3) = \boxed{0.900}$$

4. $M_{u2} = M_u - M_{u1} = \boxed{226.47}$ ft-k

5. $A_{s2} = \dfrac{M_{u2}}{\phi f_y(d - d')} = \boxed{2.013}\ \text{in.}^2$

6. $c = a/\beta_1 = \boxed{10.500}$ in.

7. $f'_s = \dfrac{c - d'}{c}(87{,}000) = \boxed{62{,}143}$ psi if $f'_s > f_y$, use $f_s = f_y$ $f'_s = \boxed{60{,}000}$ psi

8. $A_s = \dfrac{A_{s2} f_y}{f'_s} = \boxed{2.01}\ \text{in.}^2$ Select bars (No. of bars) $\boxed{3}$ (Bar size) $\boxed{\#8}$ $A'_s = 2.36\ \text{in.}^2$

9. $A'_s = A_{s1} + A_{s2} = \boxed{9.60}\ \text{in.}^2$ Select bars $\boxed{8}$ $\boxed{\#10}$ $A_s = 10.13\ \text{in.}^2$

PROBLEMS

Problem 5.1 What is the effective width of a T beam? What does it represent?

Problem 5.2 What factors affect the selection of the dimensions of T-beam stems?

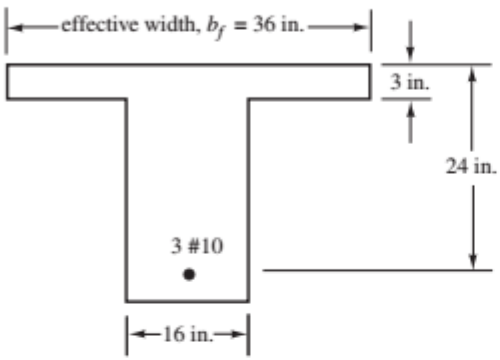
Problem 5.3 If additional reinforcing bars are placed only in the compression side of a reinforced concrete beam, will they add significantly to the beam's flexural strength? Explain your answer.

Problem 5.4 Why is compression steel reinforcing particularly important in reinforced concrete flexural members located in earthquake-prone areas?

Analysis of T Beams

For Problems 5.5 to 5.15, determine the design moment strengths ϕM_n of the sections shown. Use $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, except for Problem 5.9, where $f'_c = 5000$ psi. Check each section to see if it is ductile.

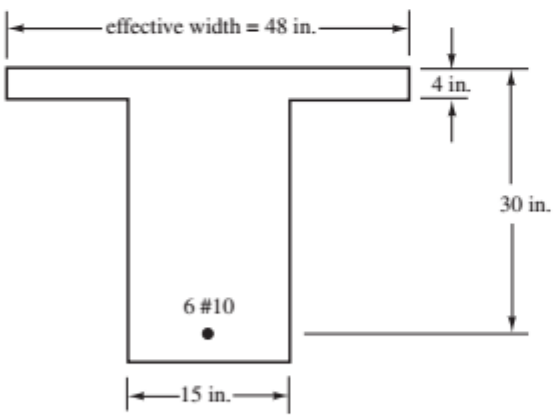
Problem 5.5 (*Ans.* 393.5 ft-k)



Problem 5.6 Repeat Problem 5.5 if 4 #10 bars are used.

Problem 5.7 Repeat Problem 5.5 if 10 #8 bars are used. (*Ans.* 775.7 ft-k)

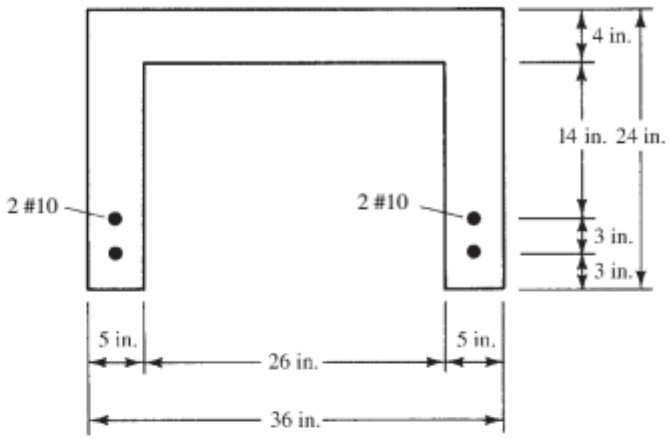
Problem 5.8



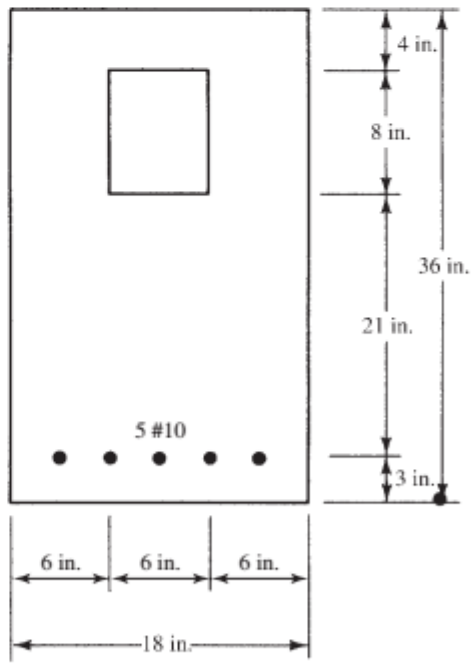
Problem 5.9 Repeat Problem 5.8 if $f'_c = 5000$ psi. (*Ans.* 986.5 ft-k)

Problem 5.10 Repeat Problem 5.8 if 8 #9 bars are used and $f'_c = 4000$ psi.

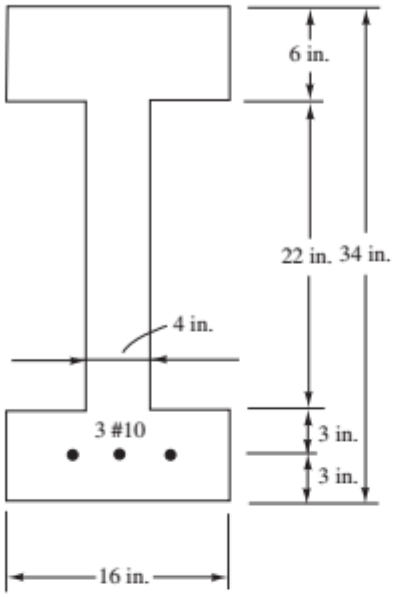
Problem 5.11 (*Ans.* 415.8 ft-k)



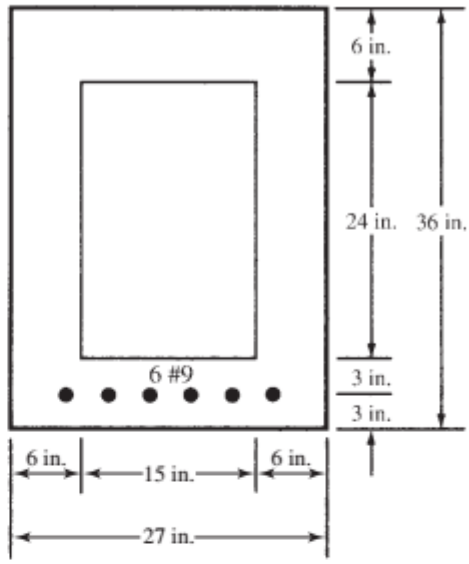
Problem 5.12



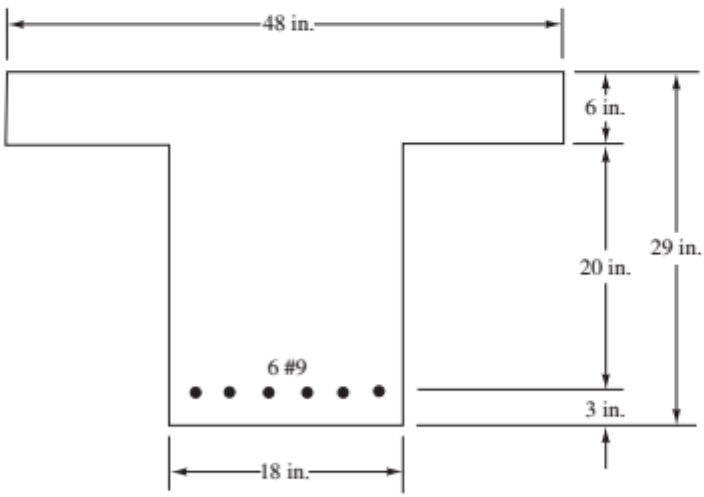
Problem 5.13 (*Ans.* 493.1 ft-k)



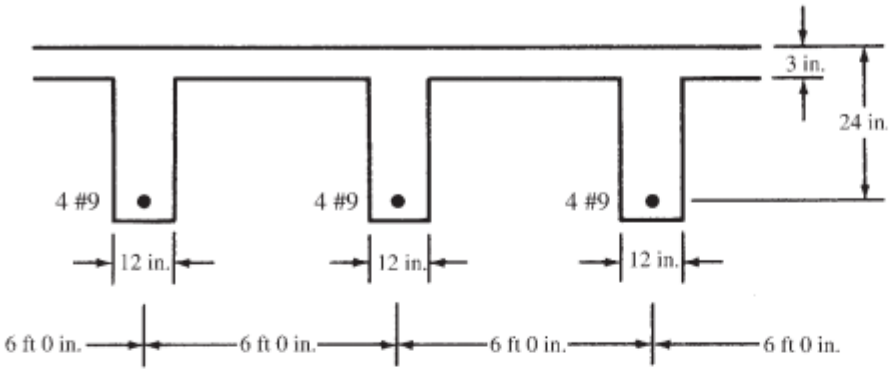
Problem 5.15 (*Ans.* 838.1 ft-k)



Problem 5.14



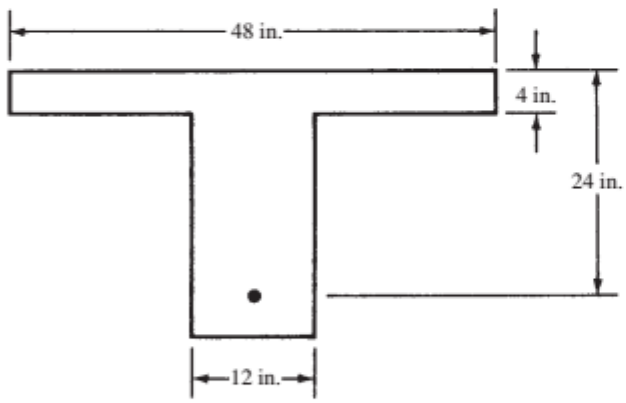
Problem 5.16 Calculate the design strength ϕM_n for one of the T beams if $f'_c = 5000$ psi, $f_y = 60{,}000$ psi, and the section has a 24-ft simple span. Is $\epsilon_t \geq 0.005$?



Problem 5.17 Repeat Problem 5.16 if $f'_c = 3000$ psi and 3 #10 bars are used in each web. (*Ans.* 396.7 ft-k)

Design of T Beams

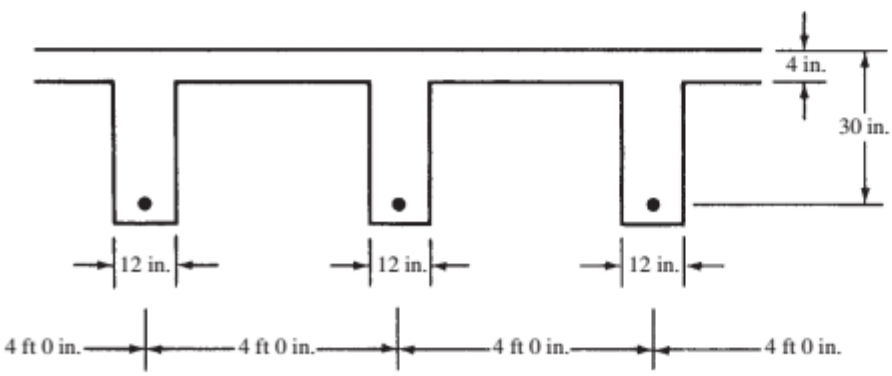
Problem 5.18 Determine the theoretical area of reinforcing steel required for the T beam shown if $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, $M_u = 400$ ft-k, and $L = 28$ ft. Clear distance between flanges = 3 ft.



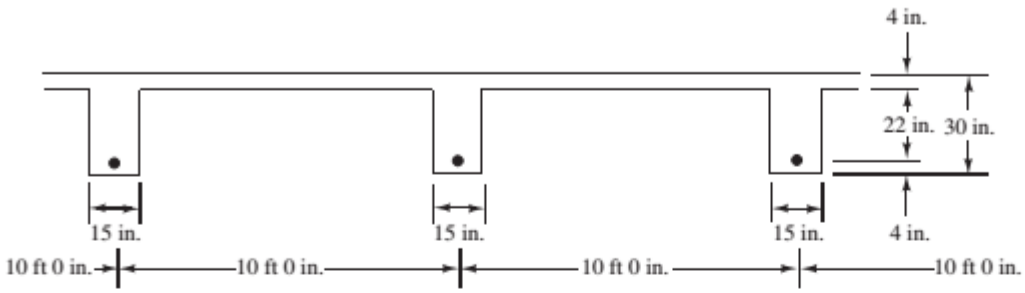
Problem 5.19 Repeat Problem 5.18 if $M_u = 500$ ft-k. (*Ans.* 4.80 in.^{2})

Problem 5.20 Repeat Problem 5.18 if $f_y = 50{,}000$ psi and $f'_c = 4000$ psi.

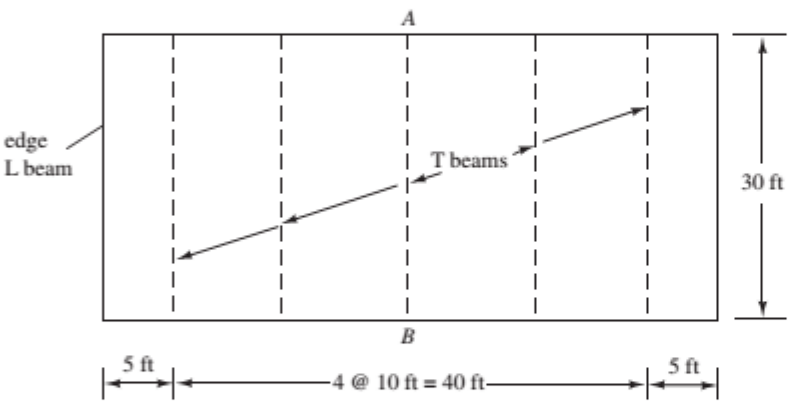
Problem 5.21 Determine the amount of reinforcing steel required for each T beam in the accompanying illustration if $f_y = 60{,}000$ psi, $f'_c = 4000$ psi, simple span = 20 ft, $M_D = 200$ ft-k (includes effect of concrete weight), and $M_L = 500$ ft-k. (*Ans.* 8.11 in.^{2})



Problem 5.22 Select the tensile reinforcement needed for the T beams if the reinforced concrete is assumed to weigh 150 lb/ft^3 and a live floor load of 140 lb/ft^2 is to be supported. Assume 40-ft simple spans, $f_y = 60{,}000$ psi, and $f'_c = 3000$ psi.



Problem 5.23 With $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, select the reinforcing for T beam AB for the floor system shown. Assume simple supports at A and B. The live load is to be 100 psf, while the dead load in addition to the concrete's weight is to be 80 psf. Concrete is assumed to weigh 150 lb/ft^3. The slab is 4 in. thick, d is 24 in., and b_w is 16 in. (*Ans.* 3.84 in.^{2}, use 4 #9 bars)



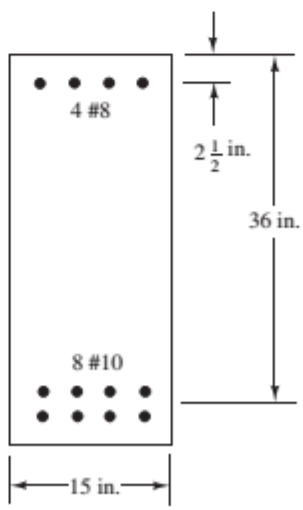
Problem 5.24 Repeat Problem 5.23 if the span is 36 ft.

Problem 5.25 Prepare a flowchart for the design of tensilely reinforced T beams with $\phi = 0.9$.

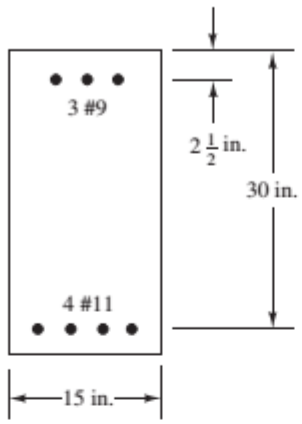
Analysis of Doubly Reinforced Beams

For Problems 5.26 to 5.32, compute the design moment strengths ϕM_n of the beams shown if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. Check the maximum permissible A_s in each case to ensure ductile behavior.

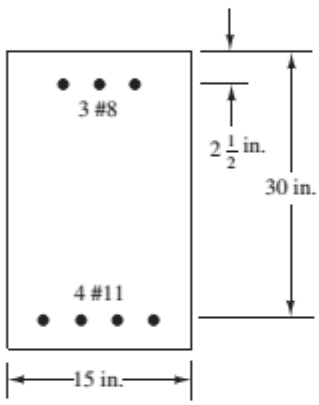
Problem 5.26



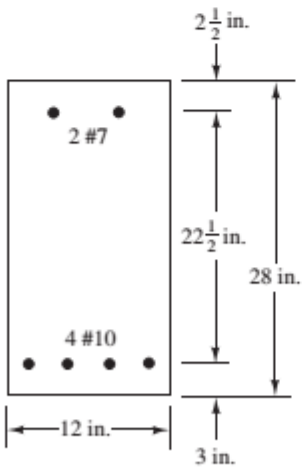
Problem 5.27 (*Ans.* 777.0 ft-k)



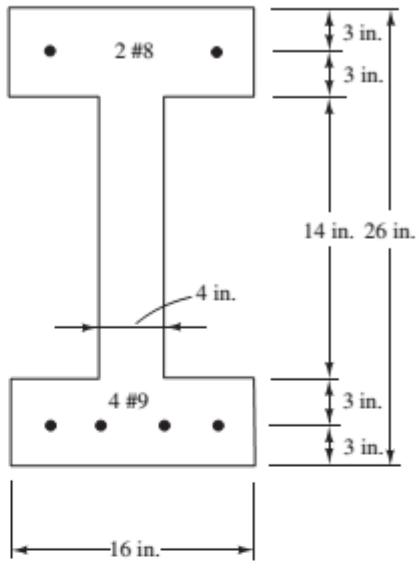
Problem 5.28



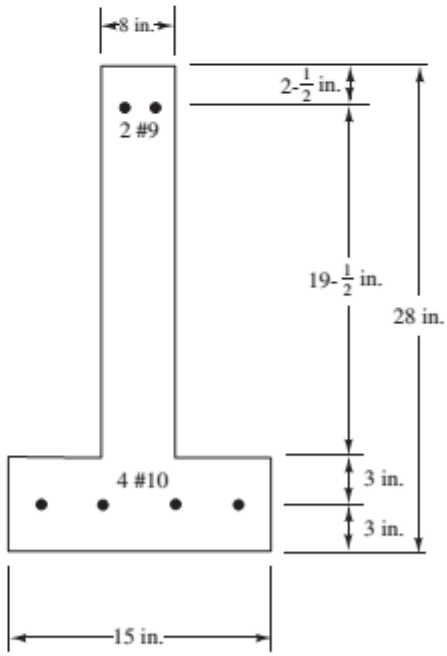
Problem 5.29 (*Ans.* 505.1 ft-k)



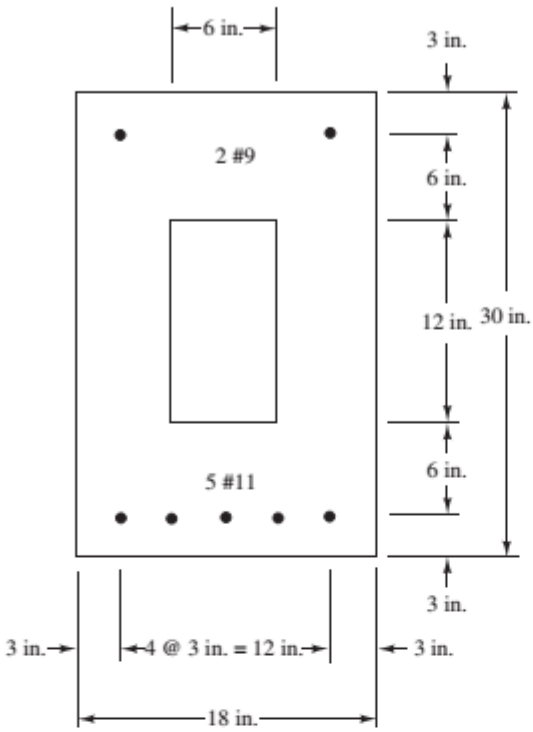
Problem 5.30



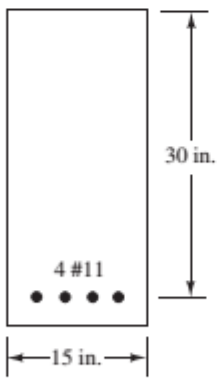
Problem 5.31 (*Ans.* 500.1 ft-k)



Problem 5.32



Problem 5.33 Compute the design moment strength, ϕM_n, of the beam shown. How much can this permissible moment be increased if 3 #9 bars are added to the top $2\frac{1}{2}$ in. from the compression face, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi? (*Ans*. 740.5 ft-k, 36.5 ft-k)

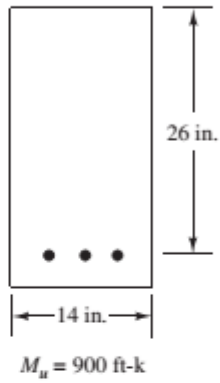


Problem 5.34 Repeat Problem 5.30 if 3 #10 bars are used in the top.

Design of Doubly Reinforced Beams

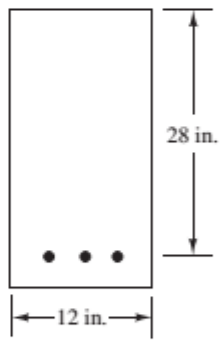
For Problems 5.35 to 5.38, determine the theoretical steel areas required for the sections shown. In each case, the dimensions are limited to the values shown. If compression steel is required, assume it will be placed 3 in. from the compression face ($d' = 3$ in.). $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi.

Problem 5.35 (*Ans*. $A_s = 9.03$ in.^{2}, $A'_s = 2.44$ in.^{2})



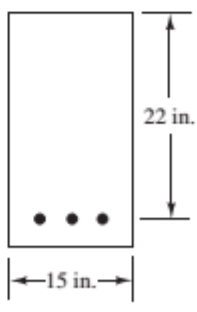
$M_u = 900$ ft-k

Problem 5.36



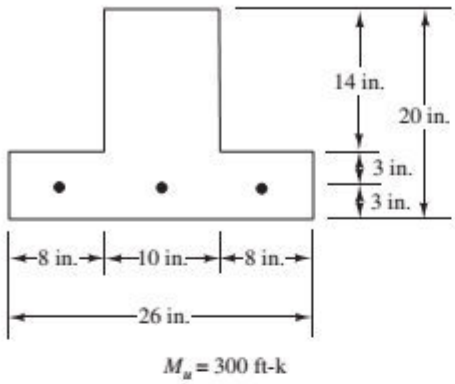
$M_u = 1000$ ft-k

Problem 5.37 (*Ans*. $A_s = 8.36$ in.^{2}, $A'_s = 2.58$ in.^{2})



$M_u = 700$ ft-k

Problem 5.38



Problem 5.39 Prepare a flowchart for the design of doubly reinforced rectangular beams.

Computer Problems

Solve Problems 5.40 to 5.45 using the Chapter 5 spreadsheet.

Problem 5.40 Problem 5.5

Problem 5.41 Problem 5.7 (*Ans.* 775.7 ft-k)

Problem 5.42 Problem 5.14

Problem 5.43 Problem 5.21 (*Ans.* $A_s = 8.11$ in.^{2})

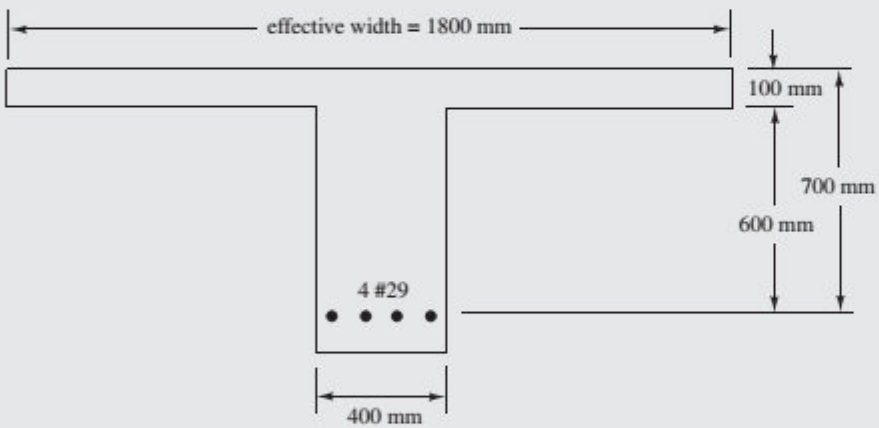
Problem 5.44 Problem 5.27

Problem 5.45 Problem 5.35 (*Ans.* $A_s = 9.02$ in.^{2}, $A_s' = 2.45$ in.^{2})

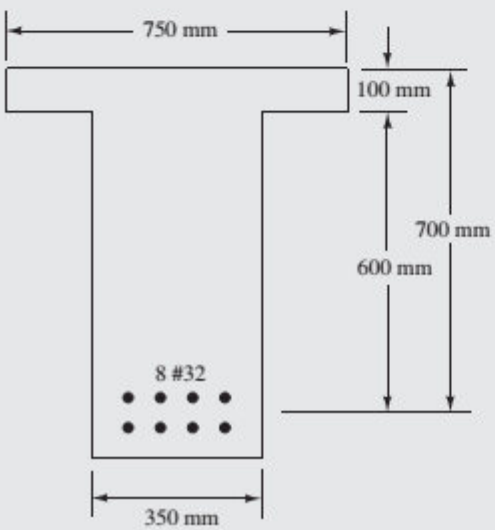
Problems in SI Units

For Problems 5.46 and 5.47, determine the design moment strengths of the beams shown in the accompanying illustrations if $f_c' = 28$ MPa and $f_y = 420$ MPa. Are the steel percentages in each case sufficient to ensure tensile behavior; that is, $\epsilon_t \geq 0.005$?

Problem 5.46

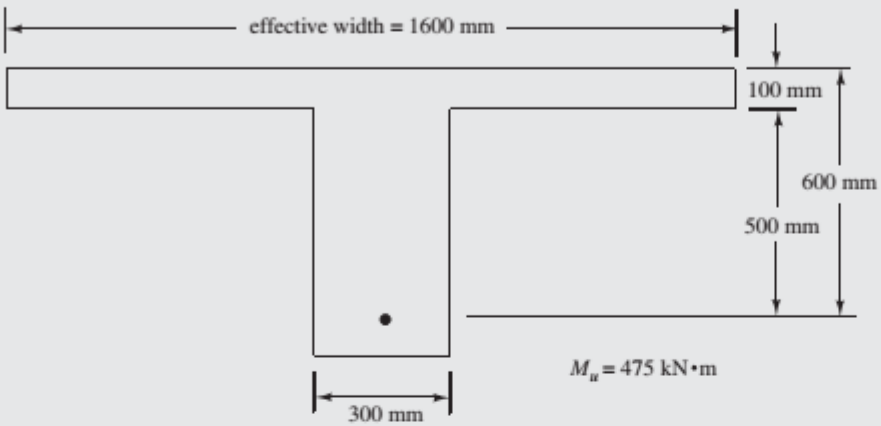


Problem 5.47 (*Ans.* 1516 kN · m)

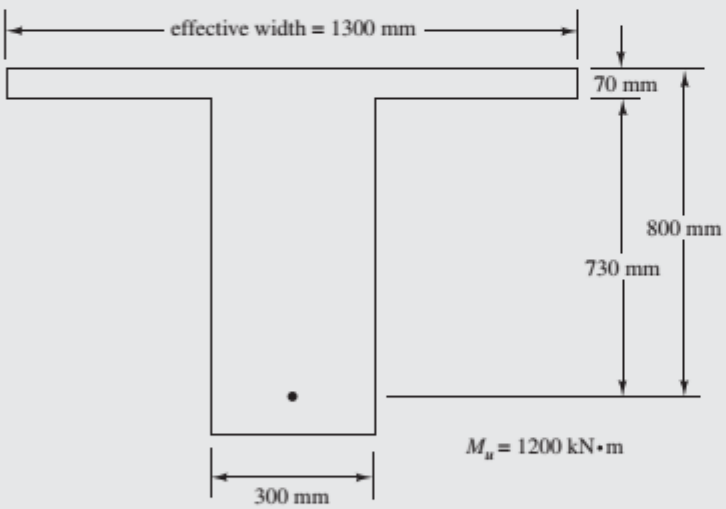


For Problems 5.48 and 5.49, determine the area of reinforcing steel required for the T beams shown if $f'_c = 28$ MPa and $f_y = 420$ MPa. Check ϵ_t to see that it is ≥ 0.005.

Problem 5.48

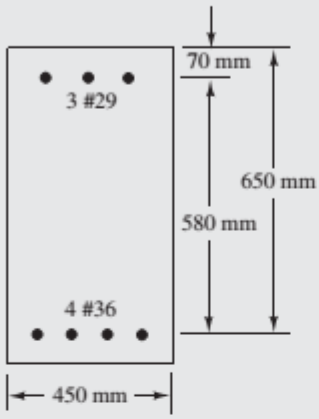


Problem 5.49 (*Ans.* 4112 mm^2)

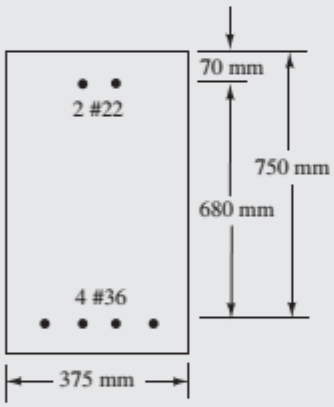


For Problems 5.50 to 5.52, compute the design moment strengths of the beams shown if $f_y = 420$ MPa and $f'_c = 21$ MPa. Check the maximum permissible A_s in each case to ensure ductile failure. $E_s = 200,000$ MPa.

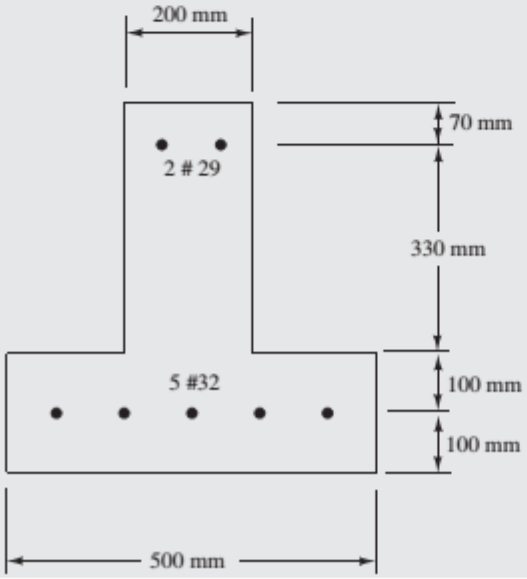
Problem 5.50



Problem 5.51 (*Ans.* 995 kN • m)

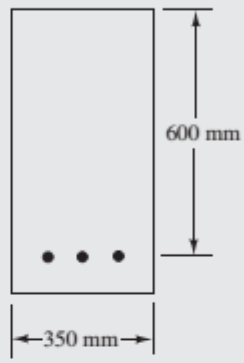


Problem 5.52



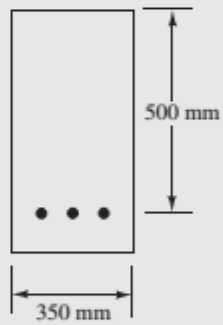
For Problems 5.53 and 5.54, determine the theoretical steel areas required for the sections shown. In each case, the dimensions are limited to the values shown. If compression steel is required, assume it will be placed 70 mm from the compression face. $f'_c = 28$ MPa, $f_y = 420$ MPa, and $E_s = 200,000$ MPa.

Problem 5.53 (*Ans.* $A_s = 5925$ mm^2, $A'_s = 2154$ mm^2)



$M_u = 1150$ kN•m

Problem 5.54



$M_u = 750$ kN•m

More Detailed Problems

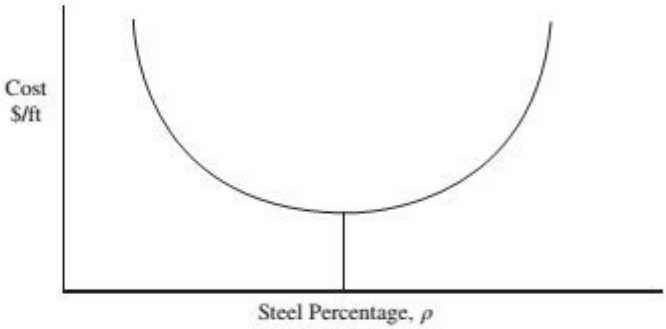
Problem 5.55 Two-foot-wide, 4-in.-deep precast reinforced concrete slabs are to be used for a flat roof deck. The slabs are to be supported at their ends by precast rectangular beams spanning the 30 ft width of the roof (measured c. to c. of the supporting masonry walls). Select f_y and f'_c, design the slabs including their length, and design one of the supporting interior beams. Assume 30-psf roof live load and 6-psf built-up roof.
(*One ans*. Use 12-in. × 24-in. beams with 3 #9 bars.)

Problem 5.56 Repeat Problem 5.55 if the beam span is 40 ft and roof live load is 40 psf.

Problem 5.57 For the same building considered in Problem 5.55, a 6-in.-deep cast-in-place concrete slab has been designed. It is to be supported by T beams cast integrally with the slabs. The architect says that the 30-ft-long T beams are to be supported by columns that are to be spaced 18 ft o.c. The building is to be used for light manufacturing (see Table 1.3 in Chapter 1 for live loads). Select f_y and f'_c, and design one of the interior T beams.
(*One ans*. Use T beam web 12 in. wide, $h = 32$ in., $f'_c = 4$ ksi, $f_y = 60$ ksi, and 4 #10 bars.)

Problem 5.58 Repeat Problem 5.57 if the building is to be used for offices. The beam spans are to be 36 ft and the columns are to be placed 20 ft. o.c.

Problem 5.59 Determine the lowest cost design for a tensilely reinforced concrete beam for the conditions that follow: $f_y = 60$ ksi, $f'_c = 4$ ksi, $M_u = 400$ ft-k, $\ell = 24$ ft, $h = d + 2.5$ in.; concrete costs \$120 per yard and weighs 150 lb/ft^3; and reinforcing bars cost \$0.95/lb and weigh 490 lb/ft^3. Design the beam for the moment given with $d = 1.5b$, and calculate its cost per linear foot. Plot the cost per linear foot of beam (y-axis) versus steel percentage (x-axis). Then change the beam size and recalculate ρ and the new cost. Limit beam sizes to increments of 1 in. for b. Find the lowest-cost design and the corresponding value of ρ. (*Ans*. Approx. $\rho = 0.0139$ and cost = \$26.03/ft.)



Problem 5.60 Repeat Problem 5.59 if $d = 2b$.

CHAPTER 6 Serviceability

6.1 Introduction

Today the structural design profession is concerned with a *limit states* philosophy. The term *limit state* is used to describe a condition at which a structure or some part of a structure ceases to perform its intended function. There are two categories of limit states: strength and serviceability.

Strength limit states are based on the safety or load-carrying capacity of structures and include buckling, fracture, fatigue, overturning, and so on. Chapters 3 to 5 have dealt with the bending limit state of various members.

Serviceability limit states refer to the performance of structures under normal service loads and are concerned with the uses and/or occupancy of structures. Serviceability is measured by considering the magnitudes of deflections, cracks, and vibrations of structures, as well as by considering the amounts of surface deterioration of the concrete and corrosion of the reinforcing bars. You will note that these items may disrupt the use of structures but do not usually involve collapse.

This chapter is concerned with serviceability limits for deflections and crack widths. The ACI Code contains very specific requirements relating to the strength limit states of reinforced concrete members but allows the designer some freedom of judgment in the serviceability areas. This doesn't mean that the serviceability limit states are not significant, but by far the most important consideration (as in all structural specifications) is the life and property of the public. As a result, public safety is not left up to the judgment of the individual designer.

Vertical vibration for bridge and building floors, as well as lateral and torsional vibration in tall buildings, can be quite annoying to users of these structures. Vibrations are not usually a problem in the average-size reinforced concrete building, but we should be on the lookout for the situations where they can be objectionable.

The deterioration of concrete surfaces can be greatly minimized by exercising good control of the mixing, placing, and curing of the concrete. When those surfaces are subjected to harsh chemicals, special cements with special additives can be used to protect the surfaces. The corrosion of reinforcing bars can be greatly minimized by giving careful attention to concrete quality, using good vibration of the concrete, using adequate cover thickness for the bars, and limiting crack sizes.

6.2 Importance of Deflections

The adoption of the strength design method, together with the use of higher-strength concretes and steels, has permitted the use of relatively slender members. As a result, deflections and deflection cracking have become more severe problems than they were a few decades ago.

The magnitudes of deflections for concrete members can be quite important. Excessive deflections of beams and slabs may cause sagging floors, ponding on flat roofs, excessive vibrations, and even interference with the proper operation of supported machinery. Such deflections may damage partitions and cause poor fitting of doors and windows. In addition, they may damage a structure's appearance or frighten the occupants of the building, even though the building may be perfectly safe. Any structure used by people should be quite rigid and relatively vibration-free so as to provide a sense of security.

Perhaps the most common type of deflection damage in reinforced concrete structures is the damage to light masonry partitions. They are particularly subject to damage because of concrete's

long-term creep. When the floors above and below deflect, the relatively rigid masonry partitions do not bend easily and are often severely damaged. The more flexible gypsum board partitions are much more adaptable to such distortions.

6.3 Control of Deflections

One of the best ways to reduce deflections is by increasing member depths—but designers are always under pressure to keep members as shallow as possible. (As you can see, shallower members mean thinner floors, and thinner floors mean buildings with less height, with consequent reductions in many costs, such as plumbing, wiring, elevators, outside materials on buildings, and so on.) Reinforced concrete specifications usually limit deflections by specifying certain minimum depths or maximum permissible computed deflections.

Minimum Thicknesses

Table 4.1 in Chapter 4, which combines Tables 7.3.1.1 and 9.3.1.1 of ACI 318-14, provides a set of minimum thicknesses for beams and one-way slabs to be used, unless actual deflection calculations indicate that lesser thicknesses are permissible. These minimum thickness values, which were developed primarily on the basis of experience over many years, should be used only for beams and slabs that are not supporting or attached to partitions or other members likely to be damaged by deflections.

Maximum Deflections

If the designer chooses not to meet the minimum thicknesses given in Table 4.1, he or she must compute deflections. If this is done, the values determined may not exceed the values specified in Table 6.1, which is Table 24.2.2 of the ACI 318 Code.

TABLE 6.1 Maximum Permissible Calculated Deflections

Member	Condition		Deflection to Be Considered	Deflection Limitation
Flat roofs	Not supporting or attached to nonstructural elements likely to be damaged by large deflections		Immediate deflection due to maximum of L_r, S, and R	$\ell/180$*
Floors			Immediate deflection due to L	$\ell/360$
Roof or floors	Supporting or attached to nonstructural elements	Likely to be damaged by large deflections	That part of the total deflection occurring after attachment of nonstructural elements, which is the sum of the time-dependent deflection due to all sustained loads and the immediate deflection due to any additional live load†	$\ell/480$‡
		Not likely to be damaged by large deflections		$\ell/240$§

*Limit not intended to safeguard against ponding. Ponding shall be checked by calculations of deflection, including added deflections due to ponded water, and considering time-dependent effects of sustained loads, camber, construction tolerances, and reliability of provisions for drainage.

†Time-dependent deflection shall be calculated in accordance with 24.2.4, but shall be permitted to be reduced by amount of deflection calculated to occur before attachment of nonstructural elements. This amount shall be calculated on basis of accepted engineering data relating to time-deflection characteristics of members similar to those being considered.

‡Limit shall be permitted to be exceeded if measures are taken to prevent damage to supported or attached elements.

§Limit shall not exceed tolerance provided for nonstructural elements.

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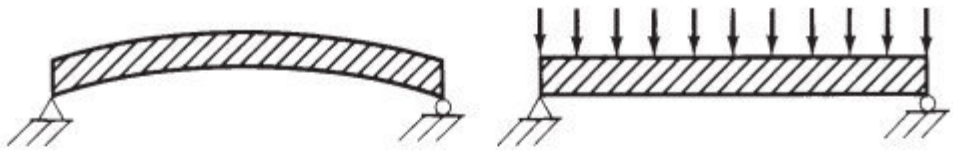

(a) Beam constructed with upward camber

(b) Beam straight under dead load plus some percentage of live load

FIGURE 6.1 Cambering.

Camber

The deflection of reinforced concrete members may also be controlled by cambering. The members are constructed of such a shape that they will assume their theoretical shape under some service loading condition (usually dead load and perhaps some part of the live load). A simple beam would be constructed with a slight convex bend, so that under certain gravity loads, it would become straight, as assumed in the calculations. (See Figure 6.1.) Some designers take into account both dead and full live loads in figuring the amount of camber. Camber is generally used only for longer-span members.

6.4 Calculation of Deflections

Deflections for reinforced concrete members can be calculated with the usual deflection expressions, several of which are shown in Figure 6.2. A few comments should be made about the magnitudes of deflections in concrete members as determined by the expressions given in this figure. It can be seen that the centerline deflection of a uniformly loaded simple beam [Figure 6.2(a)] is five times as large as the centerline deflection of the same beam if its ends are fixed [Figure 6.2(b)]. Nearly all concrete beams and slabs are continuous, and their deflections fall somewhere between the two extremes mentioned here.

Because of the very large deflection variations that occur with different end restraints, it is essential that those restraints be considered if realistic deflection calculations are to be made. For most practical purposes, it is sufficiently accurate to calculate the centerline deflection of a member as though it is simply supported and to subtract from that value the deflection caused by the average of the negative moments at the member ends. (This can be done by using a combination of expressions taken from Figure 6.2. For instance, the deflection equation of part (a) may be used together with the one of part (g) applied at one or both ends as necessary.) Loads used in these expressions are unfactored loads. In some cases, only the live load is considered; in others, both live and dead (sustained) loads are considered.

6.5 Effective Moments of Inertia

Regardless of the method used for calculating deflections, there is a problem in determining the moment of inertia to be used. The trouble lies in the amount of cracking that has occurred. If the bending moment is less than the cracking moment (i.e., if the flexural stress is less than the modulus of rupture of about $7.5\lambda\sqrt{f'_c}$), the full uncracked section provides rigidity, and the moment of inertia for the gross section I_g is available. When larger moments are present, different-size tension cracks occur and the position of the neutral axis varies.

Figure 6.3 illustrates the problem involved in selecting the moment of inertia to be used for deflection calculations. Although a reinforced concrete beam may be of constant size (or prismatic) throughout its length, for deflection calculations, it will behave as though it were composed of segments of different-size beams.[1]

[1] Leet, K., 1997, *Reinforced Concrete Design*, 3rd ed. (New York: McGraw-Hill), p. 155.

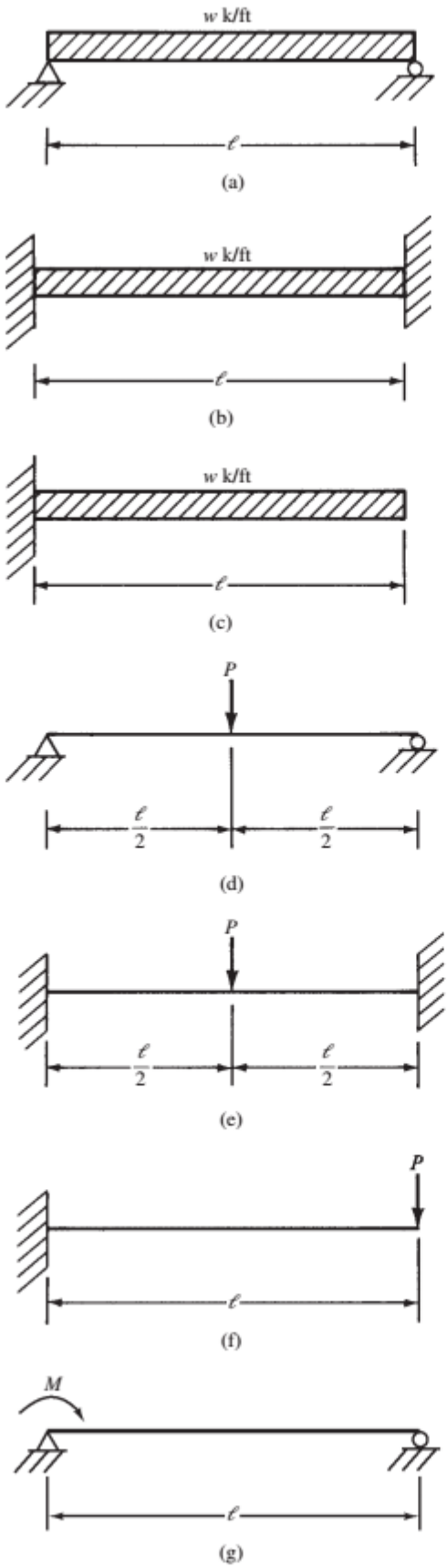

FIGURE 6.2 Some deflection expressions.

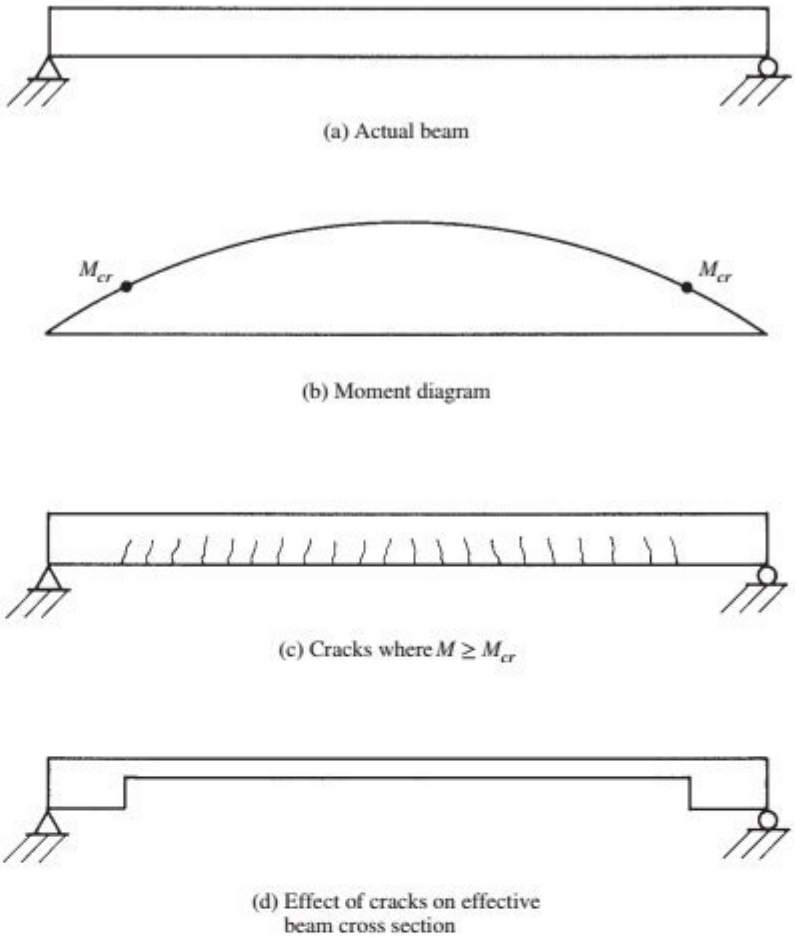

FIGURE 6.3 Effects of cracks on deflections.

For the portion of a beam where the moment is less than the cracking moment, M_{cr}, the beam can be assumed to be uncracked, and the moment of inertia can be assumed to equal I_g. When the moment is greater than M_{cr}, the tensile cracks that develop in the beam will, in effect, cause the beam cross section to be reduced, and the moment of inertia may be assumed to equal the transformed value, I_{cr}. It is as though the beam consists of the segments shown in Figure 6.3(d).

The problem is even more involved than indicated by Figure 6.3. It is true that at cross sections where tension cracks are actually located, the moment of inertia is probably close to the transformed I_{cr}, but in between cracks, it is perhaps closer to I_g. Furthermore, diagonal tension cracks may exist in areas of high shear, causing other variations. As a result, it is difficult to decide what value of I should be used.

A concrete section that is fully cracked on its tension side will have a rigidity of anywhere from one-third to three-fourths of its uncracked full section rigidity. At different sections along the beam, the rigidity varies depending on the moment present. It is easy to see that an accurate method of calculating deflections must take these variations into account.

If it is desired to obtain the immediate deflection of an uncracked prismatic member, the moment of inertia may be assumed to equal I_g along the length of the member. Should the member be cracked at one or more sections along its length, or if its depth varies along the span, a more exact value of I needs to be used.

Section 24.2.3.5 of ACI 318 gives a moment of inertia expression that is to be used for deflection calculations. This moment of inertia provides a transitional value between I_g and I_{cr} that depends upon

the extent of cracking caused by applied loads. It is referred to as I_e, the effective moment of inertia, and is based on an estimation of the probable amount of cracking caused by the varying moment throughout the span[2]:

$$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 (I_g) + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr} \qquad \text{(ACI Equation 24.2.3.5a)}$$

where M_{cr} is calculated by

$$M_{cr} = \frac{f_r I_g}{y_t} \qquad \text{(ACI Equation 24.2.3.5b)}$$

In this expression, I_g is the gross amount of inertia (without considering the steel) of the section and the modulus of rupture of concrete, as discussed under the Tensile Strength heading in Section 1.11, is $f_r = 7.5\lambda\sqrt{f'_c}$.[3] M_a is the maximum service-load moment occurring for the condition under consideration, and I_{cr} is the transformed moment of inertia of the cracked section, as described in Section 2.3.

You will note that the values of the effective moment of inertia vary with different loading conditions. This is because the service-load moment, M_a, used in the equation for I_e, is different for each loading condition. Some designers ignore this fact and use only one I_e for each member, even though different loading conditions are considered. They feel that their computed values are just as accurate as those obtained with the different I_e values. It is true that the varying conditions involved in constructing reinforced concrete members (workmanship, curing conditions, age of members when loads were first applied, etc.) make the calculation of deflections by any present-day procedure a very approximate process.

In this chapter the authors compute I_e for each different loading condition. The work is a little tedious, but it can be greatly expedited with various tables, such as the ones provided in the *ACI Design Handbook*.[4]

6.6 Long-Term Deflections

With I_e and the appropriate deflection expressions, instantaneous or immediate deflections are obtained. Long-term or sustained loads, however, cause significant increases in these deflections because of shrinkage and creep. The factors affecting deflection increases include humidity, temperature, curing conditions, compression steel content, ratio of stress to strength, and the age of the concrete at the time of loading.

If concrete is loaded at an early age, its long-term deflections will be greatly increased. Excessive deflections in reinforced concrete structures can very often be traced to the early application of loads. The creep strain after about 5 years (after which creep is negligible) may be as high as four or five times the initial strain when loads were first applied 7 to 10 days after the concrete was placed, while the ratio may only be two or three when the loads were first applied 3 or 4 months after concrete placement.

Because of the several factors mentioned in the last two paragraphs, the magnitudes of long-term deflections can only be estimated. The ACI Code 24.2.4.1.1 states that to estimate the increase in deflection due to these causes, the part of the instantaneous deflection that is due to sustained loads may be multiplied by the empirically derived factor λ and the result added to the instantaneous deflection.[5]

$$\lambda_\Delta = \frac{\xi}{1 + 50\rho'} \qquad \text{(ACI Equation 24.2.4.1.1)}$$

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[2] Branson, D. E., 1965, "Instantaneous and Time-Dependent Deflections on Simple Continuous Reinforced Concrete Beams," HPR Report No. 7, Part 1, Alabama Highway Department, Bureau of Public Roads, August 1963, pp. 1–78.

[3] $0.7\lambda\sqrt{f'_c}$ in SI.

[4] American Concrete Institute, 2009, *ACI Design Handbook* (Farmington Hills, MI: ACI), Publication SP-17 (09).

[5] Branson, D. E., 1971, "Compression Steel Effect on Long-Time Deflections," *Journal ACI*, 68(8), pp. 555–559.

TABLE 6.2 Time Factor for Sustained Loads (ACI Code 24.2.4.1.3)

Duration of Sustained Load, months	Time-Dependent Factor ξ
3	1.0
6	1.2
12	1.4
60 or more	2.0

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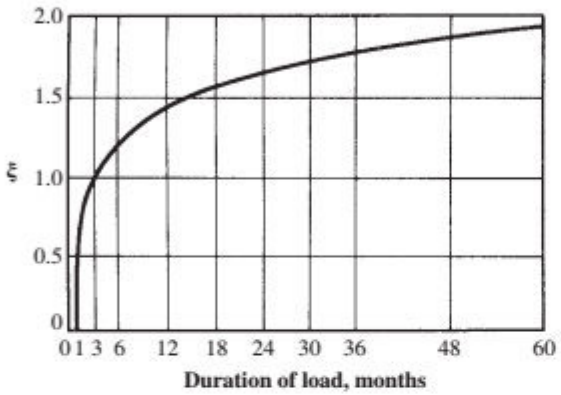

FIGURE 6.4 Multipliers for long-time deflections. (ACI Commentary Figure R24.2.4.1)
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In this expression, which is applicable to both normal- and lightweight concrete, ξ is a time-dependent factor that may be determined from Table 6.2.

Should times differing from the values given in Table 6.2 be used, values of ξ may be selected from the curve of Figure 6.4.

The effect of compression steel on long-term deflections is taken into account in the λ expression with the term ρ'. It equals A'_s/bd and is to be computed at midspan for simple and continuous spans, and at the supports for cantilevers.

The full dead load of a structure can be classified as a sustained load, but the type of occupancy will determine the percentage of live load that can be called sustained. For an apartment house or for an office building, perhaps only 20% to 25% of the service live load should be considered as being sustained, whereas perhaps 70% to 80% of the service live load of a warehouse might fall into this category.

A study by the ACI indicates that under controlled laboratory conditions, 90% of test specimens had deflections between 20% below and 30% above the values calculated by the method described in this chapter.[6] The reader should realize, however, that field conditions are not lab conditions, and deflections in actual structures will vary much more than those occurring in the lab specimens. Despite the use of plans and specifications and field inspection, it is difficult to control fieldwork adequately. Construction crews may add a little water to the concrete to make it more workable. Further, they may not obtain satisfactory mixing and compaction of the concrete, with the result that voids and honeycomb occur. Finally, the forms may be removed before the concrete has obtained its full design strength. If this is the case, the moduli of rupture and elasticity will be low, and excessive cracks may occur in beams that would not have occurred if the concrete had been stronger. All of these factors can cause reinforced concrete structures to deflect appreciably more than is indicated by the usual computations.

[6] ACI Committee 435, 1972, "Variability of Deflections of Simply Supported Reinforced Concrete Beams," *Journal ACI*, 69(1), p. 29.

It is logical to assume that the live load cannot act on a structure when the dead load is not present. As a result of this fact, we will compute an effective I_e and a deflection δ_D for the case where the dead load alone is acting. Then we will compute an I_e and a deflection δ_{D+L} for the case where both dead and live loads are acting. This will enable us to determine the initial live load part of the deflection as follows:

$$\delta_L = \delta_{D+L} - \delta_D$$

The long-term deflection will equal the initial live load deflection, δ_L, plus the infinitely long-term multiplier, λ_∞, times the dead load deflection, δ_D, plus λ_t, the live load sustained multiplier, times the initial live load deflection, δ_{SL}.

$$\delta_{LT} = \delta_L + \lambda_\infty \delta_D + \lambda_t \delta_{SL}$$

The steps involved in calculating instantaneous and long-term deflections can be summarized as follows:

(a) Compute the instantaneous or short-term deflection, δ_D, for dead load only.

(b) Compute instantaneous deflection, δ_{D+L}, for dead plus full live load.

(c) Determine instantaneous deflection, δ_L, for full live load only.

(d) Compute instantaneous deflection due to dead load plus the sustained part of the live load, $\delta_D + \delta_{SL}$.

(e) Determine instantaneous deflection, δ_{SL}, for the part of the live load that is sustained.

(f) Determine the long-term deflection for dead load plus the sustained part of the live load, δ_{LT}.

As previously mentioned, the deflections calculated as described in this chapter should not exceed certain limits, depending on the type of structure. Maximum deflections permitted by the ACI for several floor and roof situations are presented in Table 6.1.

6.7 Simple-Beam Deflections

Example 6.1 presents the calculation of instantaneous and long-term deflections for a uniformly loaded simple beam.

Example 6.1

The beam of Figure 6.5 has a simple span of 20 ft and supports a dead load excluding its own weight of 0.75 klf and a live load of 0.7 klf. $f'_c = 3000$ psi, normal weight concrete.

(a) Calculate the instantaneous deflection for $D + L$.

(b) Calculate the deflection assuming that 30% of the live load is continuously applied for 3 years.

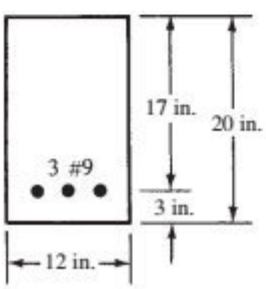


FIGURE 6.5 Beam cross section for Example 6.1.

SOLUTION

(a) Instantaneous or Short-Term Dead Load Deflection (δ_D)

$$I_g = \left(\frac{1}{12}\right)(12\text{ in.})(20\text{ in.})^3 = 8000\text{ in.}^4$$

$$M_{cr} = \frac{f_r I_g}{y_t} = \frac{(7.5\sqrt{3000}\text{ psi})(8000\text{ in.}^4)}{10\text{ in.}} = 328{,}633\text{ in.-lb} = 27.4\text{ ft-k}$$

$$w_{SW} = (b)(h)(\gamma_c) = (12\text{ in.})(20\text{ in.})(150\text{ pcf})/144\text{ in.}^2/\text{ft}^2 = 250\text{ plf}$$

$$M_a = \frac{(0.75\text{ klf} + 0.25\text{ klf})(20\text{ ft})^2}{8} = 50\text{ ft-k} = M_D$$

Should the dead load moment, M_D, be less than the cracking moment, M_{cr}, we should use $M_{cr} = M_a$ and $I_e = I_g$.

By transformed-area calculations, the values of x and I_{cr} can be determined as previously illustrated in Example 2.3.

$$x = 6.78\text{ in.}$$

$$I_{cr} = 4067\text{ in.}^4$$

Then I_e is calculated using ACI Equation 24.2.3.5a:

$$I_e = \left(\frac{27.4\text{ ft-k}}{50\text{ ft-k}}\right)^3(8000\text{ in.}^4) + \left[1 - \left(\frac{27.4\text{ ft-k}}{50\text{ ft-k}}\right)^3\right]4067\text{ in.}^4 = 4714\text{ in.}^4$$

$$E_c = 57{,}000\sqrt{3000}\text{ psi} = 3.122 \times 10^6\text{ psi}$$

$$\delta_D = \frac{5wl^4}{384E_cI_e} = \frac{(5)\left(\frac{1000\text{ plf}}{12\text{ in/ft}}\right)(12\text{ in./ft} \times 20\text{ ft})^4}{(384)(3.122 \times 10^6\text{ psi})(4714\text{ in.}^4)} = \underline{\underline{0.245\text{ in.}^*}}$$

(b) Instantaneous or Short-Term Deflection for Dead + Full Live Load (δ_{D+L})

$$M_a = \frac{(1.7\text{ klf})(20\text{ ft})^2}{8} = 85\text{ ft-k}$$

Noting that the value of I_e changes when the moments change

$$I_e = \left(\frac{27.4\text{ ft-k}}{85\text{ ft-k}}\right)^3(8000\text{ in.}^4) + \left[1 - \left(\frac{27.4\text{ ft-k}}{85\text{ ft-k}}\right)^3\right](4067\text{ in.}^4) = 4199\text{ in.}^4$$

$$\delta_{D+L} = \frac{(5)\left(\frac{1700\text{ plf}}{12\text{ in/ft}}\right)(12\text{ in/ft} \times 20\text{ ft})^4}{(384)(3.122 \times 10^6\text{ psi})(4199\text{ in.}^4)} = 0.467\text{ in.}^*$$

(c) Initial Deflection for Full Live Load (δ_L)

$$\delta_L = \delta_{D+L} - \delta_D = 0.467\text{ in.} - 0.245\text{ in.} = \underline{\underline{0.222\text{ in.}^*}}$$

This is the live load deflection that would be compared with the first or second row of Table 6.1. If the beam is part of a floor system that is "not supporting or attached to nonstructural elements

* The authors really got carried away in this chapter when they calculated deflection to thousandths of an inch. We cannot expect this kind of accuracy, and accuracy to within hundredths of an inch, and perhaps even tenths, is more realistic. These instances are denoted in this chapter by *'s.

likely to be damaged by large deflections" (left column of Table 6.1), then the deflection limit is $\ell/480 = (20\text{ ft})(12\text{ in./ft})/360 = 0.67$ in. This limit would easily be satisfied in this case, as the calculated deflection is only 0.22 in.

(d) Initial Deflection due to Dead Load + 30% Live Load ($\delta_D + \delta_{SL}$)

$$M_a = \frac{(1.0\text{ klf} + 0.30 \times 0.7\text{ klf})(20\text{ ft})^2}{8} = 60.5\text{ ft-k}$$

$$I_e = \left(\frac{27.4\text{ ft-k}}{60.5\text{ ft-k}}\right)^3 (8000\text{ in.}^4) + \left[1 - \left(\frac{27.4\text{ ft-k}}{60.5\text{ ft-k}}\right)^3\right](4067\text{ in.}^4) = 4432\text{ in.}^4$$

$$\delta_D + \delta_{SL} = \frac{(5)\dfrac{(1000\text{ plf} + 0.30 \times 700\text{ plf})}{12}(12\text{ in./ft} \times 20\text{ ft})^4}{(384)(3.122 \times 10^6\text{ psi})(4432\text{ in.}^4)} = 0.315\text{ in.}^*$$

(e) Initial Deflection due to 30% Live Load (δ_{SL})

$$\delta_{SL} = (\delta_D + \delta_{SL}) - \delta_D = 0.315\text{ in.} - 0.245\text{ in.} = \underline{\underline{0.070\text{ in.}^*}}$$

(f) Long-Term Deflection for Dead Load Plus 3 Years of 30% Sustained Live Load (δ_{LT})

$$\lambda_\infty = \frac{2.0}{1 + 50\rho'} = \frac{2.0}{1 + 0} = 2.0$$

$$\lambda_{3\text{ years}} = \frac{1.80}{1 + 0} = 1.80$$

$$\delta_{LT} = \delta_L + \lambda_\infty \delta_D + \lambda_{3\text{ years}} \delta_{SL}$$

$$= 0.222\text{ in.} + (2.0)(0.245\text{ in.}) + (1.80)(0.070\text{ in.}) = 0.838\text{ in.}^*$$

The middle column of Table 6.1 describes this deflection for the last two rows of the table. The answer is compared with allowable deflections of either $\ell/480$ or $\ell/240$, depending on whether the structural member supports elements likely to be damaged by large deflections.

6.8 Continuous-Beam Deflections

The following discussion considers a continuous T beam subjected to both positive and negative moments. As shown in Figure 6.6, the effective moment of inertia used for calculating deflections varies a great deal throughout the member. For instance, at the center of the span at Section 1-1 where the positive moment is largest, the web is cracked and the effective section consists of the hatched section plus the tensile reinforcement in the bottom of the web. At Section 2-2 in the figure, where the largest negative moment occurs, the flange is cracked and the effective section consists of the hatched part of the web (including any compression steel in the bottom of the web) plus the tensile bars in the top. Finally, near the points of inflection, the moment will be so low that the beam will probably be uncracked, and thus the whole cross section is effective, as shown at Section 3-3 in the figure. (For this case I is usually calculated only for the web, and the effect of the flanges is neglected, as shown in Figure 6.10.)

From the preceding discussion it is obvious that to calculate the deflection in a continuous beam, theoretically it is necessary to use a deflection procedure that takes into account the varying moment of inertia along the span. Such a procedure would be very lengthy, and it is doubtful that the results so obtained would be within ±20% of the actual values. For this reason the ACI Code 9.5.2.4 permits the use of a constant moment of inertia throughout the member equal to the average of the I_e values computed at the critical positive- and negative-moment sections. *The I_e values at the critical negative-moment sections are averaged with each other, and then that average is averaged with I_e at the critical positive-moment section.* It should also be noted that the multipliers for long-term deflection at these sections should be averaged, as were the I_e values.

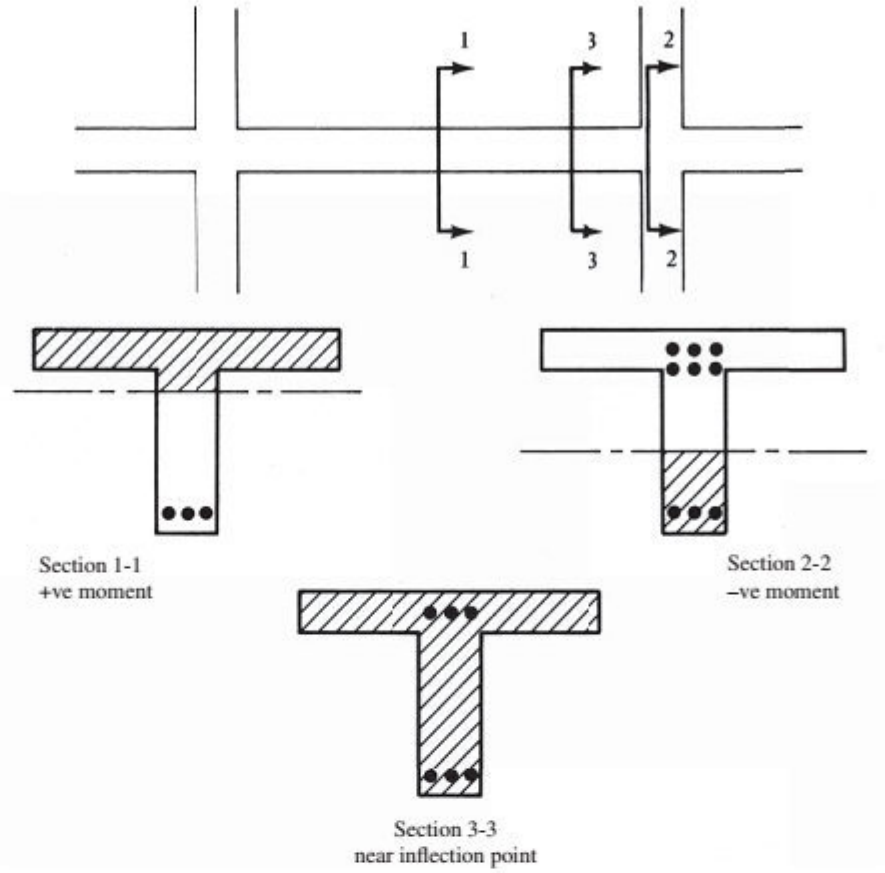


FIGURE 6.6 Deflections for a continuous T beam.

Example 6.2 illustrates the calculation of deflections for a continuous member. Although much of the repetitious math is omitted from the solution given herein, you can see that the calculations are still very lengthy, and you will understand why approximate deflection calculations are commonly used for continuous spans.

Example 6.2

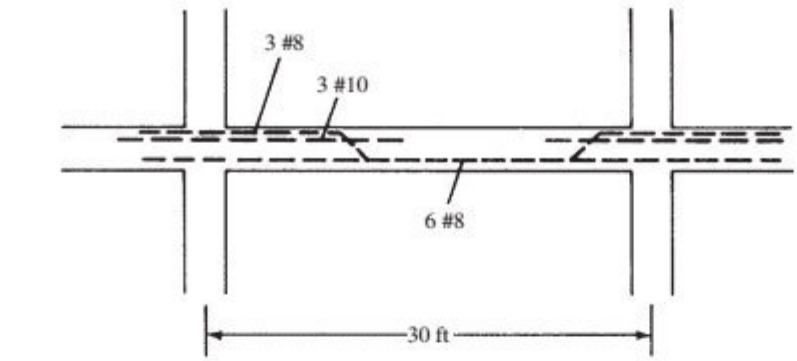
Determine the instantaneous deflection at the midspan of the continuous T beam shown in Figure 6.7(a). The member supports a dead load, including its own weight, of 1.5 k/ft, and a live load of 2.5 k/ft. $f'_c = 3000$ psi and $n = 9$. The moment diagram for full dead and live loads is shown in Figure 6.7(b), and the beam cross section is shown in Figure 6.7(c).

SOLUTION

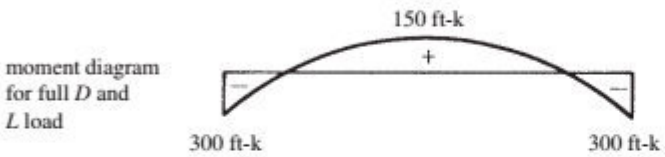
For Positive-Moment Region

1. Locating centroidal axis for uncracked section and calculating gross moment of inertia I_g and cracking moment M_{cr} for the positive-moment region (Figure 6.8). See Example 2.2 of this text.

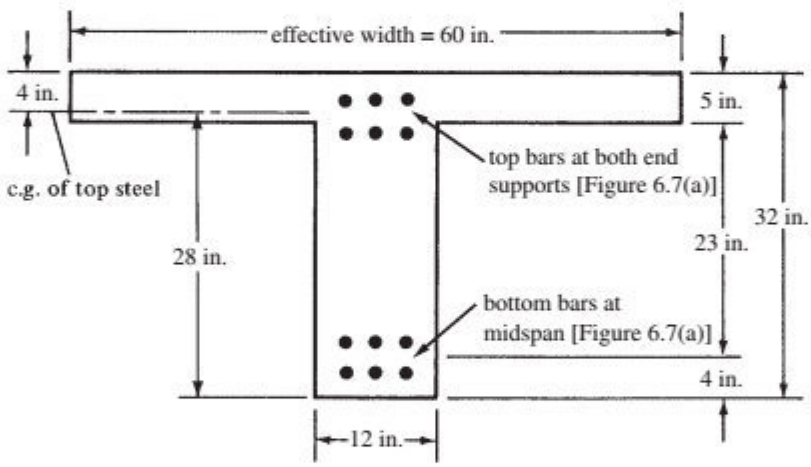
$$\bar{y} = \frac{(60 \text{ in.})(5 \text{ in.})\left(\frac{5 \text{ in.}}{2}\right) + (27 \text{ in.})(12 \text{ in.})\left(5 \text{ in.} + \frac{27 \text{ in.}}{2}\right)}{(60 \text{ in.})(5 \text{ in.}) + (27 \text{ in.})(12 \text{ in.})} = 10.81 \text{ in.}$$



(a)



(b)



(c)

FIGURE 6.7 Information for Example 6.2.

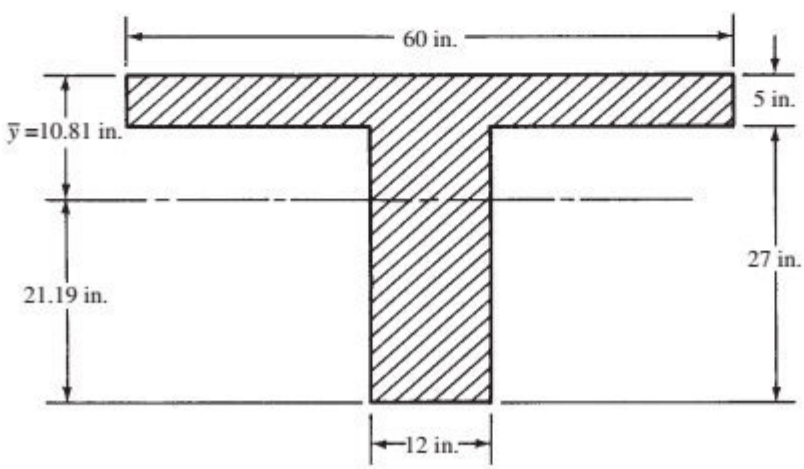


FIGURE 6.8 Centroid of uncracked section in Example 6.2.

$$I_g = \frac{(60\text{ in.})(5\text{ in.})^3}{12} + (60\text{ in.})(5\text{ in.})\left(10.81\text{ in.} - \frac{5\text{ in.}}{2}\right)^2 + \frac{(12\text{ in.})(27\text{ in.})^3}{12}$$
$$+ (12\text{ in.})(27\text{ in.})\left(21.19\text{ in.} - \frac{27\text{ in.}}{2}\right)^2 = 60{,}185\text{ in.}^4$$

$$M_{cr} = \frac{(7.5)(\sqrt{3000}\text{ psi})(60{,}185\text{ in.}^4)}{21.19\text{ in.}} = 1{,}166{,}754\text{ in.-lb} = 97.2\text{ ft-k}$$

2. Locating the centroidal axis of cracked section and calculating transformed moment of inertia I_{cr} for the positive-moment region (Figure 6.9) as previously illustrated in Example 2.6 of this text.

$$x = 5.65\text{ in.}$$
$$I_{cr} = 24{,}778\text{ in.}^4$$

3. Calculating the effective moment of inertia in the positive-moment region.

$$M_a = 150\text{ ft-k}$$
$$I_e = \left(\frac{97.2\text{ ft-k}}{150\text{ ft-k}}\right)^3 (60{,}185\text{ in.}^4) + \left[1 - \left(\frac{97.2\text{ ft-k}}{150\text{ ft-k}}\right)^3\right] 24{,}778\text{ in.}^4 = 34{,}412\text{ in.}^4$$

For Negative-Moment Region

1. Locating the centroidal axis for uncracked section and calculating gross moment of inertia I_g and cracking moment M_{cr} for the negative-moment region, considering only the hatched rectangle shown in Figure 6.10.

$$\bar{y} = \left(\frac{32\text{ in.}}{2}\right) = 16\text{ in.}$$
$$I_g = \left(\frac{1}{12}\right)(12\text{ in.})(32\text{ in.})^3 = 32{,}768\text{ in.}^4$$
$$M_{cr} = \frac{(7.5)(\sqrt{3000}\text{ psi})(32{,}768\text{ in.}^4)}{16\text{ in.}} = 841{,}302\text{ in.-lb} = 70.1\text{ ft-k}$$

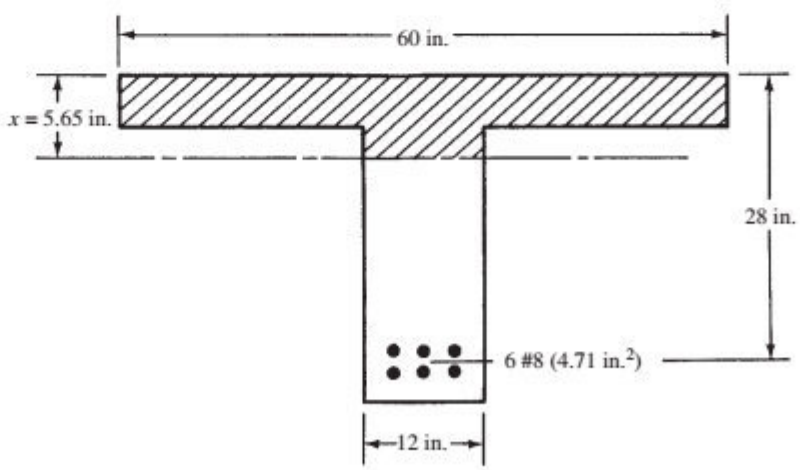


FIGURE 6.9 Centroidal axis of cracked section in Example 6.2.

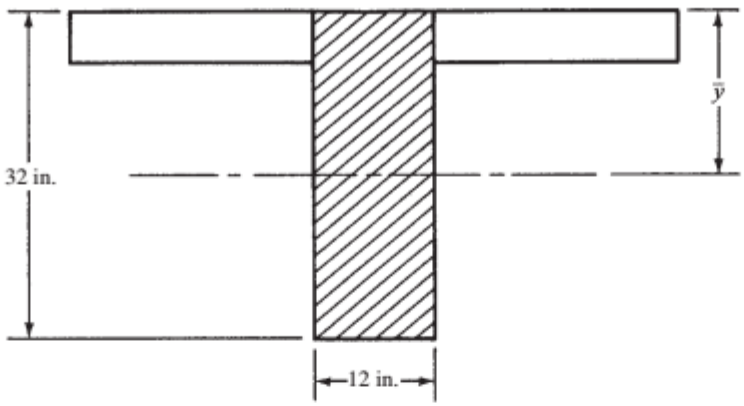


FIGURE 6.10 Centroid of uncracked section for negative moment in Example 6.2.

The code does not require that the designer ignore the flanges in tension for this calculation. The authors used this method to be conservative. If the tension flanges are considered, then the cracking moment is calculated from the section in Figure 6.8. The value of $\bar{y}$ is taken to the top of the section (10.81 in.) because the top is in tension for negative moment, so

$$M_{cr} = \frac{7.5\sqrt{3000}\text{ psi}\,(60{,}185\text{ in.}^4)}{10.81\text{ in.}} = 2{,}287{,}096\text{ in.-lb} = 190.6\text{ ft-k}$$

If this larger value for M_{cr} were used in step 3 below, the value of I_e would be 33,400 in.^{4}.

2. Locating the centroidal axis of the cracked section and calculating the transformed moment of inertia I_{tr} for the negative-moment region (Figure 6.11). See Example 2.7 for this type of calculation. The values of x and I_{cr} below were obtained by transforming the compression steel using $2n$. It is probably better to use n instead, in this case. The reason for using $2n$, as discussed prior to Example 2.7, is to account for creep in concrete. However, because long-term effects are handled by the creep factor in deflection calculations, it would be redundant to use $2n$.

$$x = 10.43\text{ in.}$$
$$I_{cr} = 24{,}147\text{ in.}^4$$

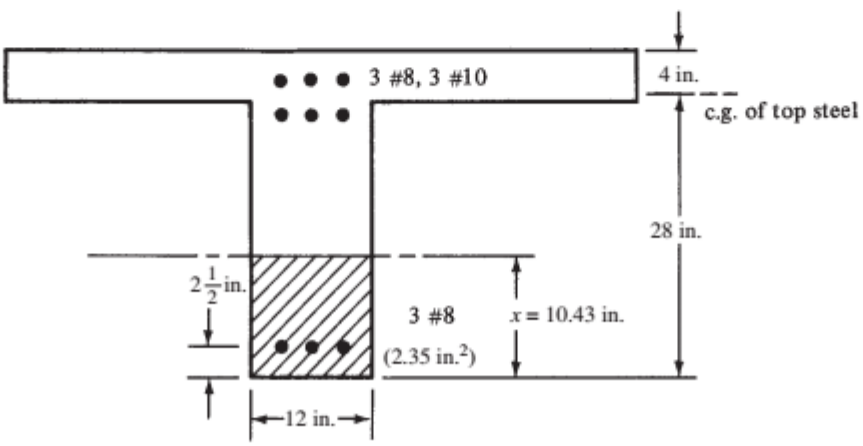


FIGURE 6.11 Centroidal axis of cracked section for negative moment in Example 6.2.

3. Calculating the effective moment of inertia in the negative-moment region.

$$M_a = 300 \text{ ft-k}$$

$$I_e = \left(\frac{70.1 \text{ ft-k}}{300 \text{ ft-k}}\right)^3 (32{,}768 \text{ in.}^4) + \left[1 - \left(\frac{70.1 \text{ ft-k}}{300 \text{ ft-k}}\right)^3\right] 24{,}147 \text{ in.}^4 = 24{,}257 \text{ in.}^4$$

Instantaneous Deflection

The I_e to be used is obtained by averaging the I_e at the positive-moment section, with the average of I_e computed at the negative-moment sections at the ends of the span.

$$\text{Average } I_e = \frac{1}{2}\left[\left(\frac{24{,}257 \text{ in.}^4 + 24{,}257 \text{ in.}^4}{2}\right) + 34{,}412 \text{ in.}^4\right] = 29{,}334 \text{ in.}^4$$

$$E_c = 57{,}000\sqrt{3000} \text{ psi} = 3.122 \times 10^6 \text{ psi}$$

Using the equation from Figure 6.2(b) and using only live loads to calculate deflections,

$$\delta_L = \frac{w_L l^4}{384 E_c I_e} = \frac{(2.5 \text{ klf})(30 \text{ ft})^4}{(384)(3122 \text{ ksi})(29{,}334 \text{ in.}^4)}(1728 \text{ in.}^3/\text{ft}^3) = 0.10 \text{ in.}$$

In this case the authors used an approximate method to calculate δ_L. Instead of the cumbersome equation ($\delta_L = \delta_{D+L} - \delta_D$) we used earlier in Example 6.1(c), we simply used w_L as the load in the above equation and averaged I_e. This approximation ignores the difference between I_e for dead load compared with I_e for dead and live load. This method gives a larger deflection, so it is conservative. Many designers have conservative approximations that they try first on many engineering calculations. If they work, there is no need to carry out the more cumbersome ones.

It has been shown that for continuous spans, the ACI 318 Code (Section 24.2.3.6) permits an averaging of the I_e values at the critical positive- and negative-moment sections. The ACI Commentary (R24.2.3.7) says that for approximate deflection calculations for continuous prismatic members, it is satisfactory to use the midspan section properties for simple and continuous spans and at supports for cantilevers. This is because these properties, which include the effect of cracking, have the greatest effect on deflections.

ACI Committee 435 has shown that better results for the deflections in continuous members can be obtained if an I_e is used that gives greater weight to the midspan values.[7] The committee suggests the use of the following expressions in which I_{em}, I_{e1}, and I_{e2} are the computed effective moments of inertia at the midspan and the two ends of the span, respectively.

Beams with two ends continuous

$$\text{Avg } I_e = 0.70I_{em} + 0.15(I_{e1} + I_{e2})$$

Beams with one end continuous

$$\text{Avg } I_e = 0.85I_{em} + 0.15(I_{\text{cont. end}})$$

For the beam of Example 6.2 with its two continuous ends, the effective moment of inertia would be

$$\text{Avg } I_e = (0.70)(34{,}412 \text{ in.}^4) + (0.15)(24{,}257 \text{ in.}^4 + 24{,}257 \text{ in.}^4)$$
$$= 31{,}365 \text{ in.}^4$$

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[7] ACI Committee 435, 1978, "Proposed Revisions by Committee 435 to ACI Building Code and Commentary Provisions on Deflections," *Journal ACI*, 75(6), pp. 229–238.

6.9 Types of Cracks

This section presents a few introductory comments concerning some of the several types of cracks that occur in reinforced concrete beams. The remainder of this chapter is concerned with the estimated widths of flexural cracks and recommended maximum spacings of flexural bars to control cracks.

Flexural cracks are vertical cracks that extend from the tension sides of beams up to the region of their neutral axes. They are illustrated in Figure 6.12(a). Should beams have very deep webs (more than 3 ft or 4 ft), the cracks will be very closely spaced, with some of them coming together above the reinforcement and some disappearing there. These cracks may be wider up in the middle of the beam than at the bottom.

Inclined cracks due to shear can develop in the webs of reinforced concrete beams either as independent cracks or as extensions of flexural cracks. Occasionally, inclined cracks will develop independently in a beam, even though no flexural cracks are in that locality. These cracks, which are called *web-shear cracks* and which are illustrated in Figure 6.12(b), sometimes occur in the webs of prestressed sections, particularly those with large flanges and thin webs.

The usual type of inclined shear cracks are the *flexure-shear cracks*, which are illustrated in Figure 6.12(c). They commonly develop in both prestressed and non-prestressed beams.

Torsion cracks, which are illustrated in Figure 6.12(d), are quite similar to shear cracks except that they spiral around the beam. Should a plain concrete member be subjected to pure torsion, it will crack and fail along 45° spiral lines due to the diagonal tension corresponding to the torsional stresses. For a very effective demonstration of this type of failure, you can take a piece of chalk in your hands and twist it until it breaks. Although torsion stresses are very similar to shear stresses, they will occur on all faces of a member. As a result, they add to the shear stresses on one side and subtract from them on the other.

Sometimes bond stresses between the concrete and the reinforcement lead to a splitting along the bars, as shown in Figure 6.12(e).

Of course, there are other types of cracks not illustrated here. Members that are loaded in axial tension will have transverse cracks through their entire cross sections. Cracks can also occur in concrete members due to shrinkage, temperature change, settlements, and so on. Considerable information concerning the development of cracks is available.[8]

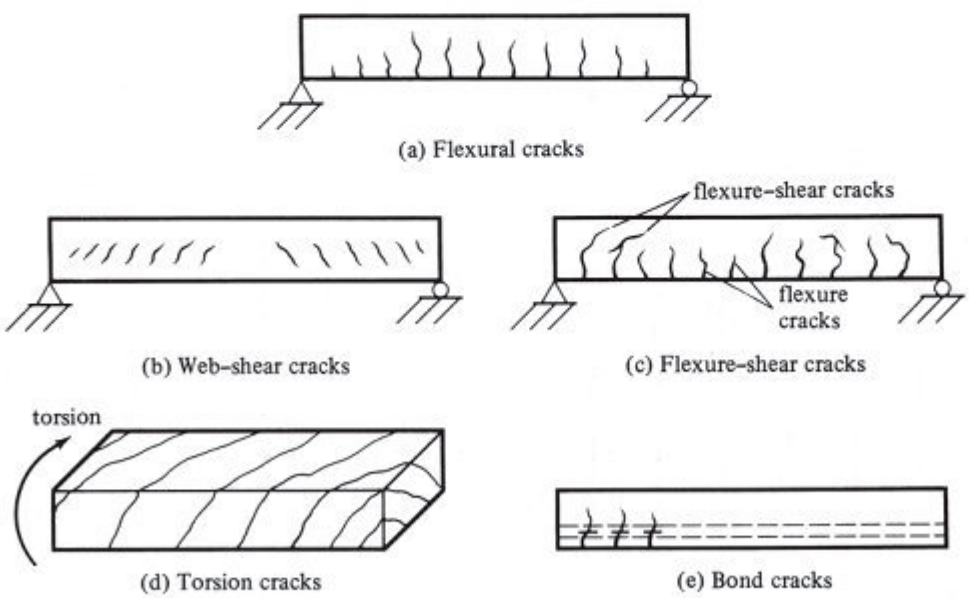


FIGURE 6.12 Some types of cracks in concrete members.

[8] Wight, J. K. and MacGregor, J. G., 2011, *Reinforced Concrete Mechanics and Design*, 6th ed. (Upper Saddle River, NJ: Prentice-Hall), pp. 434–442.

6.10 Control of Flexural Cracks

Cracks will occur in reinforced concrete structures because of concrete's low tensile strength. For members with low steel stresses at service loads, the cracks may be very small and in fact may not be visible except upon careful examination. Such cracks, called *microcracks*, are generally initiated by bending stresses.

When steel stresses are high at service load, particularly where high-strength steels are used, visible cracks will occur. These cracks should be limited to certain maximum sizes so that the appearance of the structure is not spoiled and so that corrosion of the reinforcing bars does not result. The use of high-strength bars and the strength design method have made crack control a very important item indeed. Because the yield stresses of reinforcing bars in general use have increased from 40 ksi to 60 ksi and above, it has been rather natural for designers to specify approximately the same size bars as they are accustomed to using, but fewer of them. The result has been more severe cracking of members.

Although cracks cannot be eliminated, they can be limited to acceptable sizes by spreading out or distributing the reinforcement. In other words, smaller cracks will result if several small bars are used with moderate spacings rather than a few large ones with large spacings. Such a practice will usually result in satisfactory crack control even for Grades 60 and 75 bars. An excellent rule of thumb to use regarding cracking is "*don't use a bar spacing in one-way slabs larger than about 9 in.*"

The maximum crack widths that are considered to be acceptable vary from approximately 0.004 in. to 0.016 in., depending on the location of the member in question, the type of structure, the surface texture of the concrete, illumination, and other factors. Somewhat smaller values may be required for members exposed to very aggressive environments, such as deicing chemicals and saltwater spray.

ACI Committee 224, in a report on cracking,[9] presented a set of approximately permissible maximum crack widths for reinforced concrete members subject to different exposure situations. These values are summarized in Table 6.3.

Definite data are not available as to the sizes of cracks above which bar corrosion becomes particularly serious. As a matter of fact, tests seem to indicate that concrete quality, cover thickness, amount of concrete vibration, and other variables may be more important than crack sizes in their effect on corrosion.

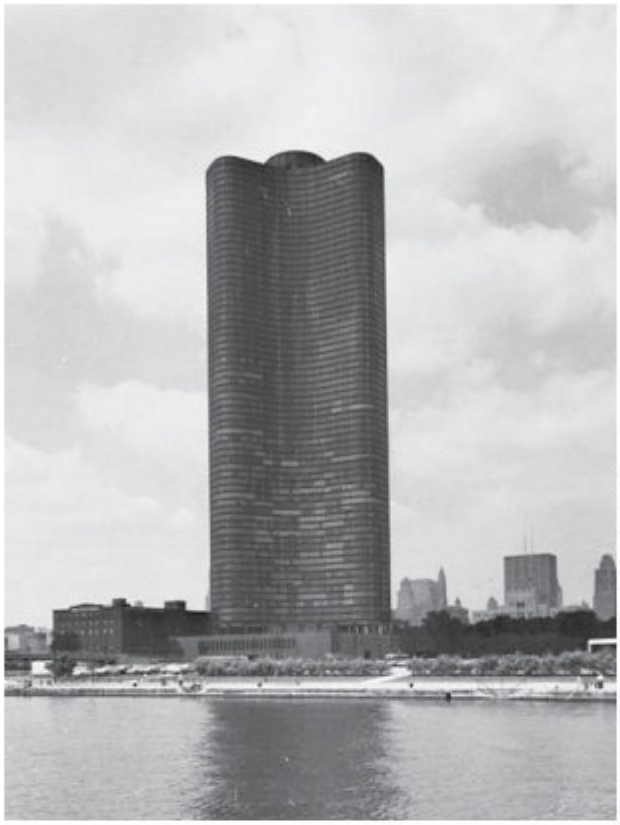
Results of laboratory tests of reinforced concrete beams to determine crack sizes vary. The sizes are greatly affected by shrinkage and other time-dependent factors. The purpose of crack-control calculations is not really to limit cracks to certain rigid maximum values but rather to use reasonable bar details, as determined by field and laboratory experience, that will in effect keep cracks within a reasonable range.

TABLE 6.3 Guide to Reasonable Crack Widths, Reinforced Concrete Under Service Loads

	Permissible Crack Widths	
Exposure Condition	**(in.)**	**(mm)**
Dry air or protective membrane	0.016	0.41
Humidity, Moist air, soil	0.012	0.30
Deicing chemicals	0.007	0.18
Seawater and seawater spray, wetting and drying	0.006	0.15
Water-retaining structures	0.004	0.10

Source: Reproduced with permission from the American Concrete Institute.

[9] ACI Committee 224 R-01, 2002, "Control of Cracking in Concrete Structures," *Journal ACI*, 69(12), Table 4.1.



Courtesy of Portland Cement Association.

Lake Point Tower, Chicago, Illinois.

The following equation was developed for the purpose of estimating the maximum widths of cracks that will occur in the tension faces of flexural members.[10] It is merely a simplification of the many variables affecting crack sizes.

$$w = 0.076\beta_h f_s \sqrt[3]{d_c A}$$

where

w = the estimated cracking width in thousandths of inches

β_h = ratio of the distance to the neutral axis from the extreme tension concrete fiber to the distance from the neutral axis to the centroid of the tensile steel (values to be determined by the working-stress method)

f_s = steel stress, in kips per square inch at service loads (designer is permitted to use $0.6 f_y$ for normal structures)

d_c = the cover of the outermost bar measured from the extreme tension fiber to the center of the closest bar or wire (for bundled bars, d_c is measured to the centroid of the bundles)

A = the effective tension area of concrete around the main reinforcement (having the same centroid as the reinforcing) divided by the number of bars

[10] Gergely, P. and Lutz, L. A., 1968, "Maximum Crack Width in Reinforced Flexural Members," *Causes, Mechanisms and Control of Cracking in Concrete*, SP-20 (Detroit: American Concrete Institute), pp. 87–117.

This expression is referred to as the Gergely–Lutz equation after its developers. In applying it to beams, reasonable results are usually obtained if β_h is set equal to 1.20. For thin one-way slabs, however, more realistic values are obtained if β_h is set equal to 1.35.

The number of reinforcing bars present in a particular member decidedly affects the value of A to be used in the equation and thus the calculated crack width. If more and smaller bars are used to provide the necessary area, the value of A will be smaller, as will the estimated crack widths.

Should all the bars in a particular group not be the same size, their number (for use in the equation) should be considered to equal the total reinforcing steel area actually provided in the group divided by the area of the largest bar size used.

Example 6.3 illustrates the determination of the estimated crack widths occurring in a tensilely reinforced rectangular beam.

Example 6.3

Assuming $\beta_h = 1.20$ and $f_y = 60$ ksi, calculate the estimated width of flexural cracks that will occur in the beam of Figure 6.13. If the beam is to be exposed to moist air, is this width satisfactory as compared to the values given in Table 6.3 of this chapter? Should the cracks be too wide, revise the design of the reinforcing structure and recompute the crack width.

SOLUTION

Substituting into the Gergely–Lutz Equation

$$d_c = 3 \text{ in.}$$

$$A = \frac{(6 \text{ in.})(16 \text{ in.})}{3} = 32 \text{ in.}^2$$

$$w = (0.076)(1.20)(0.6 \times 60 \text{ ksi})\sqrt[3]{(3 \text{ in.})(32 \text{ in.}^2)}$$

$$= \frac{15.03 \text{ in.}}{1000} = 0.015 \text{ in.} > 0.012 \text{ in.} \quad \underline{\text{No good}}$$

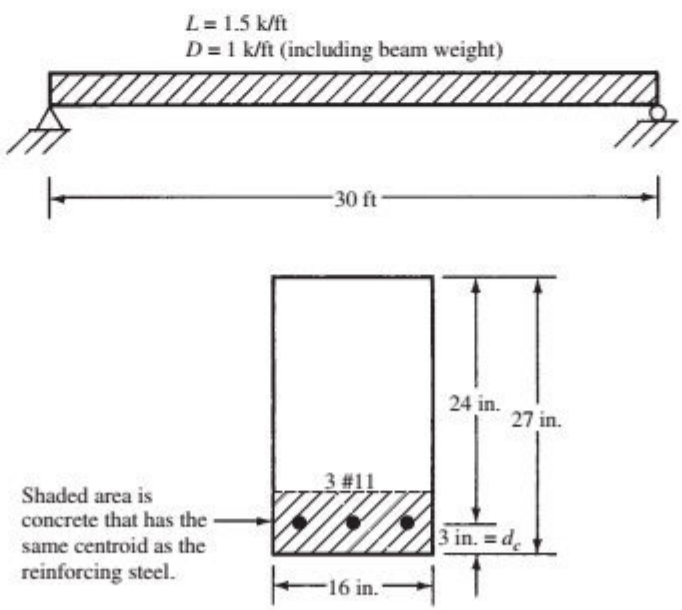

L = 1.5 k/ft
D = 1 k/ft (including beam weight)
30 ft
24 in.
27 in.
3 #11
Shaded area is concrete that has the same centroid as the reinforcing steel.
3 in. = d_c
16 in.

FIGURE 6.13 Beam properties for Example 6.3.

Replace the 3 #11 bars (4.68 in.^{2}) with five #9 bars (5.00 in.^{2}).

$$A = \frac{(6\text{ in.})(16\text{ in.})}{5} = 19.2\text{ in.}^2$$

$$w = (0.076)(1.20)(0.6 \times 60\text{ ksi})\sqrt[3]{(3\text{ in.})(19.2\text{ in.}^2)}$$

$$= \frac{12.68\text{ in.}}{1000} = 0.0127\text{ in.} > 0.012\text{ in.} \quad \underline{\underline{\text{No good}}}$$

Try 6 #8 bars (4.71 in.^{2}).

$$A = \frac{(6\text{ in.})(16\text{ in.})}{6} = 16\text{ in.}^2$$

$$w = (0.076)(1.20)(0.6 \times 60\text{ ksi})\sqrt[3]{(3\text{ in.})(16\text{ in.}^2)}$$

$$= \frac{11.93\text{ in.}}{1000} = 0.0119\text{ in.} < 0.012\text{ in.} \quad \underline{\underline{\text{OK}}}$$

Use 6 #8 bars.

If reinforced concrete members are tested under carefully controlled laboratory conditions and cracks measured for certain loadings, considerable variations in crack sizes will occur. Consequently, the calculations of crack widths described in this chapter should only be used to help the designer select good details for reinforcing bars. The calculations are clearly not sufficiently accurate for comparison with field crack sizes.

The bond stress between the concrete and the reinforcing steel decidedly affects the sizes and spacings of the cracks in concrete. When bundled bars are used, there is appreciably less contact between the concrete and the steel, as compared to the cases where the bars are placed separately from each other. To estimate crack widths successfully with the Gergely–Lutz equation when bundled bars are used, it is necessary to take into account this reduced contact surface.[11]

When bundled bars are present, some designers use a very conservative procedure in computing the value of A. For this calculation they assume each bundle is one bar, that bar having an area equal to the total area of the bars in that bundle. Certainly, the bond properties of a group of bundled bars are better than those of a single large "equivalent bar."

Particular attention needs to be given to crack control for doubly reinforced beams, where it is common to use small numbers of large-diameter tensile bars. Calculation of crack widths for such beams may result in rather large values, thus in effect requiring the use of a larger number of rather closely spaced smaller bars.

Special rules are given in ACI Code 318, Section 24.3.4 for the spacings of reinforcing to help control the amount of cracking in T beams whose flanges are in tension.

6.11 ACI Code Provisions Concerning Cracks

In the ACI 318 Code, Section 24.3.2 requires that flexural tensile reinforcement be well distributed within the zones of maximum tension so that the center-to-center spacing of the deformed bars and wires closest to a tension surface is not greater than the value computed with the following expression:

$$s = (15)\left(\frac{40{,}000}{f_s}\right) - 2.5c_c \le (12)\left(\frac{40{,}000}{f_s}\right) \qquad \text{(ACI Table 24.3.2)}$$

In this expression, f_s is the computed tensile stress at working load. It may be calculated as shown in Examples 2.3 through 2.7, or it may simply be taken equal to $\frac{2}{3}f_y$. The term c_c represents the least distance from the surface of deformed reinforcement to the tension face of the concrete, in inches.

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[11] Nawy, E. G., 2009, *Reinforced Concrete: A Fundamental Approach*, 6th ed. (Upper Saddle River, NJ: Prentice-Hall), pp. 307–309.

For beams with Grade 60 reinforcing steel and with 2-in. clear cover, the maximum code-permitted bar spacing is

$$s = (15)\left(\frac{40{,}000}{0.667 \times 60{,}000 \text{ psi}}\right) - (2.5)(2 \text{ in.})$$
$$= 10.0 \text{ in.} < (12)\left(\frac{40{,}000}{0.667 \times 60{,}000 \text{ psi}}\right) = 12.0 \text{ in.}$$

A bar spacing not more than 10.0 in. would thus be required. This limit can control the spacing of bars in one-way slabs but is not likely to control beam bar spacings.

The ACI equation for bar spacing used in this calculation applies to deformed bars and wires. In the case of bonded prestressed reinforcement, the calculated spacing maximum is reduced by $\frac{1}{3}$. If a combination of deformed bars or wires and bonded prestressed tendons is used, then the calculated spacing is reduced by $\frac{1}{6}$.

The ACI equation for maximum spacing may not be sufficient for situations where slabs or beams are subject to fatigue, designed to be watertight, or exposed to a corrosive environment. Special consideration must be given to such situations. It is probably well to use the Gergely–Lutz equation and a set of maximum crack widths, such as those of Table 6.3, for such situations.

The effect of cracks and their widths on the corrosion of reinforcing bars is not clearly understood. There does not seem to be a direct relationship between crack widths and corrosion, at least at the reinforcing stresses occurring when members are subjected to service loads. Thus the ACI Code no longer distinguishes between interior and exterior exposure, as it once did. Research seems to indicate that the total corrosion occurring in reinforcing bars is not clearly correlated to crack widths. It is true, however, that the time required for corrosion to begin in reinforcing bars is inversely related to the widths of cracks.

When using the Gergely–Lutz equation with SI units, it becomes $w = 0.0113\beta_h f_s \sqrt[3]{d_c A}$, with the resulting crack widths in mm.

The SI version of the ACI Code for the maximum spacing of tensile bars from the standpoint of crack widths is given here. To use this expression correctly, s and c_c must be used in mm, while f_s must be in MPa.

$$s = (380)\left(\frac{280}{f_s}\right) - 2.5c_c \leq (300)\left(\frac{280}{f_s}\right)$$

6.12 SI Example

Example 6.4

Is the spacing of the bars shown in Figure 6.14 within the requirements of the ACI Code from the standpoint of cracking, if $f_y = 420$ MPa?

SOLUTION

For $f_y = 420$ MPa and $c_c = 75 \text{ mm} - \frac{28.7 \text{ mm}}{2} = 60.65$ mm

$$s = (380)\left(\frac{280}{0.667 \times 420 \text{ MPa}}\right) - (2.5)(60.65 \text{ mm})$$
$$= 228 \text{ mm} < (300)\left(\frac{280}{0.667 \times 420 \text{ MPa}}\right)$$
$$= 300 \text{ mm}$$

Because the actual bar spacing of 75 mm is less than 228 mm, this spacing is acceptable.

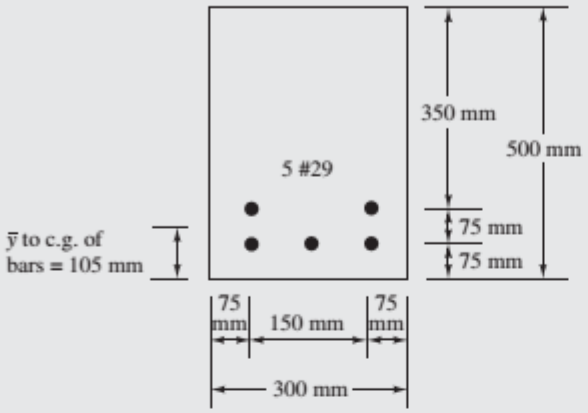


FIGURE 6.14 Beam cross section for Example 6.4.

6.13 Miscellaneous Cracks

The beginning designer will learn that it is wise to include a few reinforcing bars in certain places in some structures, even though there seems to be no theoretical need for them. Certain spots in some structures (such as in abutments, retaining walls, building walls near openings, etc.) will develop cracks. Such cracks may result from differential movement caused by concrete creep and shrinkage and temperature movement. The young designer should try to learn about such situations from more experienced people. Better structures will be the result.

6.14 Computer Example

Example 6.5

Repeat Example 6.1 using the Excel spreadsheet in Chapter 6.

Deflection Calculator for Simply Supported, Uniformly Loaded Rectangular Beam

	b =	12	in.
	d =	17	in.
	h =	20	in.
	A_s =	3.00	$in.^2$
	d' =	2.5	in.
	A'_s =	0.00	$in.^2$
	f'_c =	3	ksi
	f_y =	60	ksi
	γ_c =	145	pcf
	λ =	1	

(Continues)

	ξ = (from Table 6.2 or Figure 6.4)	2.00	
	w_D =	750	plf
	w_{SW} =	250	plf
	w_L =	700	plf
	l = span =	20	ft
	Deflection limit (denominator from Table 6.1)	180	
	% live load that is sustained	30	%
	E_c =	3156	ksi
	$n = E_s/E_c$	9.189	9
	ρ =	0.0147	
	$n\rho$ =	0.1324	
	k =	0.3989	
	$x = kd$ =	6.781	in.
	I_{cr} =	4067	in.^{4}
	I_g =	8000	in.^{4}
	f_r =	410.8	psi
	M_{cr} =	27.39	k-ft
Dead + Full Live Load			
	$M_{a,D+L}$ =	85.00	k-ft
	$(M_{cr}/M_{a,D+L})^3$ =	0.0334	
	$I_e = \left(\frac{M_{cr}}{M_a}\right)^3 I_g + \left[1 - \left(\frac{M_{cr}}{M_a}\right)^3\right] I_{cr}$ =	4198.3	in.^{4}
	δ_{D+L} =	0.462	in.
Dead Load Only			
	$M_{a,D}$ =	50.00	k-ft
	$(M_{cr}/M_{a,D})^3$ =	0.164	
	I_e =	4713	in.^{4}
	δ_D =	0.242	in.
Live Load Only			
	$\delta_L = \delta_{D+L} - \delta_D$ =	0.220	in.
Initial δ from $D + \%L$			
	$M_{a,D+\%L}$ =	60.50	
	$(M_{cr}/M_{a,D+\%L})^3$ =	0.0928	
	I_e =	4431.6	in.^{4}
	$\delta_{D+\%L}$ =	0.311	in.
Initial δ from $\%L$ Only			
	$\delta_{\%L} = (\delta_D + \delta_{\%L}) - \delta_D$ =	0.069	in.

(Continues)

Long-Term δ for D + Long-Term Sustained L			
	$\rho' =$	0	
	$\lambda_{\Delta\infty} = \xi/(1+\rho')$	2.00	
	$\lambda_{\Delta} = \xi/(1+\rho')$	2	
	$\delta_{LT} = \delta_L + \lambda_\Delta \delta_{ID} + \lambda_\Delta \delta_{\%L} =$	0.843	in.
	$\delta_{\text{Limit}} =$	1.333333	in.
	Deflection complies with Table 6.1		

PROBLEMS

Problem 6.1 What factors make it difficult to estimate accurately the magnitude of deflections in reinforced concrete members?

Problem 6.2 Why do deflections in concrete members increase as time goes by?

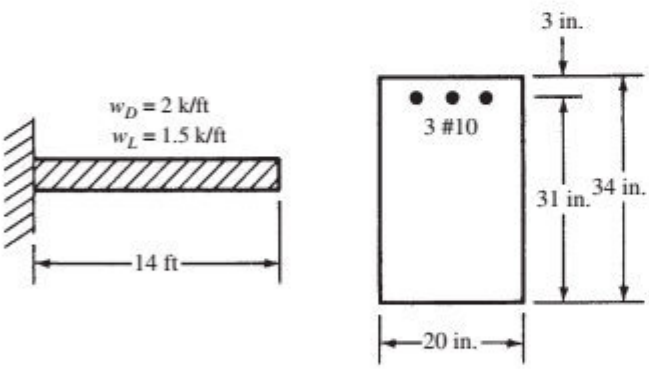
Problem 6.3 How can the deflection of concrete beams be limited?

Problem 6.4 Why is it necessary to limit the width of cracks in reinforced concrete members? How can it be done?

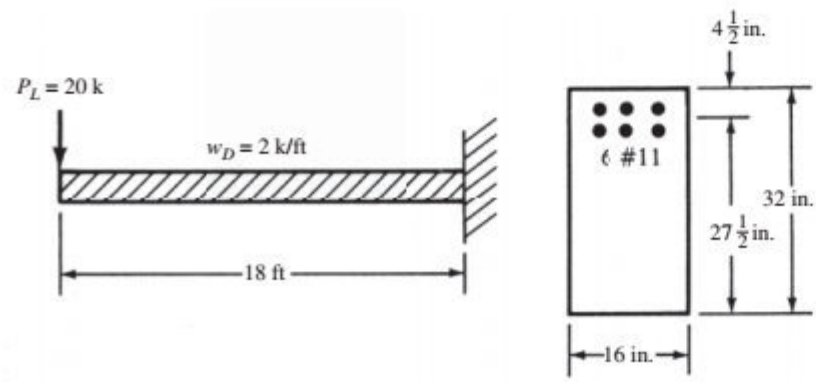
Deflections

For Problems 6.5 to 6.10, calculate the instantaneous deflections for the dead and live loads shown. Use $f_y = 60{,}000$ psi, $f_c = 4000$ psi, and $n = 8$. Beam weights are included in the w_D values.

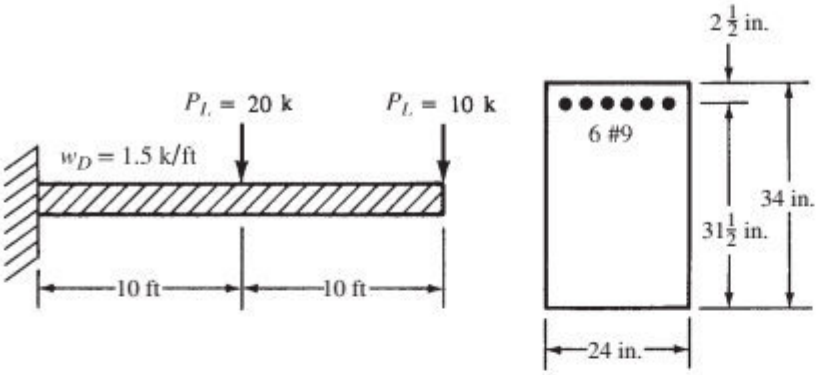
Problem 6.5 (*Ans.* 0.343 in.)



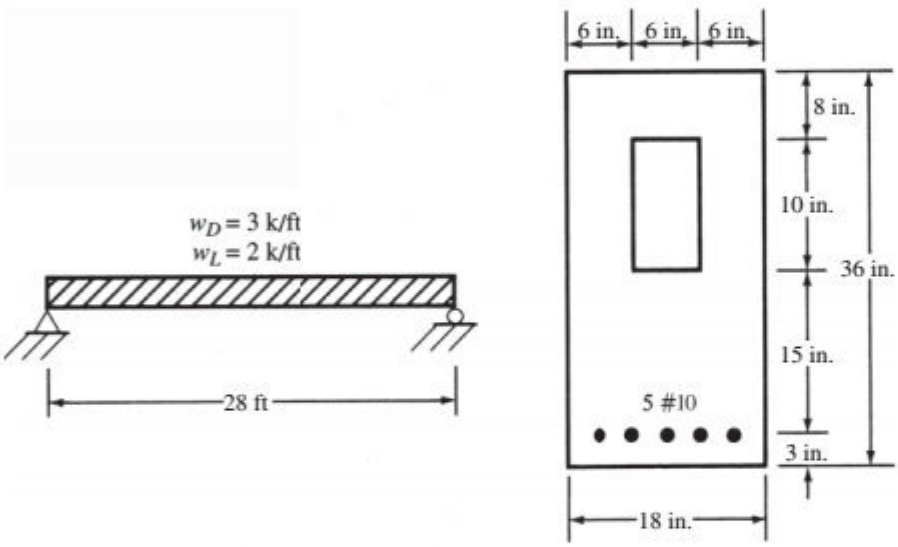
Problem 6.6



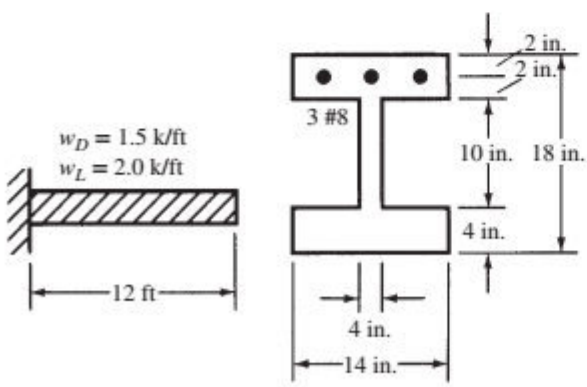
Problem 6.7 (*Ans.* 1.14 in.)



Problem 6.8



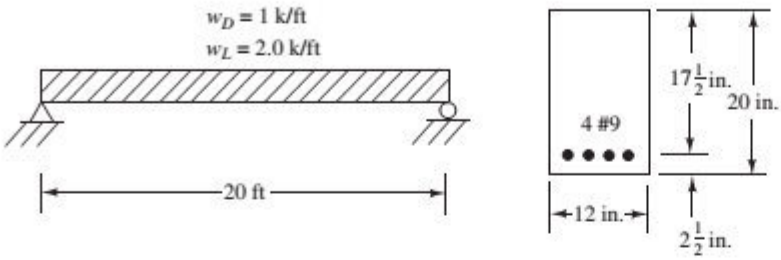
Problem 6.9 (*Ans.* 1.54 in.)



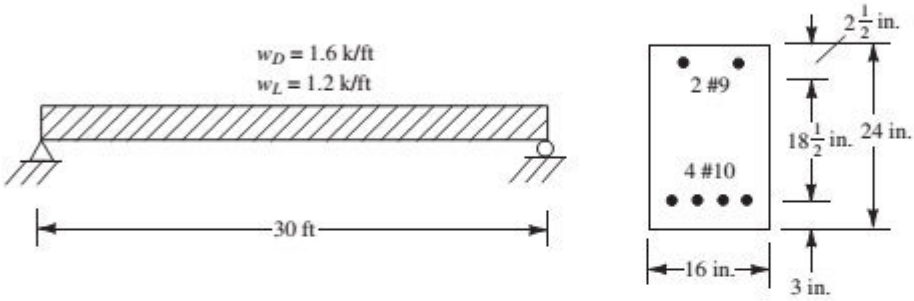
Problem 6.10 Repeat Problem 6.8 if a 25-k concentrated live load is added at the centerline of the span.

For Problems 6.11 and 6.12, calculate the instantaneous deflections and the long-term deflections after 4 years, assuming that 30% of the live loads are continuously applied for 48 months. $f_y = 60{,}000$ psi, $f'_c = 4000$ psi, and $n = 8$.

Problem 6.11 (*Ans.* Instantaneous δ for full $w_D + w_L = 0.610$ in., long-term $\delta = 1.043$ in.)



Problem 6.12



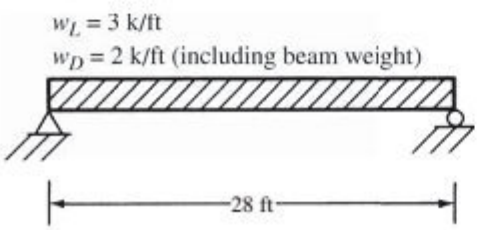
Problem 6.13 Repeat Problem 6.12 if the two top #9 compression bars are removed. (*Ans.* Instantaneous δ for full $D + L = 1.47$ in., long-term $\delta = 2.66$ in.)

Problem 6.14 Repeat Problem 6.12 using sand-lightweight concrete ($\gamma_c = 125$ pcf).

Problem 6.15 Using Chapter 6 Excel spreadsheet, repeat Problem 6.12 using all-lightweight concrete ($\gamma_c = 100$ pcf). (*Ans.* Instantaneous δ for full $w_D + w_L = 1.50$ in., long-term $\delta = 2.24$ in.)

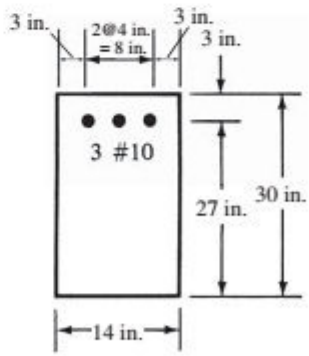
Crack Widths

Problem 6.16 Select a rectangular beam section for the span and loads shown. Use $\rho = \frac{1}{2}\rho_b$, #9 bars, $f'_c = 3000$ psi, and $f_y = 60{,}000$ psi. Compute the estimated maximum crack widths using the Gergely–Lutz equation. Are they less than the suggested maximum value given in Table 6.3 for dry air?

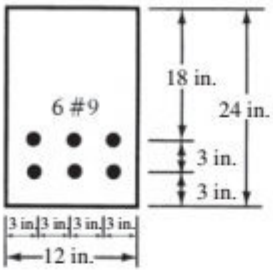


For Problems 6.17 and 6.18, estimate maximum crack widths using the Gergely–Lutz equation. Compare the results with the suggested maximum values given in Table 6.3. Assume $f_y = 60$ ksi and $\beta_h = 1.20$. Also, calculate maximum permissible bar spacings as per ACI Table 24.3.2 (the equation for maximum spacing, s, in Section 6.11). Assume moist air conditions.

Problem 6.17 (*Ans.* 0.016 in. > 0.012 in., max: ACI spacing = 9.09 in.)

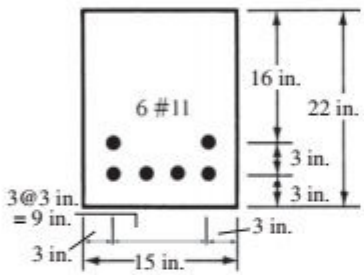


Problem 6.18

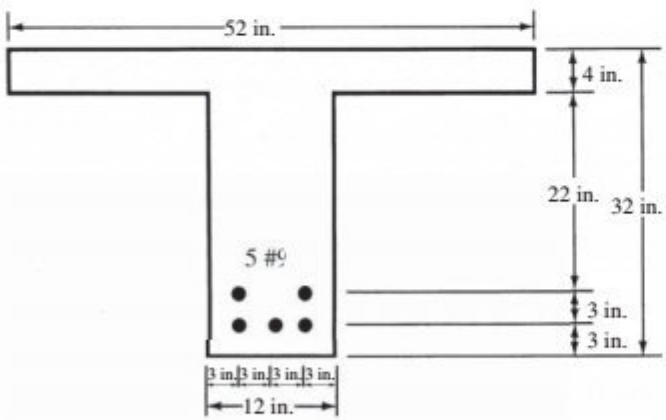


For Problems 6.19 to 6.21, same questions as for Problems 6.17 and 6.18, but assume interior exposure.

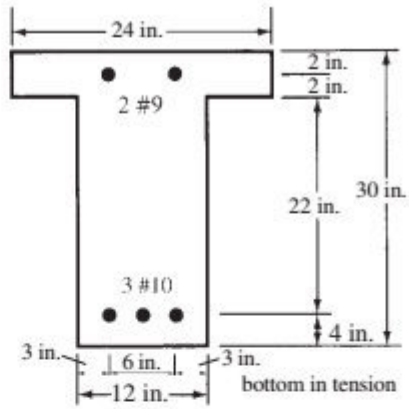
Problem 6.19 (*Ans.* 0.0129 in. < 0.016 in., max: ACI spacing = 9.26 in.)



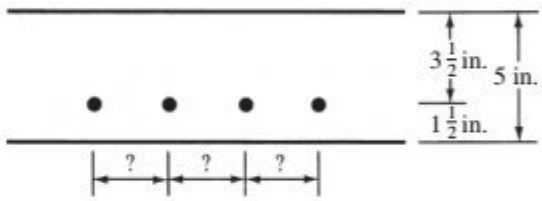
Problem 6.20



Problem 6.21 (*Ans.* 0.0165 in. > 0.016 in., 6.59 in.)



Problem 6.22 What is the maximum permissible spacing of #5 bars in the one-way slab shown that will satisfy the ACI Code crack requirements? $f_y = 60{,}000$ psi.

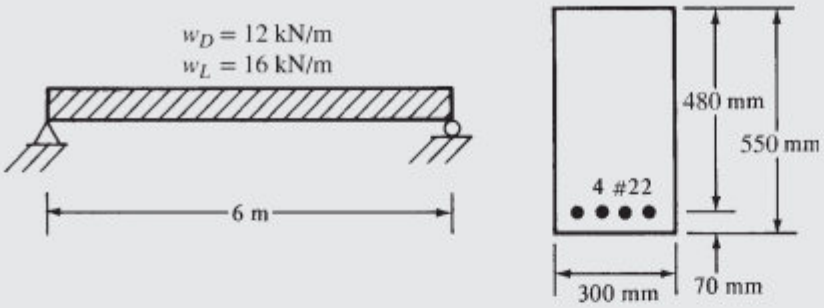


Problems in SI Units

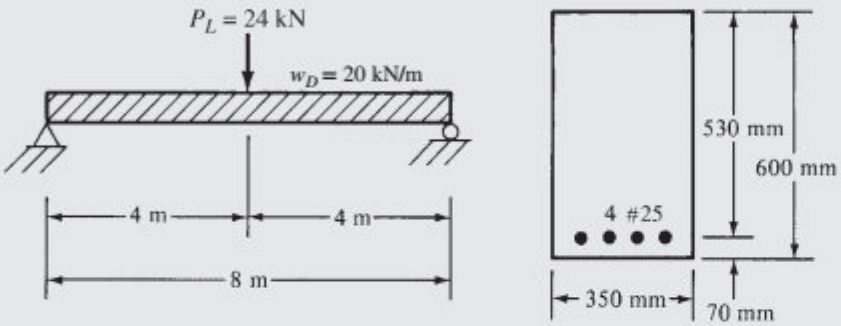
For Problems 6.23 to 6.25, calculate the instantaneous deflections. Use normal-weight concrete with $f'_c = 28$ MPa and $f_y = 420$ MPa.

Assume that the w_D values shown include the beam weights. $E_s = 200{,}000$ MPa.

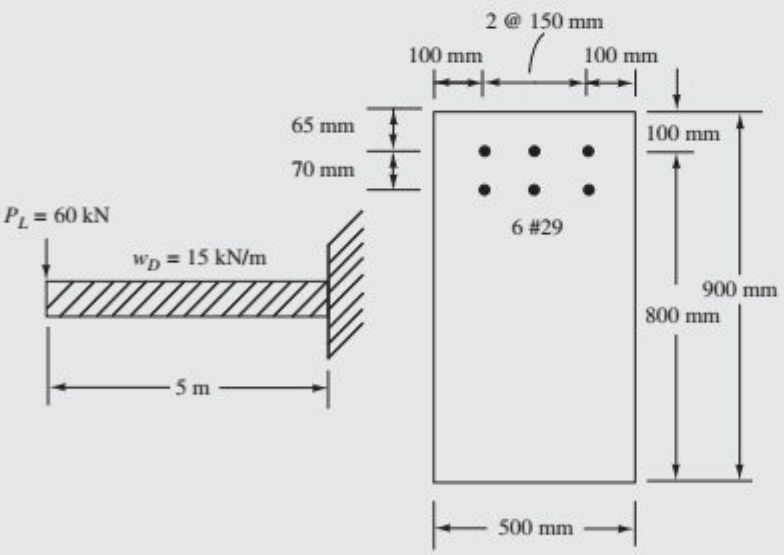
Problem 6.23 (*Ans*. 9.93 mm)



Problem 6.24



Problem 6.25 (*Ans*. 10.22 mm)



For Problems 6.26 and 6.27, do these beams meet the maximum spacing requirements of the ACI Code if $f_y = 420$ MPa?

Problem 6.26 The beam of Problem 6.24.

Problem 6.27 The beam of Problem 6.25 ($s = 253$ mm).

Problem 6.28 Rework Problem 6.11 using Chapter 6 Excel spreadsheet.

CHAPTER 7 Bond, Development Lengths, and Splices

7.1 Cutting Off or Bending Bars

The beams designed up to this point have been selected on the basis of maximum moments. These moments have occurred at or near span centerlines for positive moments and at the faces of supports for negative moments. At other points in the beams, the moments were less. Although it is possible to vary beam depths in some proportion to the bending moments, it is normally more economical to use prismatic sections and reduce or cut off some reinforcing bars when the bending moments are sufficiently small. Reinforcing steel is quite expensive, and cutting it off where possible may appreciably reduce costs.

Should the bending moment fall off 50% from its maximum, approximately 50% of the bars can be cut off or perhaps bent up or down to the other face of the beam and made continuous with the reinforcement in the other face. For this discussion, the uniformly loaded simple beam of Figure 7.1 is considered. This beam has six bars, and it is desired to cut off two bars when the moment falls off a third and two more bars when it falls off another third. For the purpose of this discussion, the maximum moment is divided into three equal parts by the horizontal lines shown. If the moment diagram is drawn to scale, a graphical method is satisfactory for finding the theoretical cutoff points.

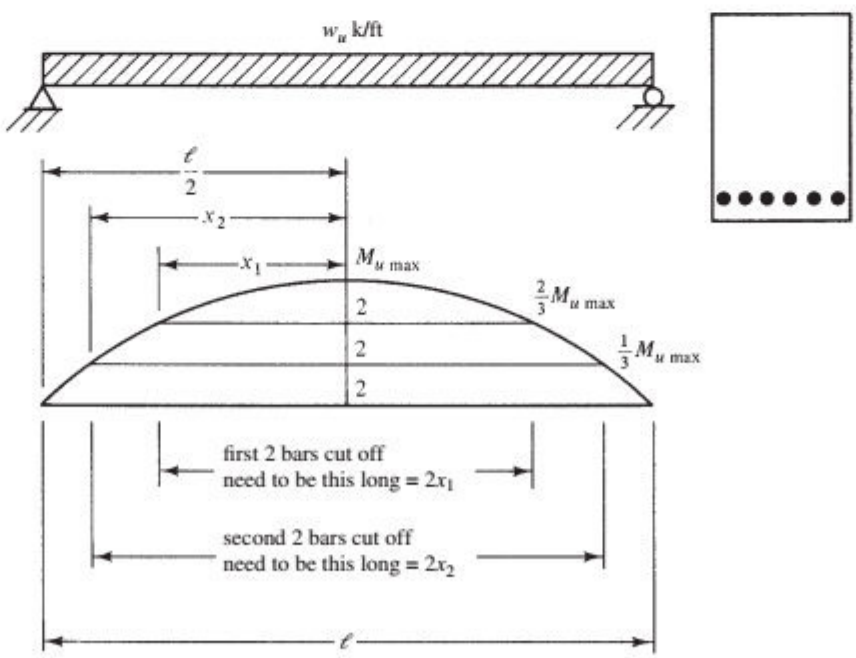


FIGURE 7.1 Theoretical cutoff locations for a simple span beam.

For the parabolic moment diagram of Figure 7.1, the following expressions can be written and solved for the bar lengths x_1 and x_2 shown in the figure:

$$\frac{x_1^2}{(\ell/2)^2} = \frac{2}{6}$$
$$\frac{x_2^2}{(\ell/2)^2} = \frac{4}{6}$$

For different-shaped moment diagrams, other mathematical expressions would have to be written, or a graphical method must be used.

Actually, the design ultimate moment capacity

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

does not vary exactly in proportion to the area of the reinforcing bars, as is illustrated in Example 7.1, because of variations in the depth of the compression block as the steel area is changed. The change is so slight, however, that for all practical purposes, the moment capacity of a beam can be assumed to be directly proportional to the steel area.

It will be shown in this chapter that the moment capacities calculated as illustrated in this example problem will have to be reduced if sufficient lengths are not provided beyond the theoretical cutoff points for the bars to develop their full stresses.

Example 7.1

For the uniformly loaded simple beam of Figure 7.2, determine the theoretical points on each end of the beam where two bars can be cut off, then determine the points where two more bars can be cut off. $f'_c = 3000$ psi, $f_y = 60{,}000$ psi.

SOLUTION

When the beam has only four bars,

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(4.00 \text{ in.}^2)\,(60 \text{ ksi})}{(0.85)\,(3 \text{ ksi})\,(18 \text{ in.})} = 5.23 \text{ in.}$$

$$\phi M_n = \phi A_s f_y \left(d - \frac{a}{2}\right) = (0.9)\,(4.00 \text{ in.}^2)\,(60 \text{ ksi}) \left(27 \text{ in.} - \frac{5.23 \text{ in.}}{2}\right) = 5267 \text{ in.-k} = 439 \text{ ft-k}$$

When the moment falls off to 439 ft-k, two of the six bars can theoretically be cut off.

When the beam has only two bars,

$$a = \frac{(2.00 \text{ in.}^2)\,(60 \text{ ksi})}{(0.85)\,(3 \text{ ksi})\,(18 \text{ in.})} = 2.61 \text{ in.}$$

$$\phi M_n = (0.9)\,(2.00 \text{ in.}^2)\,(60 \text{ ksi}) \left(27 \text{ in.} - \frac{2.61 \text{ in.}}{2}\right) = 2775 \text{ in.-k} = 231 \text{ ft-k}$$

When the moment falls off to 231 ft-k, two more bars can theoretically be cut off, leaving two bars in the beam.

(Notice that ρ with 6 bars = 6.00 in.^{2}/(18 in.)(27 in.) = 0.0123, which is less than $\rho_{max} = 0.0136$ from Appendix A, Table A.7, so the beam is ductile and $\phi = 0.9$. Also, this ρ is > ρ_{min} of 200/60,000 psi = 0.00333.)

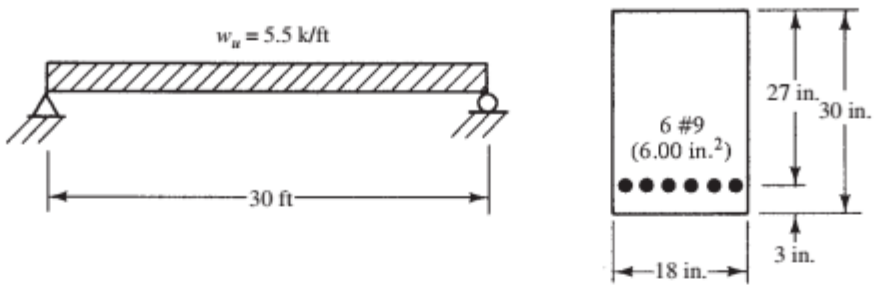


FIGURE 7.2 Given information for Example 7.1.

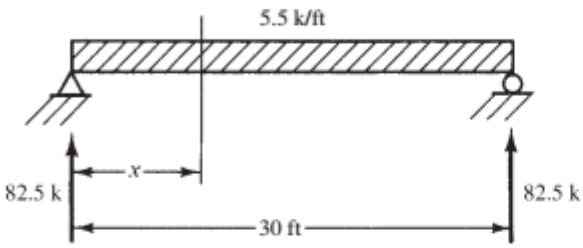


FIGURE 7.3 Beam reactions.

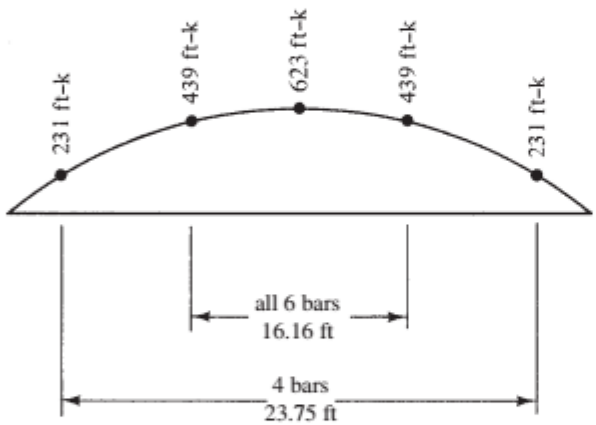


FIGURE 7.4 ϕM_n diagram for beam in Example 7.1.

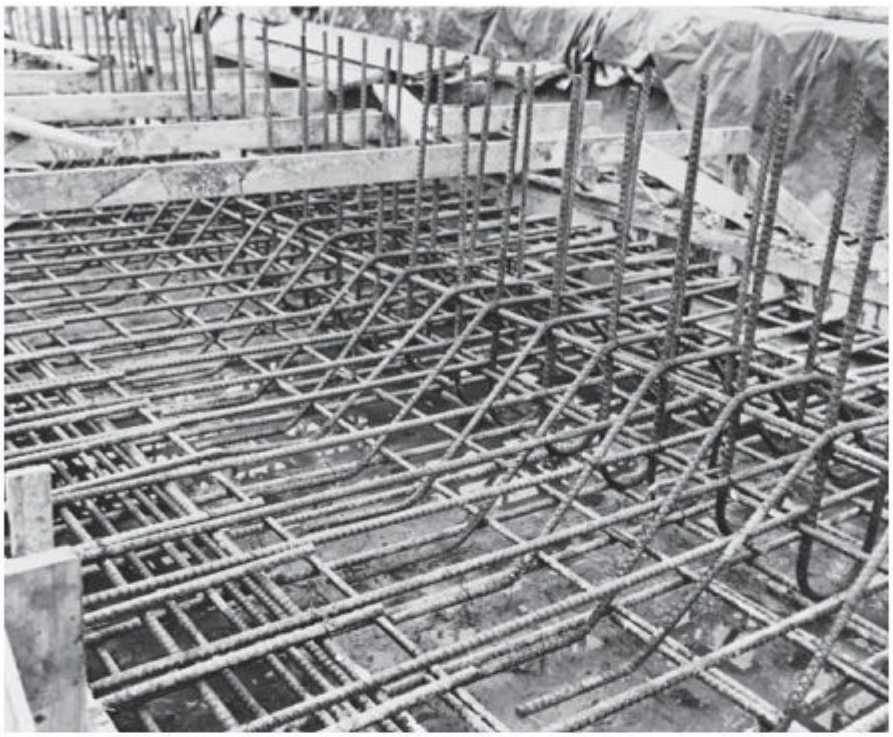
The moment at any section in the beam at a distance x from the left support is as follows, with reference being made to Figure 7.3:

$$M = 82.5x - (5.5x)\left(\frac{x}{2}\right)$$

From this expression, the location of the points in the beam where the moment is 439 ft-k and 231 ft-k can be determined. The results are shown in Figure 7.4.

Discussion: If the approximate procedure had been followed (where bars are cut off purely on the basis of the ratio of the number of bars to the maximum moment, as was illustrated with the equations on the previous page), the first two bars would have had lengths equal to 17.2 ft (as compared to the theoretically correct value of 16.16 ft), and the second two bar lengths would be equal to 24.45 ft (as compared to the theoretically correct value of 23.75 ft). It can then be seen that the approximate procedure yields fairly reasonable results.

In this section, the theoretical points of cutoff have been discussed. As will be seen in subsequent sections of this chapter, the bars will have to be run additional distances because of variations of moment diagrams, anchorage requirements of the bars, and so on.



Courtesy of Portland Cement Association.

Lab Building, Portland Cement Association, Skokie, Illinois.

7.2 Bond Stresses

A basic assumption made for reinforced concrete design is that there must be absolutely no slippage of the bars in relation to the surrounding concrete. In other words, the steel and the concrete should stick together, or *bond*, so that they will act as a unit. If there is no bonding between the two materials and if the bars are not anchored at their ends, they will pull loose from the concrete. As a result, the concrete beam will act as an unreinforced member and will be subject to sudden collapse as soon as the concrete cracks.

It is obvious that the magnitude of bond stresses will change in a reinforced concrete beam as the bending moments in the beam change. The greater the rate of bending moment change (occurring at locations of high shear), the greater will be the rate of change of bar tensions and, thus, bond stresses.

What may not be so obvious is the fact that bond stresses are also drastically affected by the development of tension cracks in the concrete. At a point where a crack occurs, all of the longitudinal tension will be resisted by the reinforcing bar. At a small distance along the bar at a point away from the crack, the longitudinal tension will be resisted by both the bar and the uncracked concrete. In this small distance, there can be a large change in bar tension due to the fact that the uncracked concrete is now resisting tension. Thus the bond stress in the surrounding concrete, which was zero at the crack, will drastically change within this small distance as the tension in the bar changes.

In the past, it was common to compute the maximum theoretical bond stresses at points in the members and to compare them with certain allowable values obtained by tests. It is the practice today, however, to look at the problem from an ultimate standpoint, where the situation is a little different. Even if the bars are completely separated from the concrete over considerable parts of their length, the ultimate strength of the beam will not be affected if the bars are so anchored at their ends that they cannot pull loose.

The bonding of the reinforcing bars to the concrete is due to several factors, including the chemical adhesion between the two materials, the friction due to the natural roughness of the bars, and the bearing of the closely spaced rib-shaped deformations on the bar surfaces against the concrete. The application of the force P to the bar shown in Figure 7.5 is considered in the discussion that follows.

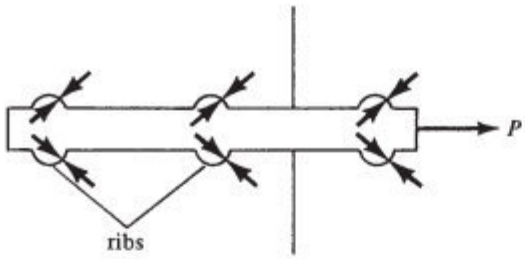


FIGURE 7.5 Bearing forces on bar and bearing of bar ribs on concrete.

When the force is first applied to the bar, the resistance to slipping is provided by the adhesion between the bar and the concrete. If plain bars were used, it would not take much tension in the bars to break this adhesion, particularly adjacent to a crack in the concrete. If this were to happen for a smooth surface bar, only friction would remain to keep the bar from slipping. There is also some Poisson's effect due to the tension in the bars. As they are tensioned, they become a little smaller, enabling them to slip more easily. If we were to use straight, plain, or smooth reinforcing bars in beams, there would be very little bond strength, and the beams would be only a little stronger than if there were no bars. Deformed bars were introduced so that in addition to the adhesion and friction, there would also be a resistance due to the bearing of the concrete on the lugs or ribs (or deformations) of the bars as well as the so-called shear-friction strength of the concrete between the lugs.

Deformed bars are used in almost all work. However, plain bars or plain wire fabrics are sometimes used for transverse reinforcement in compression members (as ties or spirals, as described in Chapter 9), for members subject to torsion, and for confining reinforcing in splices (ACI Code 318, Section 20.2.1.4).

As a result of these facts, reinforcing bars are made with rib-type deformations. The chemical adhesion and friction between the ribs are negligible, and thus bond is primarily supplied by bearing on the ribs. Based on testing, the crack patterns in the concrete show that the bearing stresses are inclined to the axis of the bars from about 45° to 80° (the angle being appreciably affected by the shape of the ribs).[1]

Equal and opposite forces develop between the reinforcing bars and the concrete, as shown in Figure 7.5. These internal forces are caused by the wedging action of the ribs bearing against the concrete. They will cause tensile stresses in a cylindrical piece of concrete around each bar. It's rather like a concrete pipe filled with water that is pressing out against the pipe wall, causing it to be placed in tension. If the tension becomes too high, the pipe will split.

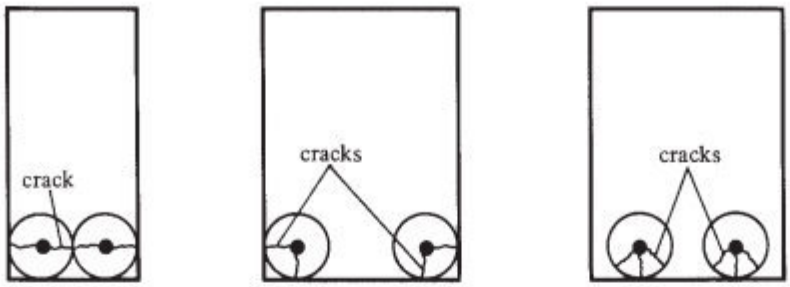
In a similar manner, if the bond stresses in a beam become too high, the concrete will split around the bars, and eventually the splits will extend to the side and/or bottom of the beam. If either of these types of splits runs all the way to the end of the bar, the bar will slip and the beam will fail. The closer the bars are spaced together and the smaller the cover, the thinner will be the concrete cylinder around each bar and the more likely that a bond-splitting failure will occur.

Figure 7.6 shows examples of bond failures that may occur for different values of concrete cover and bar spacing. These are as shown by MacGregor.[2]

Courtesy of Clemson University Communications Center.

Twisted square bar, formerly used to increase bond between concrete and steel.

[1] Goto, Y., 1971, "Cracks Formed on Concrete Around Deformed Tensioned Bar," *ACI Journal, Proceedings 68*, p. 244.
[2] MacGregor, J. G., 2005, *Reinforced Concrete Mechanics and Design*, 4th ed. (Upper Saddle River, NJ: Prentice-Hall), p. 334.



(a) Side cover and one-half clear spacing between bars < bottom cover

(b) Cover on sides and bottom equal and < one-half clear spacing between bars

(c) Bottom cover < side cover and < one-half clear spacing between bars

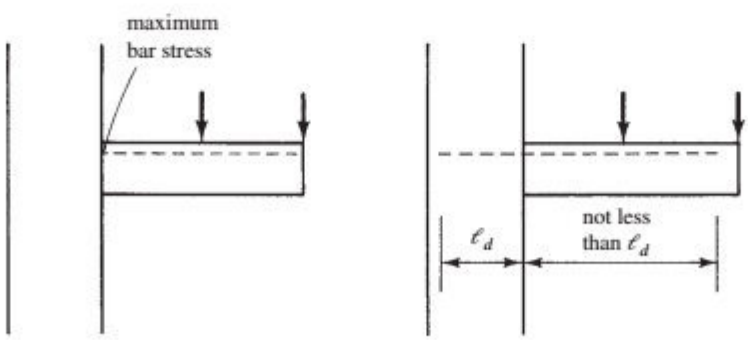
FIGURE 7.6 Types of bond failures.

Splitting resistance along bars depends on quite a few factors, such as the thickness of the concrete cover, the spacing of the bars, the presence of coatings on the bars, the types of aggregates used, the transverse confining effect of stirrups, and so on. Because there are so many variables, it is impossible to make comprehensive bond tests that are good for a wide range of structures. Nevertheless, the ACI has attempted to do just this with its equations, as will be described in the sections to follow.

7.3 Development Lengths for Tension Reinforcement

For this discussion, reference is made to the cantilever beam of Figure 7.7. It can be seen that both the maximum moment in the beam and the maximum stresses in the tensile bars occur at the face of the support. Theoretically, a small distance back into the support the moment is zero, and thus it would seem that reinforcing bars would no longer be required. This is the situation pictured in Figure 7.7(a). Obviously, if the bars were stopped at the face of the support, the beam would fail.

The bar stresses must be transferred to the concrete by bond between the steel and the concrete before the bars can be cut off. In this case the bars must be extended some distance back into the support and out into the beam to anchor them or to develop their strength. This distance called the *development length* (ℓ_d) is shown in Figure 7.7(b). It can be defined as the minimum length of embedment of bars



(a) No development length at support (beam will fail)

(b) Bars extended into the support a distance of ℓ_d

FIGURE 7.7 Development length in a cantilever support.

that is necessary to permit them to be stressed to their yield point plus some extra distance to ensure member toughness. A similar case can be made for bars in other situations and in other types of beams.

As previously mentioned, the ACI for many years required designers to calculate bond stresses with a formula that was based on the change of moment in a beam. Then the computed values were compared to allowable bond stresses in the code. Originally, bond strength was measured by means of pullout tests. A bar would be cast in a concrete cylinder and a jack would be used to see how much force was required to pull it out. The problem with such a test is that the concrete is placed in compression, preventing the occurrence of cracks. In a flexural member, however, we have an entirely different situation due to the off-again/on-again nature of the bond stresses caused by the tension cracks in the concrete. In recent years, more realistic tests have been made with beams; the ACI Code development length expressions to be presented in this chapter are based primarily on such tests at the National Institute of Standards and Technology and the University of Texas.

The development lengths used for deformed bars or wires in tension may not be less than the values computed in accordance with ACI 318 Equation 25.4.2.3a. If the equation is written as (ℓ_d/d_b), the results obtained will be in terms of bar diameters. This form of answer is very convenient to use as, say, 30 bar diameters, 40 bar diameters, and so on.

$$\ell_d = \frac{3}{40}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)}d_b \qquad \text{(ACI Equation 25.4.2.3a)}$$

or

$$\frac{\ell_d}{d_b} = \frac{3}{40}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)}$$

Or in SI units,

$$\ell_d = \frac{1}{1.1}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)}d_b$$

This expression, which seems to include so many terms, is much easier to use than it might at first appear because several of the terms are usually equal to 1.0. Even if not equal to 1.0, the factors can be quickly obtained.

In the following paragraphs, all of the terms in ACI Equation 25.4.2.3a that have not previously been introduced are described. Then their values for different situations are given in Table 7.1.

1. *Location of reinforcement*—Horizontal bars that have at least 12 in.[3] of fresh concrete placed beneath them do not bond as well to concrete as do bars placed nearer the bottom of the concrete. These bars are referred to as *top bars*. During the placing and vibration of the concrete, some air and excess water tend to rise toward the top of the concrete, and some portion may be caught under the higher bars. In addition, there may be some settlement of the concrete below. As a result, the reinforcement does not bond as well to the concrete underneath, and increased development lengths will be needed. To account for this effect, the *reinforcement location factor*, ψ_t, is used.

2. *Coating of bars*—Epoxy-coated or zinc and epoxy dual-coated reinforcing bars are frequently used today to protect the steel from severe corrosive situations, such as where deicing chemicals are used. Bridge decks and parking garage slabs in the colder states fit into this class. When bar

[3] 300 mm in SI.

coatings are used, bonding is reduced and development lengths must be increased. To account for this fact, the term ψ_e—the *coating factor*—is used in the equation.

3. *Sizes of reinforcement*—If small bars are used in a member to obtain a certain total cross-sectional area, the total surface area of the bars will be appreciably larger than if fewer but larger bars are used to obtain the same total bar area. As a result, the required development lengths for smaller bars with their larger surface bonding areas (in proportion to their cross-sectional areas) are less than those required for larger-diameter bars. This factor is accounted for with the *reinforcement size factor*, ψ_s.

4. *Lightweight aggregates*—The dead weight of concrete can be substantially reduced by substituting lightweight aggregate for the regular stone aggregate. The use of such aggregates (expanded clay or shale, slag, etc.) generally results in lower-strength concretes. Such concretes have lower splitting strengths, and so development lengths will have to be larger. In the equation, λ is the *lightweight concrete modification factor* as discussed in Section 1.12.

5. *Spacing of bars or cover dimensions*—Should the concrete cover or the clear spacing between the bars be too small, the concrete may very well split, as was previously shown in Figure 7.6. This situation is accounted for with the $(c_b + K_{tr})/d_b$ term in the development length expression. It is called the *confinement term*. In the equation, c_b represents the smaller of the distance from the center of the tension bar or wire to the nearest concrete surface, or one-half the center-to-center spacing of the reinforcement.

TABLE 7.1 Factors for Use in the Expressions for Determining Required Development Lengths for Deformed Bars and Deformed Wires in Tension (ACI Code, Section 25.4.2.4)

(1) ψ_t = *reinforcement location factor*

Horizontal reinforcement so placed that more than 12 in. of fresh concrete is cast in the member below the development length or splice	1.3
Other reinforcement	1.0

(2) ψ_e = *coating factor*

Epoxy-coated or zinc and epoxy dual-coated reinforcement with cover less than $3d_b$, or clear spacing less than $6d_b$	1.5
Epoxy-coated or zinc and epoxy dual-coated reinforcement for all other conditions	1.2
Uncoated and zinc-coated (galvanized) reinforcement	1.0

However, the product of $\psi_t\psi_e$ need not be taken as greater than 1.7.

(3) ψ_s = *reinforcement size factor*

No. 6 and smaller bars and deformed wires	0.8
No. 7 and larger bars	1.0
In SI units	
No. 19 and smaller bars and deformed wires	0.8
No. 22 and larger bars	1.0

(4) λ (*lambda*) = *lightweight aggregate concrete factor*

When lightweight aggregate concrete is used, λ shall not exceed 0.75

However, when f_{ct} is specified, λ shall be permitted to be taken as $\dfrac{f_{ct}}{6.7\sqrt{f_{cm}}}$, where f_{cm} is the measured average compressive strength of concrete (ACI Code, Section 19.2.4.3).

It's $\dfrac{f_{ct}}{0.56\sqrt{f_{cm}}}$ in SI.

But not greater than	1.0
When normal-weight concrete is used	1.0

(5) c_b = *spacing or cover dimension, in.*

Use the smaller of either the distance from the center of the bar or wire to the nearest concrete surface, or one-half the center-to-center spacing of the bars or wires being developed.

In this expression, K_{tr} is a factor called the *transverse reinforcement index*. It is used to account for the contribution of confining reinforcement (stirrups or ties) across possible splitting planes.

$$K_{tr} = \frac{40A_{tr}}{sn}$$

where

A_{tr} = the total cross-sectional area of all transverse reinforcement having the center-to-center spacing s and a yield strength f_{yt}

n = the number of bars or wires being developed or lap spliced along the plane of splitting. If steel is in two layers, n is the largest number of bars in a single layer.

s = center-to-center spacing of transverse reinforcement

The ACI Code, Section 25.4.2.3 conservatively permits the use of $K_{tr} = 0$ to simplify the calculations, even if transverse reinforcement is present. ACI Code, Section 25.4.2.3 limits the value of $(c_b + K_{tr})/d_b$ used in the equation to a maximum value of 2.5. (It has been found that if values larger than 2.5 are used, the shorter development lengths resulting will increase the danger of pullout-type failures.)

The calculations involved in applying ACI Equation 25.4.2.3a are quite simple, as is illustrated in Example 7.2.

In SI units, $K_{tr} = \dfrac{40A_{tr}}{sn}$

Example 7.2

Determine the development length required for the #8 uncoated bottom bars shown in Figure 7.8,

(a) assuming $K_{tr} = 0$ and

(b) using the computed value of K_{tr}.

SOLUTION

From Table 7.1

$\psi_t = 1.0$ for bottom bars

$\psi_e = 1.0$ for uncoated bars

$\psi_s = 1.0$ for #8 bars

$\lambda = 1.0$ for normal-weight concrete

c_b = side cover of bars measured from center of bars = $2\frac{1}{2}$ in.

or

c_b = one-half of c. to c. spacing of bars = $1\frac{1}{2}$ in. ←

(a) Using ACI Equation 25.4.2.3a with $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{1.50 \text{ in.} + 0 \text{ in.}}{1.00 \text{ in.}} = 1.50 < 2.50 \quad \underline{\text{OK}}$$

$$\frac{\ell_d}{d_b} = \frac{3}{40}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\left(\dfrac{c_b + K_{tr}}{d_b}\right)}$$

$$= \left(\frac{3}{40}\right)\left[\frac{60{,}000 \text{ psi}}{(1.0)\sqrt{3000} \text{ psi}}\right]\frac{(1.0)\,(1.0)\,(1.0)}{1.50} = \underline{55 \text{ diameters}}$$

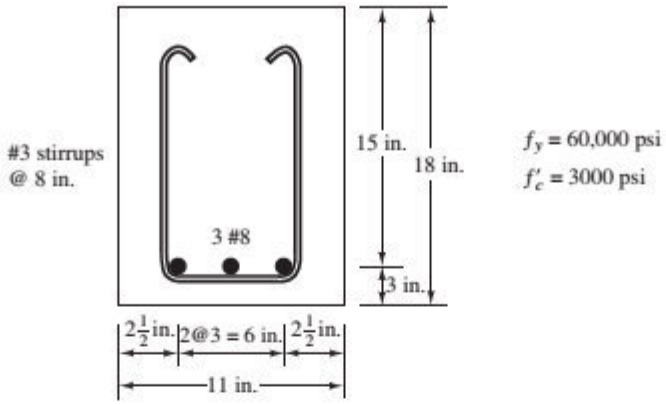


FIGURE 7.8 Beam cross section for Example 7.2.

(b) Using Computed Value of K_{tr} and ACI Equation 25.4.2.3a

$$K_{tr} = \frac{40A_{tr}}{sn} = \frac{(40)\,(2)\,(0.11\text{ in.}^2)}{(8\text{ in.})\,(3)} = 0.367\text{ in.}$$

$$\frac{c_b + K_{tr}}{d_b} = \frac{1.50\text{ in.} + 0.367\text{ in.}}{1.0\text{ in.}} = 1.867 < 2.5 \quad \underline{\underline{\text{OK}}}$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{60{,}000\text{ psi}}{\sqrt{3000}\text{ psi}}\right)\frac{(1.0)\,(1.0)\,(1.0)\,(1.0)}{1.867} = \underline{\underline{44\text{ diameters}}}$$

In determining required development lengths, there are two more ACI requirements to keep in mind:

1. Section 25.4.1.4 states that values of $\sqrt{f'_c}$ used in the equations cannot be greater than 100 psi or $\frac{25}{3}$ MPa in SI. (This limit is imposed because there has not been a sufficient amount of research on the development of bars in higher-strength concretes to justify higher $\sqrt{f'_c}$ values, which would result in smaller ℓ_d/d_b values.)
2. When the amount of flexural reinforcement provided exceeds the theoretical amount required, and where the specifications being used do not specifically require that the development lengths be based on f_y, the value of ℓ_d/d_b may be multiplied by $(A_{s\text{ required}}/A_{s\text{ provided}})$, according to ACI Code, Section 25.4.10. This reduction factor may not be used for the development of reinforcement at the face of non-continuous supports for positive reinforcement, at locations where the full yield strength is required to be developed, for the development of shrinkage and temperature reinforcement, for headed or mechanically anchored deformed reinforcement or for a few other situations referenced in ACI Code, Section 25.4.10.2. This reduction also is not permitted in regions of high seismic risk (seismic design categories D, E, and F).

Instead of using ACI Equation 25.4.2.3a for computing development lengths, the ACI in its Section 25.4.4.2 permits the use of a somewhat simpler and more conservative approach (as shown in Table 7.2 herein) for certain conditions. With this approach, the ACI recognizes that in a very large percentage of cases, designers use spacing and cover values and confining reinforcement that result in a value of $(c_b + K_{tr})/d_b$ equal to at least 1.5. Based on this value and the appropriate values of ψ_s, the expressions in Table 7.2 were determined.

For SI values, see Section 25.4.2.2 of the 318M-14 Code.

TABLE 7.2 Simplified Development Length Equations

	#6 and Smaller Bars and Deformed Wires	#7 and Larger Bars
Clear spacing of bars being developed or spliced not less than d_b, clear cover not less than d_b, and stirrups or ties throughout ℓ_d not less than the code minimum or Clear spacing of bars being developed or spliced not less than $2d_b$ and clear cover not less than d_b	$\frac{\ell_d}{d_b} = \frac{f_y \psi_t \psi_e}{25\lambda\sqrt{f'_c}}$	$\frac{\ell_d}{d_b} = \frac{f_y \psi_t \psi_e}{20\lambda\sqrt{f'_c}}$
Other cases	$\frac{\ell_d}{d_b} = \frac{3f_y \psi_t \psi_e}{50\lambda\sqrt{f'_c}}$	$\frac{\ell_d}{d_b} = \frac{3f_y \psi_t \psi_e}{40\lambda\sqrt{f'_c}}$

If a minimum cover equal to d_b and a minimum clear spacing between bars of $2d_b$ (or a minimum clear spacing of bars equal to d_b, along with a minimum of ties or stirrups) are used, the expressions in Table 7.2 can be used. Otherwise, it is necessary to use the more rigorous ACI Equation 25.4.2.3a.

The authors feel that the application of the so-called simplified equations requires almost as much effort as is needed to use the longer equation. Furthermore, the development lengths computed with the "simpler" equations are often so much larger than the ones determined with the regular equation as to be uneconomical.

For these reasons the authors recommend the use of Equation 25.4.2.3a for computing development lengths. In using this long-form equation, however, you may very well like to assume that $K_{tr} = 0$, as the results obtained usually are only slightly more conservative than those obtained with the full equation. *The authors use Equation 25.4.2.3a with $K_{tr} = 0$ for all applications after this chapter.*

Examples 7.3 and 7.4, which follow, present the determination of development lengths using each of the methods that have been described in this section.

Example 7.3

The #7 bottom bars shown in Figure 7.9 are epoxy coated. Assuming normal-weight concrete, $f_y = 60{,}000$ psi, and $f'_c = 3500$ psi, determine required development lengths

(a) Using the simplified equations of Table 7.2
(b) Using the full ACI Equation 25.4.2.3a with the calculated value of K_{tr} and
(c) Using ACI Equation 25.4.2.3a with $K_{tr} = 0$.

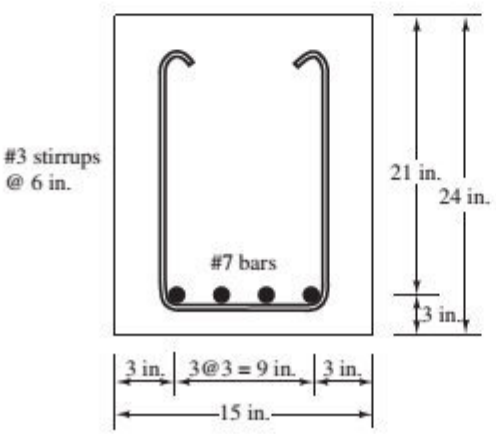


FIGURE 7.9 Beam cross section for Example 7.3.

SOLUTION

With reference to Table 7.1

$\psi_t = 1.0$ for bottom bars

$\psi_e = 1.5$ for epoxy-coated bars with clear spacing $< 6d_b$

$\psi_t\psi_e = (1.0)(1.5) = 1.5 < 1.7$ OK

$\psi_s = 1.0$ for #7 and larger bars

$\lambda = 1.0$ for normal-weight concrete

c_b = cover = 3 in.

or

c_b = one-half of c. to c. spacing of bars = $1\frac{1}{2}$ in. ← controls

(a) Using Simplified Equation

$$\frac{\ell_d}{d_b} = \frac{f_y\psi_t\psi_e}{20\lambda\sqrt{f'_c}} = \frac{(60{,}000 \text{ psi})\,(1.0)\,(1.5)}{20\,(1.0)\sqrt{3500}\text{ psi}} = 76 \text{ diameters}$$

(b) Using ACI Equation 25.4.2.3a with Computed Value of K_{tr}

$$K_{tr} = \frac{40A_{tr}}{sn} = \frac{(40)\,(2)\,(0.11 \text{ in.}^2)}{(6 \text{ in.})\,(4)} = 0.367 \text{ in.}$$

$$\frac{c_b + K_{tr}}{d_b} = \frac{1.5 \text{ in.} + 0.367 \text{ in.}}{0.875 \text{ in.}} = 2.13 < 2.50 \quad \text{OK}$$

$$\frac{\ell_d}{d_b} = \frac{3}{40}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\dfrac{c_b + K_{tr}}{d_b}}$$

$$= \left(\frac{3}{40}\right)\left(\frac{60{,}000 \text{ psi}}{(1.0)\sqrt{3500}\text{ psi}}\right)\frac{(1.0)\,(1.5)\,(1.0)}{2.13}$$

$$= \underline{54 \text{ diameters}}$$

(c) Using ACI Equation 25.4.2.3a with $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{1.5 \text{ in.} + 0 \text{ in.}}{0.875 \text{ in.}} = 1.71 < 2.50 \quad \text{OK}$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{60{,}000 \text{ psi}}{(1.0)\sqrt{3500}\text{ psi}}\right)\frac{(1.0)\,(1.5)\,(1.0)}{1.71}$$

$$= \underline{67 \text{ diameters}}$$

Example 7.4

The required reinforcing steel area for the lightweight concrete beam of Figure 7.10 is 2.88 in.^{2} The #8 top bars shown are uncoated. Compute development lengths if $f_y = 60{,}000$ psi and $f'_c = 3500$ psi

(a) Using simplified equations

(b) Using the full ACI Equation 25.4.2.3a and

(c) Using ACI Equation 25.4.2.3a with $K_{tr} = 0$.

SOLUTION

With reference to Table 7.1

$\psi_t = 1.3$ for top bars

$\psi_e = 1.0$ for uncoated bars

$\psi_t\psi_e = (1.3)(1.0) < 1.7$ OK

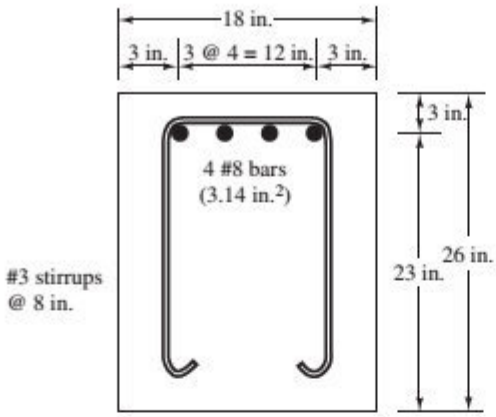


FIGURE 7.10 Cross section of cantilever beam for Example 7.4.

$\psi_s = 1.0$ for #7 and larger bars
$\lambda = 0.75$ for lightweight concrete
c_b = cover = 3 in.

or

c_b = one-half of c. to c. spacing of bars = 2 in. ← controls

(a) Using Simplified Equations

$$\frac{\ell_d}{d_b} = \frac{f_y \psi_t \psi_e}{20\lambda\sqrt{f'_c}} = \frac{(60{,}000 \text{ psi})\,(1.3)\,(1.0)}{20\,(0.75)\sqrt{3500 \text{ psi}}} = 88 \text{ diameters}$$

$\frac{\ell_d}{d_b}$ reduced for excess reinforcement to $\left(\frac{2.88 \text{ in.}^2}{3.14 \text{ in.}^2}\right)(88) = \underline{\underline{81 \text{ diameters}}}$

(b) Using ACI Equation 25.4.2.3a with Computed Value of K_{tr}

$$K_{tr} = \frac{40A_{tr}}{sn} = \frac{(40)\,(2)\,(0.11 \text{ in.}^2)}{(8 \text{ in.})\,(4)} = 0.275$$

$$\frac{c_b + K_{tr}}{d_b} = \frac{2.0 \text{ in.} + 0.275 \text{ in.}}{1.0 \text{ in.}} = 2.275 < 2.5 \quad \underline{\underline{\text{OK}}}$$

$$\frac{\ell_d}{d_b} = \frac{3}{40}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\dfrac{c_b + K_{tr}}{d_b}}$$

$$= \left(\frac{3}{40}\right)\left[\frac{60{,}000 \text{ psi}}{(0.75)\sqrt{3500 \text{ psi}}}\right]\frac{(1.3)\,(1.0)\,(1.0)}{2.275}$$

$$= 58 \text{ diameters}$$

$\frac{\ell_d}{d_b}$ reduced for excess reinforcement to $\left(\frac{2.88 \text{ in.}^2}{3.14 \text{ in.}^2}\right)(58) = \underline{\underline{53 \text{ diameters}}}$

(c) Using ACI Equation 25.4.2.3a with $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{2.0 \text{ in.} + 0 \text{ in.}}{1.0 \text{ in.}} = 2.0 < 2.5$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left[\frac{60{,}000 \text{ psi}}{(0.75)\sqrt{3500 \text{ psi}}}\right]\frac{(1.3)\,(1.0)\,(1.0)}{2.0}$$

$$= 66 \text{ diameters}$$

$\frac{\ell_d}{d_b}$ reduced for excess reinforcement to $\left(\frac{2.88 \text{ in.}^2}{3.14 \text{ in.}^2}\right)(66) = \underline{\underline{61 \text{ diameters}}}$

7.4 Development Lengths for Bundled Bars

When bundled bars are used, greater development lengths are needed because there is not a "core" of concrete between the bars to provide resistance to slipping. The ACI Code, Section 25.6.1.5 states that development lengths for bundled bars are to be determined by first computing the lengths needed for the individual bars and then by increasing those values by 20% for three-bar bundles and 33% for four-bar bundles.

When the factors relating to cover and clear spacing are being computed for a particular bundle, the bars are treated as though their area were furnished by a single bar (ACI 25.6.1.6). In other words, it is necessary to replace the bundle of bars with a fictitious single bar with a diameter such that its cross-sectional area equals that of the bundle of bars. This is conservative because the bond properties of the bundled bars are actually better than for the fictitious single bar.

The diameter of the equivalent bar shall be used for d_b when determining

(a) Spacing limitations based on d_b

(b) Cover requirements based on d_b

(c) Spacing and cover values in text Table 7.2

(d) Confinement term in ACI Equation 25.4.2.3

(e) ψ_e factor in text Table 7.1

Example 7.5 presents the calculation of the development length needed for a three-bar bundle of #8 bars.

Example 7.5

Compute the development length required for the uncoated bundled bars shown in Figure 7.11, if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi with normal-weight concrete. Use ACI Equation 25.4.2.3a and assume $K_{tr} = 0$.

SOLUTION

With reference to Table 7.1

$\psi_t = \psi_e = \psi_s = \lambda = 1.0$

Area of 3 #8 bars = 2.35 in.^{2}

Diameter d_{bf} of a single bar of area 2.35 in.^{2}

$$\frac{\pi d_{bf}^2}{4} = 2.35$$

$$d_{bf} = 1.73 \text{ in.}$$

Find the lowest value for c_b [Figure 7.11(b)].

$$c_{b1} = \text{side cover of bars} = 2 \text{ in.} + \frac{3}{8} \text{ in.} + 1.00 \text{ in.} = 3.38 \text{ in.}$$

$$c_{b2} = \text{bottom cover of bars} = 2 \text{ in.} + \frac{3}{8} \text{ in.} + 0.79 d_b^{[4]} = 2 \text{ in.} + \frac{3}{8} \text{ in.} + 0.79(1.00 \text{ in.}) = 3.16 \text{ in.} \leftarrow$$

[4] See Figure 7.11(b) for this dimension.

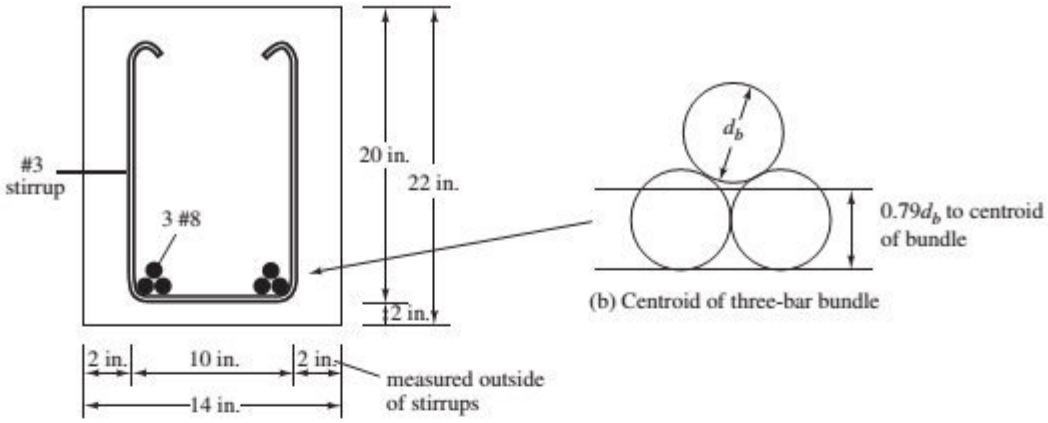


FIGURE 7.11 Beam cross section for Example 7.5.

where d_b is the actual (not the fictitious) bar diameter.

$$c_{b3} = \frac{1}{2}\text{ c. to c. spacing of bars} = \frac{10\text{ in.} - (2)\left(\frac{3}{8}\text{ in.}\right) - (2)\,(1.00\text{ in.})}{2} = 3.62\text{ in.}$$

Using ACI Equation 25.4.2.3a with $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_{bf}} = \frac{3.16\text{ in.} + 0\text{ in.}}{1.73\text{ in.}} = 1.83 < 2.5$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{60{,}000\text{ psi}}{(1.0)\sqrt{4000}\text{ psi}}\right)\frac{(1.0)\,(1.0)\,(1.0)}{1.83}$$

$$= 39\text{ diameters}$$

But should be increased 20% for a three-bar bundle according to ACI Code, Section 25.6.1.5.

$$\frac{\ell_d}{d_b} = (1.20)\,(39) = 47\text{ diameters}$$

$$\ell_d = (47)\,(1.0\text{ in.}) = 47\text{ in.}$$

Note that the actual bar diameter is used in the last equation, not the fictitious bar (ACI R25.6.1.6).

7.5 Hooks

When sufficient space is not available to anchor *tension* bars by running them straight for their required development lengths, as described in Section 7.3, hooks may be used. (*Hooks are considered ineffective for compression bars for development length purposes.*)

Figure 7.12 shows details of the standard 90° and 180° hooks specified in ACI Code 318, Section 25.3.1 for deformed bars in tension. Either the 90° hook with an extension of 12 bar diameters ($12d_b$) at the free end or the 180° hook with an extension of 4 bar diameters ($4d_b$) but not less than $2\frac{1}{2}$ in. may be used at the free end. The radii and diameters shown are measured on the inside of the bends.

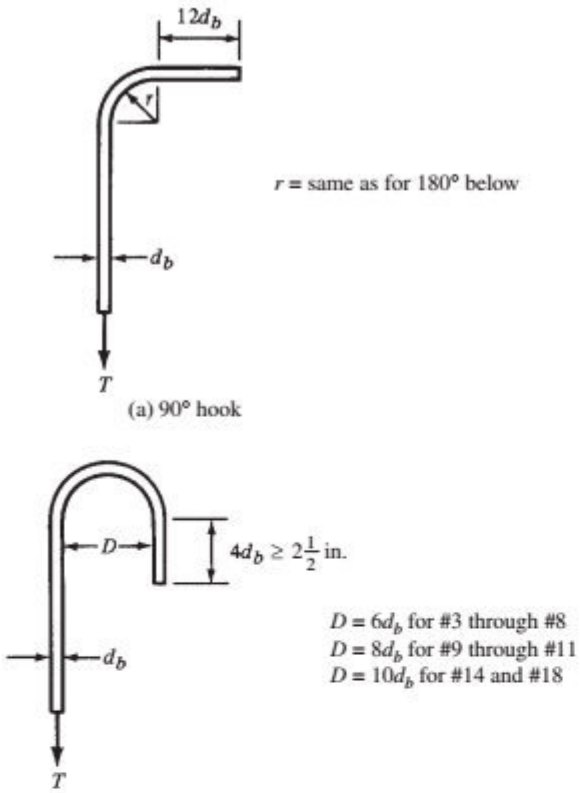


FIGURE 7.12 Hook configurations.

The dimensions given for hooks were developed to protect members against splitting of the concrete or bar breakage, no matter what concrete strengths, bar sizes, or bar stresses are used. Actually, hooks do not provide an appreciable increase in anchorage strength because the concrete in the plane of the hook is somewhat vulnerable to splitting. This means that adding more length (i.e., more than the specified $12d_b$ or $4d_b$ values) onto bars beyond the hooks doesn't really increase their anchorage strengths.

The development length needed for a hook is directly proportional to the bar diameter. This is because the magnitude of compressive stresses in the concrete on the inside of the hook is governed by d_b. To determine the development lengths needed for standard hooks, the ACI Code, Section 25.4.3.1 requires the calculation of the following:

$$\ell_{dh} = \left(\frac{f_y \psi_e \psi_c \psi_r}{50\lambda\sqrt{f'_c}} \right) d_b$$

$$\text{In SI units, } \ell_{dh} = \left(\frac{f_y \psi_e \psi_c \psi_r}{4\lambda\sqrt{f'_c}} \right) d_b$$

Modification factors, λ, ψ_e, ψ_c, and ψ_r, which appear in the equation for ℓ_{dh}, are shown in Table 7.3.

The development length, ℓ_{dh}, is measured from the critical section of the bar to the outside end or edge of the hooks, as shown in Figures 7.13 and 7.14.

TABLE 7.3 Modification Factors for Development of Hooked Bars in Tension (ACI Code 318-14, Table 25.4.3.2)

Modification Factor	Condition	Value of Factor
Lightweight λ	Lightweight concrete	0.75
	Normal-weight concrete	1.0
Epoxy ψ_e	Epoxy-coated or zinc and epoxy dual-coated reinforcement	1.2
	Uncoated or zinc-coated (galvanized) reinforcement	1.0
Cover ψ_c	For #11 bar and smaller hooks with side cover (normal to plane of hook) $\geq 2\frac{1}{2}$ in. and for 90° hook with cover on bar extension beyond hook ≥ 2 in.	0.7
	Other	1.0
Confining reinforcement ψ_r †	For 90° hooks of #11 and smaller bars (1) enclosed along ℓ_{dh} within ties or stirrups* perpendicular to ℓ_{dh} at $s \leq 3d_b$ or (2) enclosed along the bar extension beyond hook including the bend within ties or stirrups* perpendicular to ℓ_{ext} at $s \leq 3d_b$	0.8
	For 180° hooks of #11 and smaller bars enclosed along ℓ_{dh} within ties or stirrups* perpendicular to ℓ_{dh} at $s \leq 3d_b$	
	Other	1.0

*The first tie or stirrup shall enclose the bent portion of the hook within $2d_b$ of the outside of the bend.
†d_b is the nominal diameter of the hooked bar.
Source: Reproduced with permission from the American Concrete Institute.

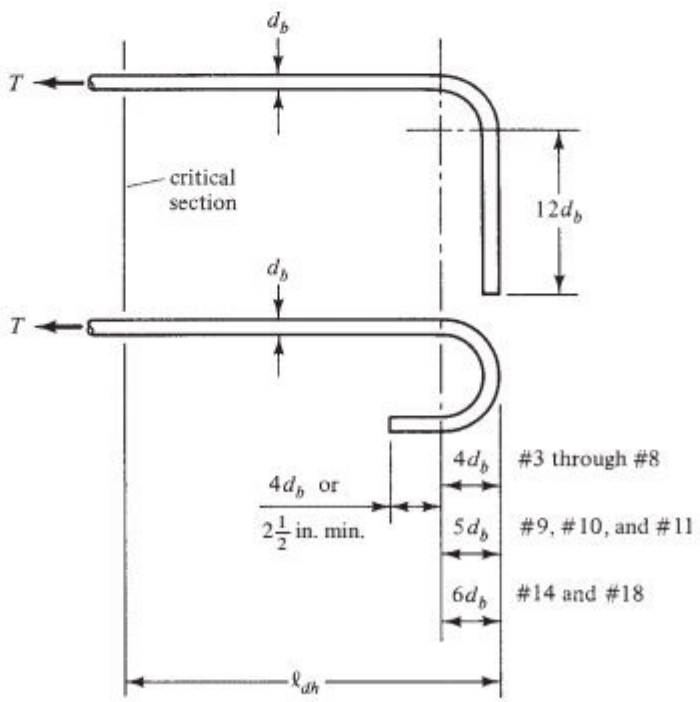


FIGURE 7.13 Hooked-bar details for development of standard hooks.

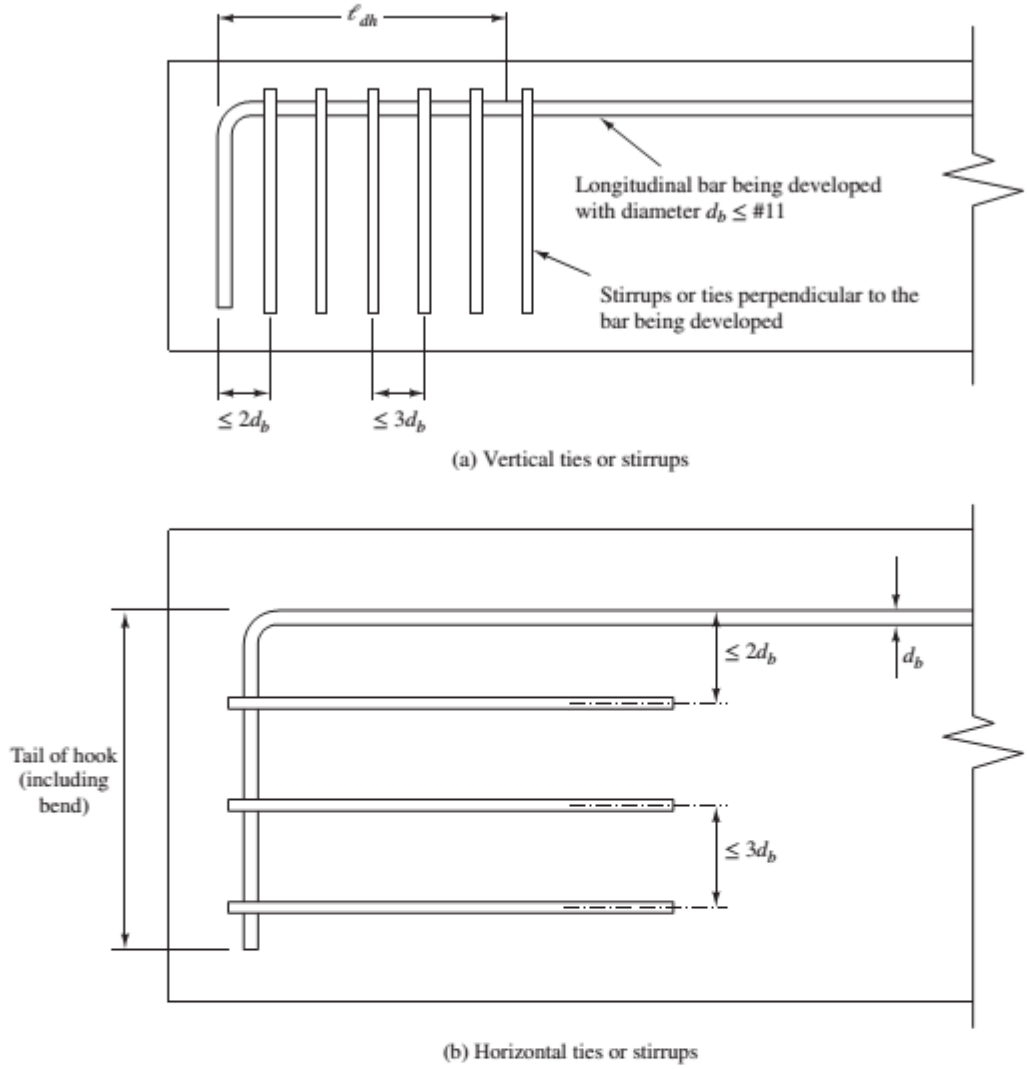


FIGURE 7.14 Ties or stirrups parallel to the bar being developed along the length of the tail extension of a 90° hook plus bend complying with the $\psi_r = 0.8$ multiplier. The stirrups or ties shown can be either vertical or horizontal [ACI Code 318, Figure R25.4.3.2 (a) and (b)]. *Source*: Reproduced with permission from the American Concrete Institute.

The modification factors that appear in Table 7.3 are discussed in subparagraphs (a) through (f). For the design of hooks, no distinction is made between top bars and other bars. (It is difficult to distinguish top from bottom anyway when hooks are involved.)

(a) Lightweight, λ—Because the tensile capacity of standard hooks depends on the tensile strength of the concrete in which the hooks are embedded, the effect of lightweight aggregate, and its associated reduction in concrete tensile strength, must be considered. The λ factor is used in the denominator of the equation for ℓ_{dh} because a reduced tensile strength of concrete increases the hook's development length.

(b) Epoxy, ψ_e—Epoxy-coated or zinc and epoxy dual-coated bars do not bond as well to concrete as uncoated bars. They are used in situations where corrosion is a major exposure consideration. However, the resulting development length of such coated bars is increased by the use of the ψ_e modification factor.

(c) Cover, ψ_c—When hooks are made with #11 or smaller bars and have side cover values normal to the plane of the hooks (C_s) no less than $2\frac{1}{2}$ in. and where the cover on the bar extensions beyond 90° hooks (C_e) is not less than 2 in., multiply by $\psi_c = 0.7$. Figure 7.15 illustrates the locations of C_s and C_e.

(d) *Ties or stirrups*, ψ_r—When hooks made of #11 or smaller bars are enclosed either vertically or horizontally within ties or stirrup ties along their full development length ℓ_{dh}, and the stirrups or ties are spaced no farther apart than $3d_b$ (where d_b is the diameter of the hooked bar), multiply by $\psi_r = 0.8$. This situation is shown in Figure 7.14. (Detailed dimensions are given for stirrup and tie hooks in Table 25.3.1 of the ACI Code.)

(e) When 180° hooks consisting of #11 or smaller bars are used and are enclosed within ties or stirrups placed perpendicular to the bars being developed, and spaced no further than 3d apart along the development length ℓ_{dh} of the hook, multiply by $\psi_r = 0.8$. If the 90° hook shown in Figure 7.14 is replaced with a 180° hook and ties or stirrups are perpendicular (not parallel) to the longitudinal bar being developed, Figure 7.14 applies to this case as well.

(f) Reduction of development lengths for deformed bars in tension, ℓ_{dh}, is permitted by use of the ratio $(A_{s\ \text{required}})/(A_{s\ \text{provided}})$, except where prohibited by ACI Code, Section 25.4.10.2 (for areas of high seismic risk). The reduced development lengths shall not be less than the minimums of $8d_b$ and 6 in., however.

The danger of a concrete splitting failure is quite high if both the side cover (perpendicular to the hook) and the top and bottom cover (in the plane of the hook) are small. The ACI Code, Section 25.4.3.3, therefore, states that when standard hooks with less than $2\frac{1}{2}$ in. side, C_s, and top, C_t, or bottom cover are used at discontinuous ends of members, the hooks shall be enclosed within ties or stirrups spaced no farther than $3d_b$ for the full development length, ℓ_{dh} (Figure 7.15). The first tie or stirrup must enclose the bent part of the hook within a distance of $2d_{bh}$ of the outside of the bend. Furthermore, the modification factor, ψ_r, shall be taken as 1.0. If the longitudinal bar being developed with the hook, as shown in Figure 7.15, were at a discontinuous end of a member, such as the free end of a cantilever beam, the ties or stirrups shown in that figure would be *required*, unless side and top cover both were at least $2\frac{1}{2}$ in. Figure 7.15 shows a cover dimension C_t, but not the corresponding dimension C_b (bottom cover). If the longitudinal bar being developed with a hook were on the bottom of the beam shown, then C_b would be measured from the bottom of the bar to the bottom of the beam.

The procedures described to determine development lengths for hooked bars apply only for cases where standard hooks are used. The effect of hooks with larger radii is not covered by the ACI Code 318.

Example 7.6, which follows, illustrates the calculations necessary to determine the development lengths required at the support for the tensile bars of a cantilever beam. The lengths for straight or hooked bars are determined.

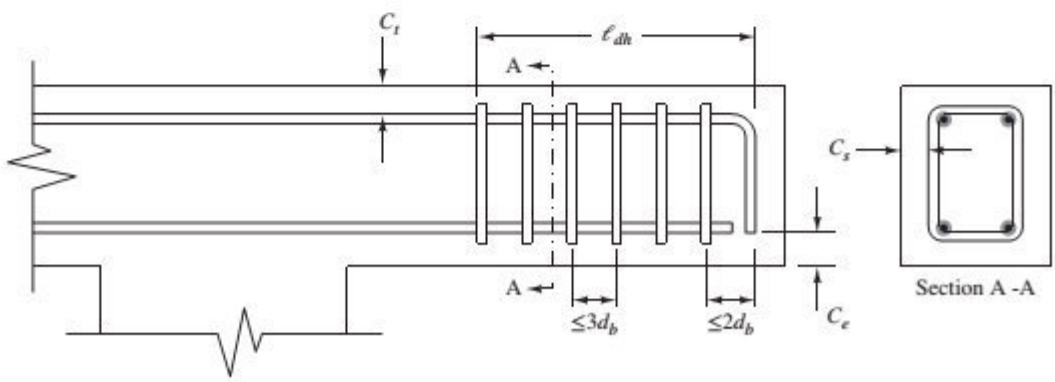


FIGURE 7.15 Cover requirements for use of ψ_c and for ACI Code 318, Section 25.4.3.3.
Source: Reproduced with permission from the American Concrete Institute.

Example 7.6

Determine the development or embedment length required for the epoxy-coated bars of the cantilever beam connected to a column shown in Figure 7.16.

(a) If the bars are straight, assuming $K_{tr} = 0$.
(b) If a 180° hook is used.
(c) If a 90° hook is used.

The six #9 bars shown are considered to be top bars. $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

(a) Straight Bars

$\psi_t = 1.3$ for top bars
$\psi_e = 1.5$ for coated bars with cover $< 3d_b$ or clear spacing $< 6d_b$
$\psi_t\psi_e = (1.3)\,(1.5) = 1.95 > 1.7 \quad \therefore$ Use 1.7
$\psi_s = 1.0$ for 9 bars
$\lambda = 1.0$ for normal-weight concrete
$c_b =$ side cover = top cover = 2.5 in.
$c_b =$ one-half of c. to c. spacing of bars = 2.25 in. ← controls

$$\frac{c_b + K_{tr}}{d_b} = \frac{2.25 \text{ in.} + 0 \text{ in.}}{1.128 \text{ in.}} = 1.99 < 2.5 \quad \text{OK}$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left(\frac{60{,}000 \text{ psi}}{(1.0)\sqrt{4000} \text{ psi}}\right)\frac{(1.7)\,(1.0)}{1.99} = 61 \text{ diameters}$$

$$\ell_d = (61)\,(1.128 \text{ in.}) = 69 \text{ in.}$$

(b) Using 180° hooks [see Figure 7.17(a) and note that $\psi_e = 1.2$ as required in Table 7.3 for epoxy-coated bars]

$$\ell_{dh} = \left(\frac{f_y\psi_e\psi_c\psi_r}{50\lambda\sqrt{f'_c}}\right)d_b = \frac{(60{,}000 \text{ psi})\,(1.2)\,(1.0)\,(1.0)\,(1.128 \text{ in.})}{(50)\,(1.0)\sqrt{4000} \text{ psi}} = 25.68 \text{ in.} \quad \text{Say 26 in.}$$

ψ_c was taken as 1.0 because the side cover is less than 2.5 in. ψ_r was taken as 1.0 because no confining ties or stirrups are specified.

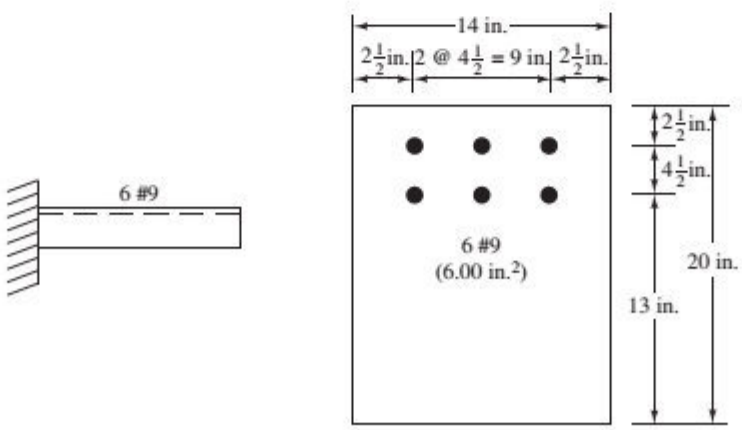


FIGURE 7.16 Given information for Example 7.6.

Note: The dimensions shown in the beam cross section (Figure 7.16) indicate $2\frac{1}{2}$ in. from the bar center to the top and side of the beam. The cover is 2.5 in. $- d_b/2 = 1.936$ in. < 2.5 in. If this hook were in the free end of a cantilever beam, ties or stirrups would be required, and the 0.8 reduction factor would not be applicable. In this example, the hook is in a column, so special ties are not required. If they were provided, a reduction of 0.8 would apply. In this example, they are not provided. The development length of 26 in. for both the 90° and 180° hooks dictate a minimum column width of about 28 in. If the column is not that wide, the designer might use a larger number of smaller bars.

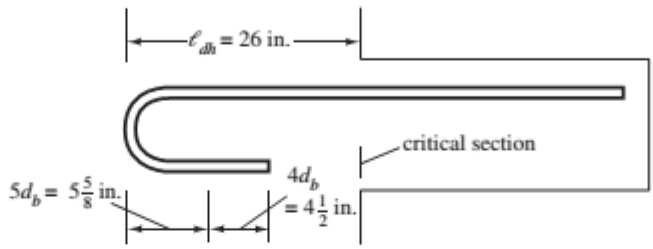


FIGURE 7.17(a) Details for 180° hook.

(c) Using 90° Hooks [see Figure 7.17(b)]

$$\ell_{dh} = 26 \text{ in.}$$

Again the 0.8 reduction factor does not apply because ties or stirrups are not provided.

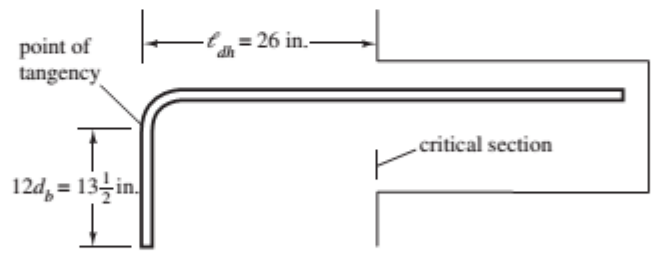


FIGURE 7.17(b) Details for 90° hook.

7.6 Development Lengths for Welded Wire Fabric in Tension

Section 25.4.6 of the ACI Code provides minimum required development lengths for deformed welded wire fabric, whereas Section 25.4.7 provides minimum values for plain welded wire fabric.

The minimum required development length for deformed welded wire fabric in tension measured from the critical section equals the value determined for ℓ_d, as per ACI Code, Section 25.4.2.2 or 25.4.2.3, multiplied by a wire fabric factor, ψ_w, from ACI Code, Section 25.4.6.3 or 25.4.6.4.

This factor, which follows, contains the term s, which is the spacing of the wire to be developed. The resulting development length may not be less than 8 in. except in the computation of lap splices. You might note that epoxy coatings seem to have little effect on the lengths needed for welded wire fabric, and it is thus permissible to use $\psi_e = 1.0$.

The wire fabric factor, ψ_w, for welded wire fabric with at least one crosswire within the development length not less than 2 in. from the critical section is

$$\psi_w = \frac{f_y - 35{,}000}{f_y} \text{ not less than } \frac{5d_b}{s}$$

but need not be taken > 1.0.

In SI units for welded wire fabric with at least one crosswire within the development length and not less than 50 mm from the point of the critical section, the wire fabric factor, ψ_w, is $(f_y - 240)/f_y$, not less than $5d_b/s$ but need not be taken > 1.0.

The yield strength of welded plain wire fabric is considered to be adequately developed by two crosswires if the closer one is not less than 2 in. from the critical section. The ACI Code, Section 25.4.7.2, however, says that the development length, ℓ_d, measured from the critical section to the outermost crosswire may not be less than the value computed from the following equation, in which A_w is the area of the individual wire to be developed.

$$\ell_d = 0.27\frac{A_w}{s}\left(\frac{f_y}{\lambda\sqrt{f'_c}}\right) \text{ but not} < 6 \text{ in.}$$

Or in SI units

$$\ell_d = 3.3\frac{A_w}{s}\left(\frac{f_y}{\lambda\sqrt{f'_c}}\right) \text{ but not} < 150 \text{ mm}$$

The development lengths obtained for either plain or deformed wire may be reduced, as were earlier development lengths, by multiplying them by $(A_{s\ \text{required}}/A_{s\ \text{furnished}})$ (ACI Code, Section 25.4.10.2), but the modified results may not be less than the minimum values given in this section.

7.7 Development Lengths for Compression Bars

There is not a great deal of experimental information available about bond stresses and needed embedment lengths for compression steel. It is obvious, however, that embedment lengths will be smaller than those required for tension bars. For one reason, there are no tensile cracks present to encourage slipping. For another, there is some bearing of the ends of the bars on concrete, which also helps develop the load.

The ACI Code, Section 25.4.9.2 states that the minimum basic development length provided for compression bars (ℓ_{dc}) may not be less than the value computed from the following expression.

$$\ell_{dc} = \left(\frac{f_y\psi_r}{50\lambda\sqrt{f'_c}}\right)d_b, \text{ but not less than } 0.0003\,f_y\psi_r d_b$$

Or in SI units

$$\ell_{dc} = \left(\frac{f_y \psi_r}{4\lambda\sqrt{f'_c}}\right) d_b, \text{ but not less than } 0.043 f_y \psi_r d_b$$

If more compression steel is used than is required by analysis, ℓ_{dc} may be multiplied by $(A_{s\ \text{required}}/A_{s\ \text{provided}})$ as per ACI Code, Section 25.4.10.1. When bars are enclosed in spirals for any kind of concrete members, the members become decidedly stronger due to the confinement or lateral restraint of the concrete. The normal use of spirals is in spiral columns, which are discussed in Chapter 9. Should compression bars be enclosed by spirals or by circular, continuously wound ties of not less than $\frac{1}{4}$ in. diameter and with a pitch not greater than 4 in., or within #4 ties or D20 wire ties spaced at not more than 4 in. on center or hoops spaced less than or equal to 4 in., the value of ℓ_{dc} may be multiplied by $\psi_r = 0.75$ (ACI Code, Section 25.4.9.3). The value of λ is as defined earlier in Section 1.12 of the text, except there is no provision for semi-lightweight concrete. The value is simply 0.75 for lightweight and 1.0 for normal weight. In no case can the development length be less than 8 in.

An introductory development length problem for compression bars is presented in Example 7.7. The forces in the bars at the bottom of the column of Figure 7.18 are to be transferred down into a reinforced concrete footing by means of dowels. Dowels such as these are usually bent at their bottoms (as shown in the figure) and set on the main footing reinforcement where they can be tied securely in place. The bent or hooked parts of the dowels, however, do not count as part of the required development lengths for compression bars (ACI Code, Section 25.4.1.2), because they are ineffective.

In a similar fashion, the dowel forces must be developed up into the column. In Example 7.7, the required development lengths up into the column and down into the footing are different because the f'_c values for the footing and the column are different in this case. The topic of dowels and force transfer from walls and columns to footings is discussed in some detail in Chapter 12. (The development lengths determined in this example are for compression bars, as would normally be the case at the base of columns. If uplift is possible, however, it will be necessary to consider tension development lengths, which could very well control.)

Example 7.7

The forces in the column bars of Figure 7.18 are to be transferred into the footing with #9 dowels. Determine the development lengths needed for the dowels (a) down into the footing and (b) up into the column, if $f_y = 60{,}000$ psi. The concrete in both the column and the footing is normal weight.

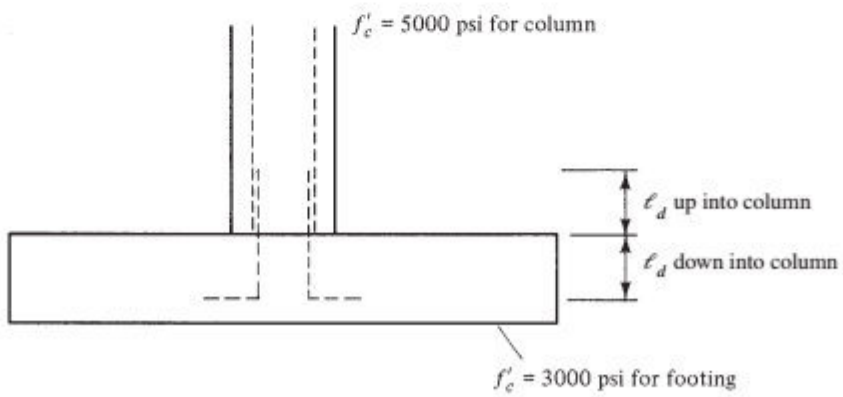


FIGURE 7.18 Information for Example 7.7.

SOLUTION

(a) Down into the Footing

$$\ell_{dc} = \left(\frac{f_y \psi_r}{50\lambda\sqrt{f'_c}}\right) d_b = \frac{(60,000 \text{ psi})\,(1.0)}{(50)\,(1.0)\sqrt{3000 \text{ psi}}}(1.128 \text{ in.}) = 24.71 \text{ in.} \leftarrow$$

$$\ell_{dc} = 0.0003 f_y \psi_r d_b = (0.0003)\,(60,000 \text{ psi})\,(1.0)\,(1.128 \text{ in.}) = 20.30 \text{ in.}$$

Hence $\ell_d = 24.71$ in., say 25 in., as there are no applicable modification factors. Under no circumstances may ℓ_d be less than 8 in.

(b) Up into Column

$$\ell_{dc} = \left(\frac{f_y \psi_r}{50\lambda\sqrt{f'_c}}\right) d_b = \frac{(60,000 \text{ psi})\,(1.0)}{(50)\,(1.0)\sqrt{5000 \text{ psi}}}(1.128 \text{ in.}) = 19.14 \text{ in.}$$

$$\ell_{dc} = 0.0003 f_y \psi_r d_b = (0.0003)\,(60,000 \text{ psi})\,(1.0)\,(1.128 \text{in.}) = 20.30 \text{ in.} \leftarrow$$

Hence $\ell_d = 20.30$ in., say 21 in. (can't be < 8 in.), as there are no applicable modification factors. (*Ans:* Extend the dowels 25 in. down into the footing and 21 in. up into the column.)

Note: The bar details shown in Figure 7.18 are unsatisfactory for seismic areas, because the bars should be bent inward and not outward. The reason for this requirement is that the ACI Code, Chapter 21, on seismic design, stipulates that hooks must be embedded in confined concrete.

7.8 Critical Sections for Development Length

Before the development length expressions can be applied in detail, it is necessary to understand clearly the critical points for tensile and compressive stresses in the bars along the beam.

First, it is obvious that the bars will be stressed to their maximum values at those points where maximum moments occur. Thus, those points must be no closer in either direction to the bar ends than the ℓ_d values computed.

There are, however, other critical points for development lengths. As an illustration, a critical situation occurs whenever there is a tension bar whose neighboring bars have just been cut off or bent over to the other face of the beam. Theoretically, if the moment is reduced by a third, one-third of the bars are cut off or bent, and the remaining bars would be stressed to their yield points. The full development lengths would be required for those bars.

This could bring up another matter in deciding the development length required for the remaining bars. The ACI Code, Section 9.7.3.3 requires that bars that are cut off or bent be extended a distance beyond their theoretical flexure cutoff points by d or 12 bar diameters, whichever is greater. In addition, the point where the other bars are bent or cut off must also be at least a distance ℓ_d from their points of maximum stress (ACI Code, Section 9.7.3.4). Thus, these two items might very well cause the remaining bars to have a stress less than f_y, thus permitting their development lengths to be reduced somewhat. A conservative approach is normally used, however, in which the remaining bars are assumed to be stressed to f_y.

7.9 Effect of Combined Shear and Moment on Development Lengths

The ACI Code does not specifically consider the fact that shear affects the flexural tensile stress in the reinforcement. The ACI Code, Section 9.7.3.3 does require bars to be extended a distance beyond their theoretical cutoff points by a distance no less than the effective depth of the member d or 12

bar diameters, whichever is larger. The commentary (R9.7.3.3) states that this extension is required to account for the fact that the locations of maximum moments may shift due to changes in loading, support settlement, and other factors. It can be shown that a diagonal tension crack in a beam without stirrups can shift the location of the computed tensile stress a distance approximately equal to d toward the point of zero moment. When stirrups are present, the effect is still there but is somewhat less severe.

The combined effect of shear and bending acting simultaneously on a beam may produce premature failure due to overstress in the flexural reinforcement. Professor Charles Erdei[5,6,7] has done a great deal of work on this topic. His work demonstrates that web reinforcement participates in resisting bending moment. He shows that the presence of inclined cracks increases the force in the tensile reinforcement at all points in the shear span except in the region of maximum moment. The result is just as though we have a shifted moment diagram, which leads us to the thought that we should measure ℓ_d from the shifted moment diagram rather than from the basic one. He clearly explains the moment shift and the relationship between the development length and the shift in the moment diagram.

The late Professor P. M. Ferguson[8] stated that regardless of whether we decide to use the shifted moment concept, it is nevertheless desirable to stagger the cutoff points of bars (and it is better to bend them than to cut them).

7.10 Effect of Shape of Moment Diagram on Development Lengths

A further consideration of development lengths will show the necessity of considering the shape of the moment diagram. To illustrate this point, the uniformly loaded beam of Figure 7.19 with its parabolic moment diagram is considered. *It is further assumed that the length of the reinforcing bars on each side*

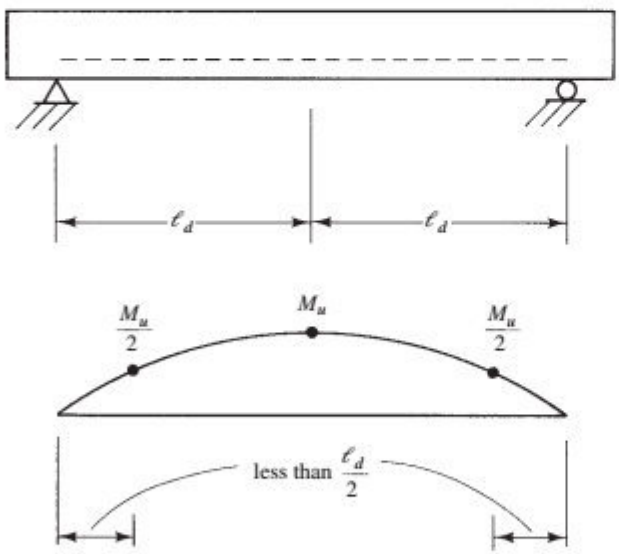


FIGURE 7.19 Effects of shape of moment diagram.

[5] Erdei, C. K., 1961, "Shearing Resistance of Beams by the Load-Factor Method," *Concrete and Constructional Engineering*, 56(9), pp. 318–319.
[6] Erdei, C. K., 1962, "Design of Reinforced Concrete for Combined Bending and Shear by Ultimate Lead Theory," *Journal of the Reinforced Concrete Association*, 1(1).
[7] Erdei, C. K., 1963, "Ultimate Resistance of Reinforced Concrete Beams Subjected to Shear and Bending," *European Concrete Committees Symposium on Shear*, Wiesbaden, West Germany, pp. 102–114.
[8] Ferguson, P. M., 1979, *Reinforced Concrete Fundamentals*, 4th ed. (New York: John Wiley & Sons), p. 187.

of the beam centerline equals the computed development length ℓ_d. The discussion to follow will prove that this distance is not sufficient to properly develop the bars for this moment diagram.[9]

At the centerline of the beam of Figure 7.19, the moment is assumed to equal M_u, and the bars are assumed to be stressed to f_y. Thus the development length of the bars on either side of the beam centerline must be no less than ℓ_d. If one then moves along this parabolic moment diagram on either side to a point where the moment has fallen off to a value of $M_u/2$, it is correct to assume a required development length from this point to be equal to $\ell_d/2$.

The preceding discussion clearly shows that the bars will have to be extended farther out from the centerline than ℓ_d. For the moment to fall off 50%, one must move more than halfway toward the end of the beam.

7.11 Cutting Off or Bending Bars (Continued)

This section presents a few concluding remarks concerning the cutting off of bars, a topic that was introduced in Section 7.1. The last several sections have offered considerable information that affects the points where bars may be cut off. Here we give a summary of the previously mentioned requirements, together with some additional information. First, a few comments concerning shear are in order.

When some of the tensile bars are cut off at a point in a beam, a sudden increase in the tensile stress will occur in the remaining bars. For this increase to occur, there must be a rather large increase in strain in the beam. Such a strain increase quite possibly may cause large tensile cracks to develop in the concrete. If large cracks occur, there will be a reduced beam cross section left to provide shear resistance—and thus a greater possibility of shear failure.

To minimize the possibility of a shear failure, Section 9.7.3.5 of the ACI Code states that *at least one* of the following conditions must be met if bars are cut off in a tension zone:

1. The shear at the cutoff point must not exceed two-thirds of the design shear strength, ϕV_n, in the beam, including the strength of any shear reinforcement provided [ACI Code, Section 9.7.3.5(a)].
2. An area of shear reinforcement in excess of that required for shear and torsion must be provided for a distance equal to $\frac{3}{4}d$ from the cutoff point. The minimum area of this reinforcement and its maximum spacing are provided in Section 9.7.3.5(b) of the ACI Code.
3. A stirrup or hoop area exceeding that required for shear and torsion is provided along each terminated bar or wire over a distance $\frac{3}{4}d$ from the termination point. Excess stirrup or hoop area shall be at least $\frac{60b_w s}{f_{yt}}$. Spacing s shall not exceed $d/(8\beta_b)$.

The moment diagrams used in design are only approximate. Variations in loading, settlement of supports, the application of lateral loads, and other factors may cause changes in those diagrams. In Section 7.9 of this chapter, we saw that shear forces could appreciably offset the tensile stresses in the reinforcing bars, thus in effect changing the moment diagrams. As a result of these factors, the ACI Code, Section 9.7.3.3 says that reinforcing bars should be continued for a distance of 12 bar diameters or the effective depth d of the member, whichever is greater (except at the supports of simple spans and the free ends of cantilevers), beyond their theoretical cutoff points.

Various other rules for development lengths apply specifically to positive-moment reinforcement, negative-moment reinforcement, and continuous beams. These are addressed in Chapter 14 of this text. Another item presented in that chapter, which is usually of considerable interest to students, are the rules of thumb that are frequently used in practice to establish cutoff and bend points.

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[9] Ibid., pp. 191–193.

Courtesy of Aluma Systems/Brand Energy & Infrastructure Services.

Los Angeles County water project.

Another rather brief development length example is presented in Example 7.8. A rectangular section and satisfactory reinforcement have been selected for the given span and loading condition. It is desired to determine where two of the four bars may be cut off, considering both moment and development length.

Example 7.8

The rectangular beam with four #8 bars shown in Figure 7.20(b) has been selected for the span and loading shown in part (a) of the figure. Determine the cutoff point for two of the bars, considering both the actual moment diagram and the required development length.

The design moment capacity (ϕM_n) of this beam has been computed to equal 359.7 ft-k when it has 4 #8 bars and 185.3 ft-k when it has 2 #8 bars. (Notice that ρ with 2 bars = 1.57 in.^{2}/(18 in.)(27 in.) = 0.00323 < ρ_{min} = 200/60,000 psi = 0.00333, but is considered to be close enough.) In addition, ℓ_d for the bars has been determined to equal 41 in., using ACI Equation 25.4.2.3a with $\lambda = 0.75$ and K_{tr} = 0.275 in. based on #3 stirrups at s = 8 in. (similar to Example 7.4).

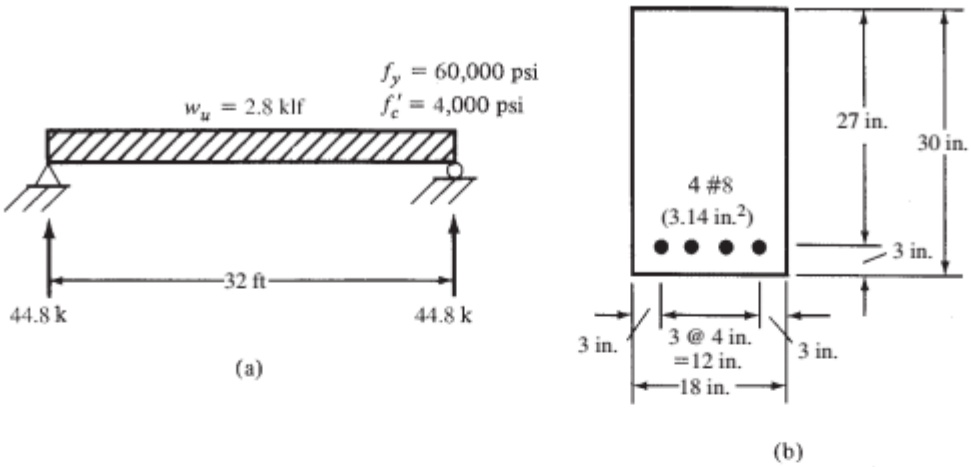

FIGURE 7.20 Given information for Example 7.8.

SOLUTION

The solution for this problem is shown in Figure 7.21. There are two bars beginning at the left end of the beam. Because no development length is available, the design moment capacity of the member is zero. If we move a distance ℓ_d from point A at the left end of the beam to point B, the design moment capacity can be approximated as a straight line from 0 to 185.3 ft-k. From point B to point C, it will remain equal to 185.3 ft-k.

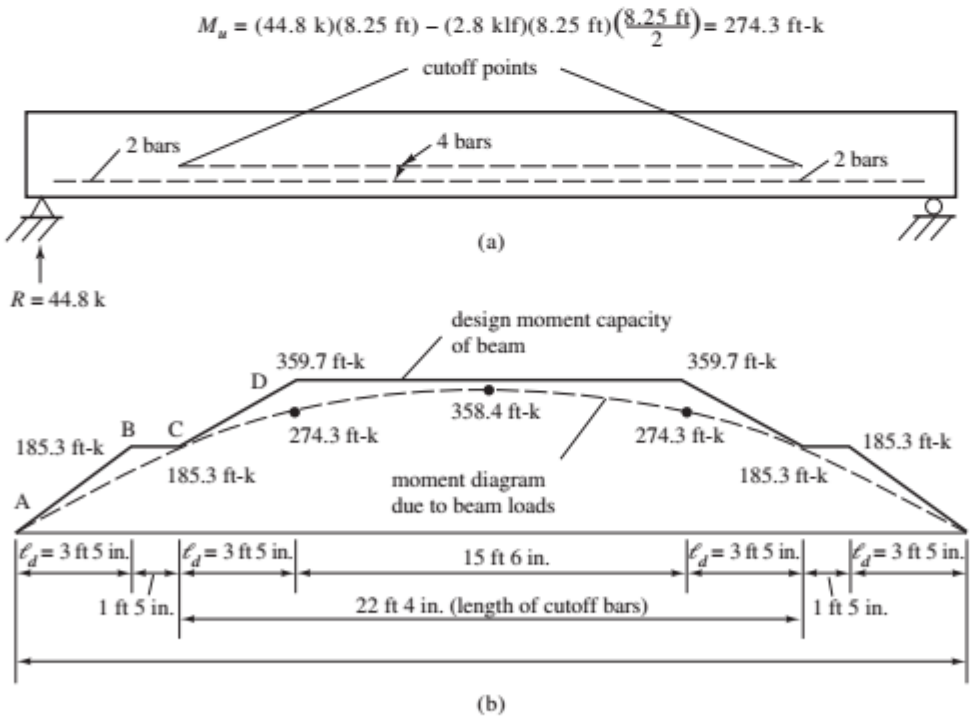

FIGURE 7.21 Comparison of moment diagram to moment capacities.

At point C, we reach the cutoff point of the bars, and from C to D (a distance equal to ℓ_d), the design moment capacity will increase from 185.3 ft-k to 359.7 ft-k. [In Figure 7.21(a) the bars seem to be shown in two layers. They are actually on one level, but the authors have shown them this way so that the reader can get a better picture of how many bars there are at any point along the beam.]

At no point along the span can the design strength of the beam be less than the actual bending moment caused by the loads. We can then see that point C is located where the actual bending moment equals 185.3 ft-k. The left reaction for this beam is 44.8 k, as shown in Figure 7.20(a). Using this value, an expression is written for the moment at point C (185.3 ft-k) at a distance x from the left support. The resulting expression can be solved for x.

$$44.8x - (2.8x)\left(\frac{x}{2}\right) = 185.3 \text{ ft-k}$$

$$x = 4.88 \text{ ft} \quad \text{Say, 4 ft 10 in.}$$

By the time we reach point D (3 ft 5 in. to the right of C and 8 ft 3 in. from the left support), the required moment capacity is

$$M_u = (44.8 \text{ k})(8.25 \text{ ft}) - (2.8 \text{ klf})(8.25 \text{ ft})\left(\frac{8.25 \text{ ft}}{2}\right) = 274.3 \text{ ft-k}$$

Earlier in this section, reference was made to ACI Code, Section 9.7.3.5 in which shear at bar cutoff points was considered. It is assumed that this beam will be properly designed for shear as described in the next chapter and will meet the ACI shear requirements.

7.12 Bar Splices in Flexural Members

Field splices of reinforcing bars are often necessary because of the limited bar lengths available, requirements at construction joints, and changes from larger bars to smaller bars. Although steel fabricators normally stock reinforcing bars in 60-ft lengths, it is often convenient to work in the field with bars of shorter lengths, thus necessitating the use of rather frequent splices.

The reader should carefully note that the ACI Code, Section 26.6.1.1(d) clearly states that the designer is responsible for specifying the types and locations for splices for reinforcement.

The most common method of splicing #11 or smaller bars is simply to lap the bars one over the other. Lapped bars may be either separated from each other or placed in contact, with the contact splices being much preferred because the bars can be wired together. Such bars also hold their positions better during the placing of the concrete. Although lapped splices are easy to make, the complicated nature of the resulting stress transfer and the local cracks that frequently occur in the vicinity of the bar ends are disadvantageous. Obviously, bond stresses play an important role in transferring the forces from one bar to another. Thus the required splice lengths are closely related to development lengths. It is necessary to understand that the minimum specified clear distances between bars also apply to the distances between contact lap splices and adjacent splices or bars (ACI Code, Section 25.5.1.2).

Lap splices are not very satisfactory for several situations, including (1) where they would cause congestion; (2) where the laps would be very long, as they are for #9 through #11 Grade 60 bars; (3) where #14 or #18 bars are used because the ACI Code, Section 25.5.1.1 does not permit them to be lap spliced except in a few special situations; and (4) where very long bar lengths would be left protruding from existing concrete structures for purposes of future expansion. For such situations, other types of splices, such as those made by welding or by mechanical devices, may be used. Welded splices, from the view of stress transfer, are the best splices, but they may be expensive and may cause metallurgical problems. The result may be particularly disastrous in high seismic zones. The ACI Code, Section 25.5.7.1 states that welded splices must be accomplished by welding the bars together so that the connection will be able to develop at least 125% of the specified yield strength of the bars. It is

considered desirable to butt the bars against each other, particularly for #7 and larger bars. Splices not meeting this strength requirement can be used at points where the bars are not stressed to their maximum tensile stresses. It should be realized that welded splices are usually the most expensive because of the high labor costs and the costs of proper inspection.

Mechanical connectors usually consist of some type of sleeve splice, which fits over the ends of the bars to be joined and into which a metallic grout filler is placed to interlock the grooves inside the sleeve with the bar deformations. From the standpoint of stress transfer, good mechanical connectors are next best to welded splices. They do have the disadvantage that some slippage may occur in the connections; as a result, there may be some concrete cracks in the area of the splices.

Before the specific provisions of the ACI Code are introduced, the background for these provisions should be explained briefly. The following remarks are taken from a paper by George F. Leyh of the CRSI.[10]

1. Splicing of reinforcement can never reproduce exactly the same effect as continuous reinforcement.
2. The goal of the splice provisions is to require a ductile situation where the reinforcement will yield before the splices fail. Splice failures occur suddenly without warning and with dangerous results.
3. Lap splices fail by splitting of the concrete along the bars. If some type of closed reinforcement is wrapped around the main reinforcement (such as ties and spirals, described for columns in Chapter 9), the chances of splitting are reduced and smaller splice lengths are needed.
4. When stresses in reinforcement are reduced at splice locations, the chances of splice failure are correspondingly reduced. For this reason, the code requirements are less restrictive where stresses are low.

Splices should be located away from points of maximum tensile stress. Furthermore, not all of the bars should be spliced at the same locations—that is, the splices should be staggered. Should two bars of different diameters be lap spliced, the lap length used shall be the splice length required for the smaller bar or the development length required for the larger bar, whichever is greater (ACI Code, Section 25.5.2.2).

The length of lap splices for bundled bars must be equal to the required lap lengths for individual bars of the same size, but increased by 20% for three-bar bundles and 33% for four-bar bundles (ACI Code, Section 25.6.1.5) because there is a smaller area of contact between the bars and the concrete, and thus less bond. Furthermore, individual splices within the bundles are not permitted to overlap each other.

7.13 Tension Splices

The ACI Code, Section 25.5.2.1 divides tension lap splices into two classes, A and B. The class of splice used is dependent on the level of stress in the reinforcing and on the percentage of steel that is spliced at a particular location. Table 7.4 (which is similar to ACI Table 25.5.2.1) shows the requirements for both Class A and B splices.

Class A splices are those where (1) the area of reinforcing steel provided is not less than twice that required by analysis and (2) one-half or less of the reinforcing steel is spliced at any one location. For this case, the required minimum value of $\ell_{st} = 1.0\,\ell_d$, but not less than 12 in.

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[10] Portland Cement Association, 1972, *Proceedings of the PCA-ACI Teleconference on ACI 318-71 Building Code Requirements* (Skokie, IL: Portland Cement Association), p. 14-1.

TABLE 7.4 Lap Splice Lengths of Deformed Bars and Deformed Wires in Tension (ACI Code 318, Table 25.5.2.1)

$A_{s\ provided} / A_{s\ required}$* over Length of Span	Maximum Percentage of A_s Spliced within Required Lap Length	Splice Type	ℓ_{st}
≥ 2.0	50	Class A	Greater of 1.0 ℓ_d and 12 in.
	100	Class B	Greater of 1.3 ℓ_d and 12 in.
< 2.0	All cases	Class B	

*Ratio of area of reinforcement provided to area of reinforcement required by analysis at splice location.
Source: Reproduced with permission from the American Concrete Institute.

Class B splices do not comply with both of the requirements listed for Class A splices. Class B splices are 30% longer than Class A splices, however, both have a 12 in. minimum length requirement. Class B splices are lapped for a minimum distance of $1.3\ell_d$ (but not less than 12 in.).

In calculating the value of ℓ_d to be multiplied by 1.0 or 1.3, the reduction for excess reinforcing furnished, $A_{s\ provided}/A_{s\ required}$, cannot be used because the class of splice (A or B) already reflects any excess reinforcing steel at the splice location (ACI Code, Section 25.5.1.4).

7.14 Compression Splices

Compression bars may be spliced by lapping, end bearing, and welding or by mechanical devices. (Mechanical devices consist of bars or plates or other pieces welded or otherwise attached transversely to the flexural bars in locations where sufficient anchorage is not available.) The ACI Code, Section 25.5.5.1 says that the minimum splice length of such bars should equal $0.0005f_y d_b$ for bars with f_y of 60,000 psi or less, $(0.0009f_y - 24)d_b$ for bars with higher f_y values, but not less than 12 in. Should the concrete strengths be less than 3000 psi, it is necessary to increase the computed laps

Courtesy of Dywidag Systems International, USA, Inc.

Picture of #7 GR75 Dywidag THREADBAR (R) reinforcing bar including a couple for transferring tension loads.

by one-third. Reduced values are given in Section 10.7.5.2.1 of the ACI Code for cases where the bars are enclosed by ties or spirals.

The required length of lap splices for compression bars of different sizes is the larger of the computed compression lap splice length (ℓ_{sc}) of the smaller bars or the compression development length, ℓ_{dc} of the larger bars. It is permissible to lap splice #14 and #18 compression bars to #11 and smaller bars (ACI Code, Section 25.5.5.3).

The transfer of forces between bars that are always in compression can be accomplished by end bearing, according to Section 25.5.6 of the ACI Code. For such transfer to be permitted, the bars must have their ends square cut (within $1\frac{1}{2}°$ of a right angle), must be fitted within 3° of full bearing after assembly, and must be suitably confined (by closed ties, closed stirrups, hoops, or spirals). Section 10.7.5.3.1 of the ACI Code further states that when end-bearing splices are used in columns, in each face of the column more reinforcement has to be added that is capable of providing a tensile strength at least equal to 25% of the yield strength of the vertical reinforcement provided in that face.

The ACI Code, Section 25.5.1.1, with one exception, prohibits the use of lap splices for #14 or #18 bars. When column bars of those sizes are in compression, it is permissible to connect them to footings by means of dowels of smaller sizes with lap splices, as described in Section 16.3.5.4 of the ACI Code.

7.15 Headed and Mechanically Anchored Bars

Headed deformed bars (Figure 1.3 in Chapter 1) were added to the code in the 2008 edition. Such devices transfer force from the bar to the concrete through a combination of bearing force at the head and bond forces along the bar. There are several limitations to the use of headed bars, as follows:

(a) Bar shall conform to ASTM A970 (ACI Code 318, Section 20.2.1.6).

(b) Bar f_y shall not exceed 60,000 psi.

(c) Bar size shall not exceed #11.

(d) Concrete shall be normal weight.

(e) Net bearing area of head, A_{brg}, shall not be less than four times the area of the bar, A_b.

(f) Clear cover for bar shall not be less than $2d_b$.

(g) Clear spacing between bars shall not be less than $4d_b$.

Clear cover and clear spacing requirements in (e) and (f) are measured to the bar, not to the head.

Headed bars are limited to those types that meet the requirements of HA heads in ASTM A970 because a number of methods are used to attach heads to bars, some involving significant obstructions or interruptions of the bar deformations. Headed bars with significant obstructions or interruptions of the bar deformations were not evaluated in the tests used to formulate the provisions in ACI Code, Section 25.4.4.

The development length in tension for headed deformed bars that comply with the ASTM A970 and other special requirements pertaining to obstructions (ACI Section R20.2.1.6) is given as follows:

$$\ell_{dt} = \frac{0.016\psi_e f_y}{\sqrt{f'_c}} d_b$$

In applying this equation, f'_c cannot be taken greater than 6000 psi, and ψ_e is 1.2 for epoxy-coated or zinc and epoxy dual-coated bars and 1.0 for uncoated or zinc-coated (galvanized) bars. The calculated value of ℓ_{dt} cannot be less than $8d_b$ or 6 in., whichever is larger. The multiplier used earlier for deformed bars

without heads, $A_{s\ \text{required}}/A_{s\ \text{provided}}$, is not permitted. There are no λ, ψ_t, or ψ_s terms in this expression.

In SI units, $\ell_{dt} = \dfrac{0.19\psi_e f_y}{\sqrt{f'_c}} d_{bt}$

In applying the equation for ℓ_{dt} in SI units, f'_c cannot be taken greater than 40 MPa.

The ACI Code, Section 25.4.5.1 also permits other mechanical devices shown by tests to be effective and approved by the building official.

Example 7.9

Repeat Example 7.6 using a headed bar, and compare with the results of Example 7.6.

$$\ell_{dt} = \frac{0.016\psi_e f_y}{\sqrt{f'_c}} d_b = \frac{(0.016)\,(1.2)\,(60{,}000\text{ psi})}{\sqrt{4000}\text{ psi}}(1.128\text{ in.}) = 20.54\text{ in.} \qquad \text{Say 21 in.}$$

This value compares with 69 in. for a straight bar and 26 in. for a 90° or 180° hooked bar.

7.16 SI Example

Example 7.10

Determine the development length required for the epoxy-coated bottom bars shown in Figure 7.22

(a) assuming $K_{tr} = 0$ and
(b) computing K_{tr} with the appropriate equation, for $f_y = 420$ MPa and $f'_c = 21$ MPa.

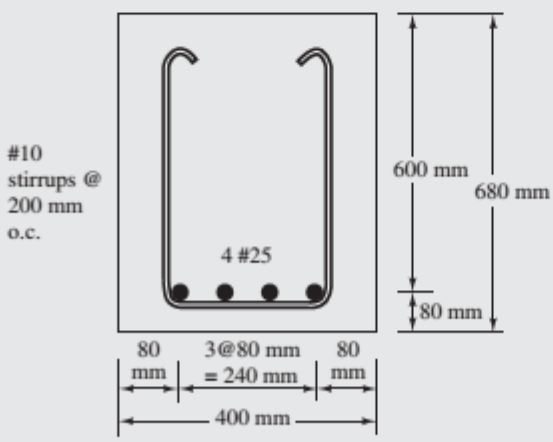


FIGURE 7.22 Beam cross section for Example 7.10.

SOLUTION

From Table 7.1 in this chapter

$$\psi_t = 1.0 \text{ for bottom bars}$$

$$\psi_e = 1.5 \text{ for epoxy-coated bars with clear spacing} < 6d_b$$

$$\psi_t\psi_e = (1.0)\,(1.5) = 1.5 < 1.7 \quad \underline{\underline{\text{OK}}}$$

$$\psi_s = \lambda = 1.0$$

$$c_b = \text{side cover of bars} = 80 \text{ mm}$$

$$c_b = \frac{1}{2} \text{ of c. to c. spacing of bars} = 40 \text{ mm} \leftarrow \text{controls}$$

(a) Using SI Equation 25.4.2.3a with $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{40 \text{ mm} + 0 \text{ mm}}{25.4 \text{ mm}} = 1.575 < 2.5 \quad \underline{\underline{\text{OK}}}$$

$$\frac{\ell_d}{d_b} = \frac{1}{1.1}\frac{f_y}{\lambda\sqrt{f'_c}}\frac{\psi_t\psi_e\psi_s}{\frac{c_b+K_{tr}}{d_b}}$$

$$= \left(\frac{1}{1.1}\right)\left(\frac{420 \text{ MPa}}{(1.0)\sqrt{21} \text{ MPa}}\right)\frac{(1.0)\,(1.5)\,(1.0)}{1.575} = \underline{\underline{79.4 \text{ diameters}}}$$

(b) Using Computed Value of K_{tr} and SI Equation 25.4.2.3a

$$K_{tr} = \frac{40A_{tr}}{sn} = \frac{(40)(2)(71 \text{ mm}^2)}{(200 \text{ mm})(4)} = 7.1 \text{ mm}$$

$$\frac{c_b + K_{tr}}{d_b} = \frac{40 \text{ mm} + 7.1 \text{ mm}}{25.4 \text{ mm}} = 1.85 < 2.5 \quad \underline{\underline{\text{OK}}}$$

$$\frac{\ell_d}{d_b} = \left(\frac{9}{10}\right)\left(\frac{420}{(1.0)\sqrt{21}}\right)\frac{(1.0)\,(1.5)\,(1.0)}{1.85} = \underline{\underline{66.9 \text{ diameters}}}$$

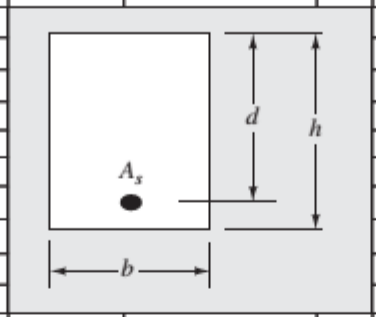
7.17 Computer Example

Example 7.11

Using the worksheet entitled "devel length tens - calc As" in the spreadsheet for Chapter 7, determine the required tension development length ℓ_d of the beam shown in Figure 7.20 if lightweight aggregate concrete and #3 stirrups at 8 in. centers are used.

SOLUTION

Input the values of the cells highlighted in yellow (only in the Excel spreadsheets, not the printed example). Some cells are optional (see note marked with * in the printout for Example 7.11).

	Development Length, Tension							
	f'_c =	4000	psi					
	f_y =	60,000	psi					
	f_{yt} =	60,000	psi					
	b =	18	in.*					
	d =	27	in.*					
	h =	30	in.*					
	A_s =	3.14	in.^{2}*					
	A_{tr} =	0.22	in.^{2}*					
	d_b =	1	in.					
	n =	4	*					
	s =	8	in.*					
	ψ_t =	1.00						
	ψ_e =	1.00		*Cells indicate that this information is optional.				
	ψ_s =	1.00		M_u, b, d, h, and A_s are needed only to calculate.				
	λ =	0.75		$A_{s\ required}$, A_{tr}, n, and s are needed only if the K_{tr} term is				
	c_b =	2.00	in.	to be used. All terms with * can be omitted, and				
	M_u =	358.40	ft-k*	a conservative value of l_d will result.				
	$\psi_t \psi_e$ =	1						
	K_{tr} =	0.275						
	$(c_b + K_{tr})/d_b$ =	2.28						
	$l_d = \frac{3}{40} \frac{f_y}{\lambda \sqrt{f'_c}} \frac{\psi_t \psi_e \psi_s}{\left[\frac{c_b + K_{tr}}{d_b}\right]} d_b$ =				41.7	diameters		
					41.7	in. (not adjusted for $A_{s\ required}/A_{s\ provided}$)		
		$A_{s\ required}$ =	3.12736	in.^{2}				
	$A_{s\ required}/A_{s\ provided}$ =		0.995975	l_d =	41.5	in. (adjusted for $A_{s\ required}/A_{s\ provided}$)		
					but not less than	12	in.	

Printout of Example 7.11 results.

Pass the cursor over cells for comments explaining what is to be input. Note that two answers are given, one with the $A_{s\ required}/A_{s\ provided}$ reduction and one without. In this example, there is little difference because this ratio is nearly 1.0.

PROBLEMS

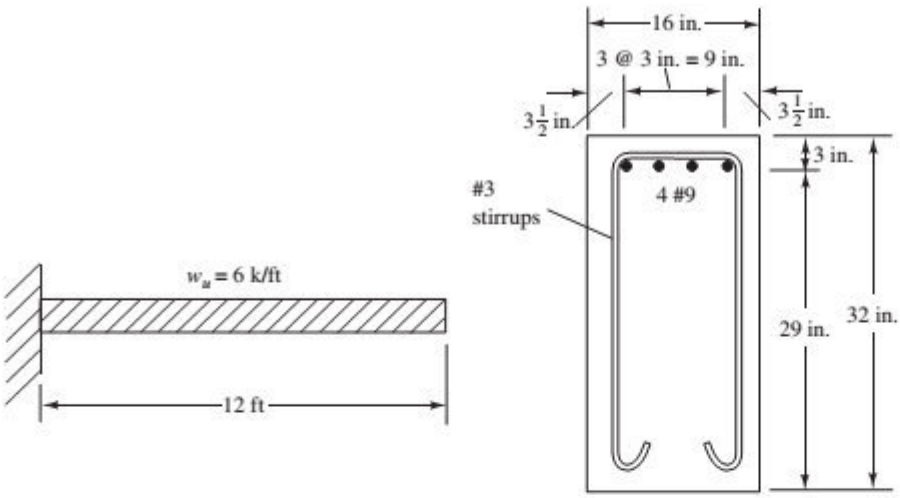
Problem 7.1 Why is it difficult to calculate actual bond stresses?

Problem 7.2 What are top bars? Why are the required development lengths greater than they would be if they were not top bars?

Problem 7.3 Why do the cover of bars and the spacing of those bars affect required development lengths?

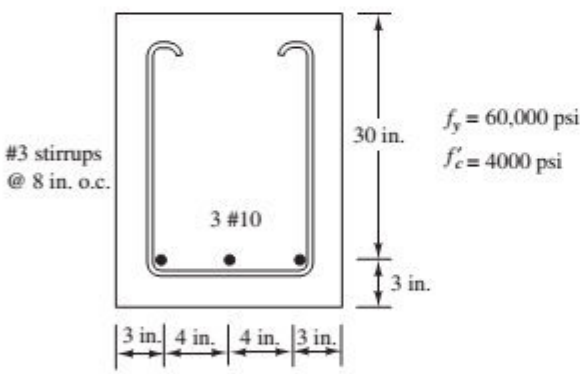
Problem 7.4 Why isn't the anchorage capacity of a standard hook increased by extending the bar well beyond the end of the hook?

Problem 7.5 For the cantilever beam shown, determine the point where two bars theoretically can be cut off from the standpoint of the calculated moment strength, ϕM_n, of the beam. $f_y = 60{,}000$ psi and $f'_c = 3000$ psi. (*Ans.* 9.09 ft from free end)

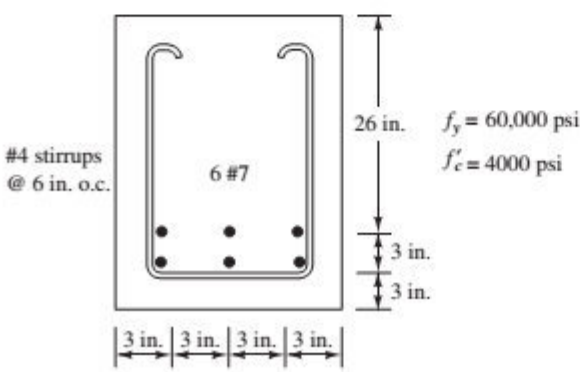


For Problems 7.6 to 7.9, determine the development lengths required for the tension bar situations described using ACI Equation 25.4.2.3a (a) assuming $K_{tr} = 0$ and (b) using the calculated value of K_{tr}.

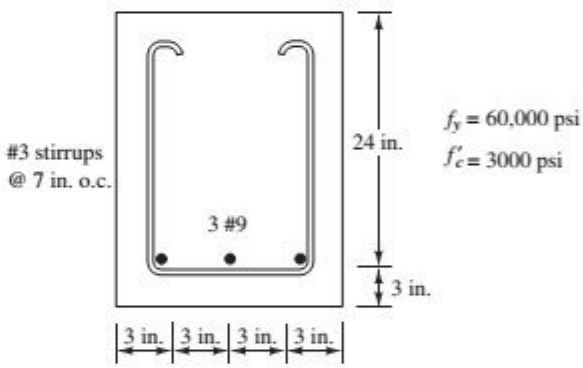
Problem 7.6 Uncoated bars in normal-weight concrete. $A_{s\,\text{required}} = 3.44$ in.^{2}.



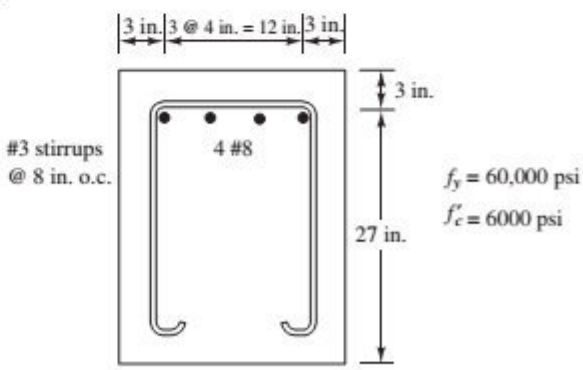
Problem 7.7 Uncoated bars in normal-weight concrete. $A_{s\,\text{required}} = 3.40$ in.^{2}. (*Ans.* 34.2 in., 23.5 in.)



Problem 7.8 Epoxy-coated bars in lightweight concrete, $A_{s\,\text{required}} = 2.76$ in.^{2}.



Problem 7.9 Uncoated top bars in normal-weight concrete. $A_{s\,\text{required}} = 2.92$ in.^{2}. (*Ans.* 35.1 in., 30.9 in.)



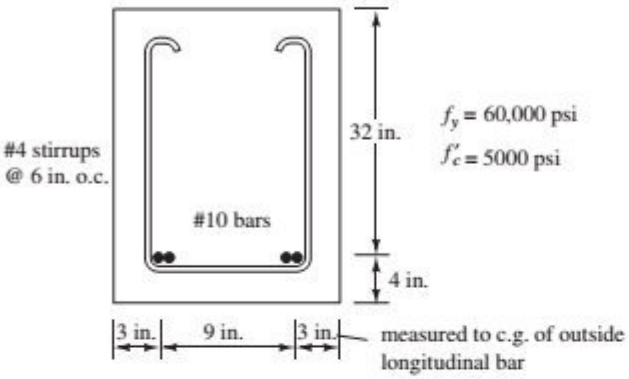
Problem 7.10 Repeat Problem 7.6 if the bars are epoxy coated.

Problem 7.11 Repeat Problem 7.7 if all-lightweight concrete with $f'_c = 3000$ psi and epoxy-coated bars are used. (*Ans.* 79.0 in., 54.2 in.)

Problem 7.12 Repeat Problem 7.8 if three uncoated #6 bars are used and $A_{s\ \text{required}} = 1.20\ \text{in.}^2$.

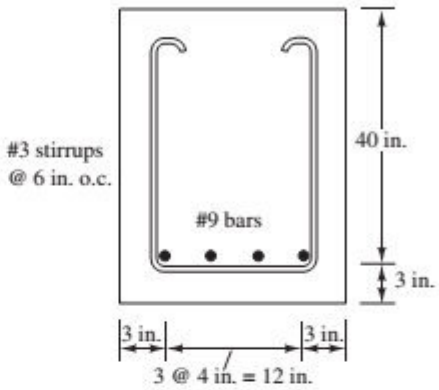
Problem 7.13 Repeat Problem 7.9 if the bars are epoxy coated and all-lightweight concrete is used. (*Ans*. 61.2 in., 53.8 in.)

Problem 7.14 The bundled #10 bars shown are uncoated and used in normal-weight concrete. $A_{s\ \text{required}} = 4.44\ \text{in.}^2$.



Problem 7.15 Repeat Problem 7.14 if the bars are epoxy coated and used in lightweight concrete with $f'_c = 4000$ psi. (*Ans*. 78.3 in., 63.4 in., etc.)

Problem 7.16 Set up a table for required development lengths for the beam shown, using $f_y = 60{,}000$ psi and f'_c values of 3000 psi, 3500 psi, 4000 psi, 4500 psi, 5000 psi, 5500 psi, and 6000 psi. Assume that the bars are uncoated and normal-weight concrete is used. Use ACI Equation 25.4.2.3a and assume $K_{tr} = 0$.



Problem 7.17 Repeat Problem 7.16 if #8 bars are used. (*Ans*. 41.1 in., 38.0 in., 35.6 in., 33.5 in., 31.8 in., etc.)

Problem 7.18 Repeat Problem 7.16 if #7 bars are used.

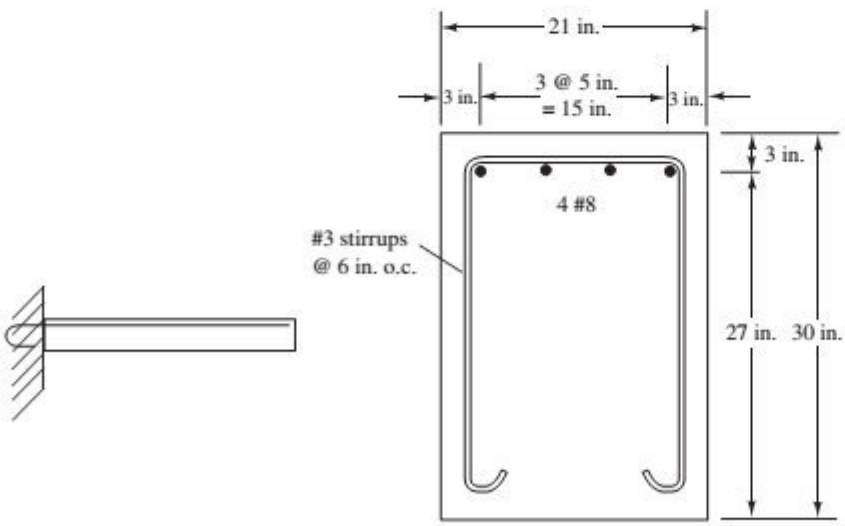
Problem 7.19 Repeat Problem 7.16 if #6 epoxy-coated bars are used in lightweight concrete. (*Ans*. 39.4 in., 36.5 in., 34.2 in., 32.2 in., 30.6 in., etc.)

Problem 7.20

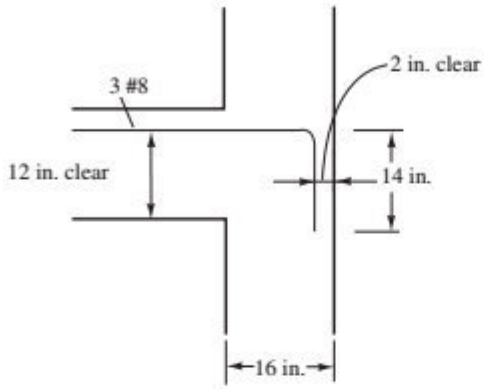
(a) Determine the tensile development length required for the uncoated #8 bars shown if normal-weight concrete is used and the bars are straight. Use ACI Equation 25.4.2.3a and compute the value of K_{tr}. $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

(b) Repeat part (a) if 180° hooks are used.

Assume side, top, and bottom cover in all cases to be at least $2\frac{1}{2}$ in.



Problem 7.21 Are the uncoated #8 bars shown anchored sufficiently with their 90° hooks? $f'_c = 4000$ psi and $f_y = 60$ ksi. Side and top cover is $2\frac{1}{2}$ in. on bar extensions. Normal-weight concrete is used. $A_{s\,\text{required}} = 2.20$ in.^{2}. (*Ans.* $\ell_{dh} = 12.4$ in., sufficient)

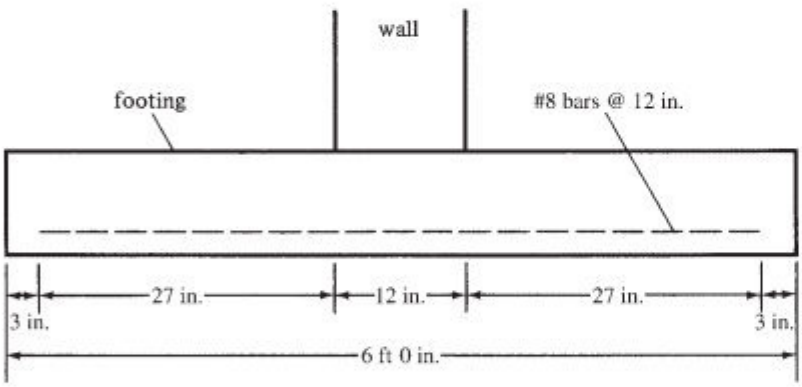


Problem 7.22 Repeat Problem 7.21 if headed bars are used instead of 90° hooks and $f'_c = 5000$ psi.

Problem 7.23 Repeat Problem 7.7 if the bars are in compression. (*Ans.* 15.68 in.)

For Problems 7.24 to 7.29, use ACI Equation 25.4.2.3a and assume $K_{tr} = 0$.

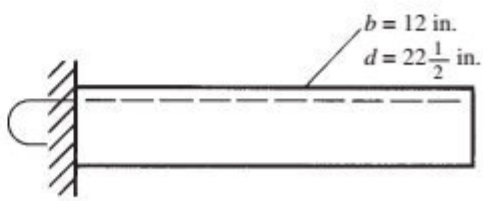
Problem 7.24 The required bar area for the wall footing shown is 0.65 in.^{2} per foot of width and #8 epoxy-coated bars 12 in. on center are used. Maximum moment is assumed to occur at the face of the wall. If $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, do the bars have sufficient development lengths? Assume $c_b = 3$ in.



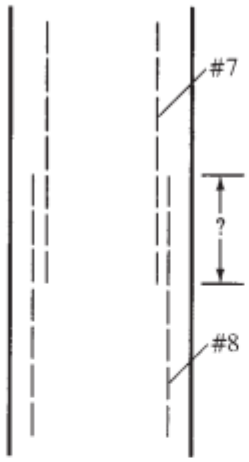
Problem 7.25 Repeat Problem 7.24 using #8 @ 9 in. and without epoxy coating. (*Ans.* $\ell_d = 23.6$ in. < 27 in. OK)

Problem 7.26 Problem 7.24 has insufficient embedment length. List four design modifications that would reduce the required development length.

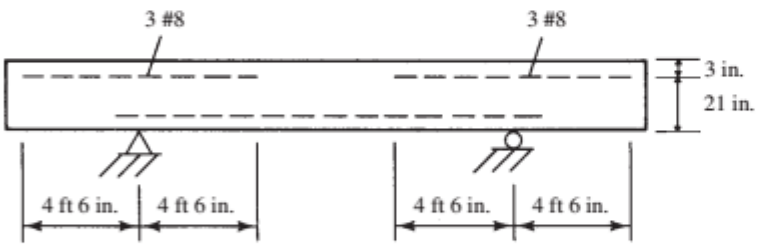
Problem 7.27 The beam shown is subjected to an M_u of 200 ft-k at the support. If $c_b = 1.5$, $K_{tr} = 0$, the concrete is lightweight, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi, do the following: (a) select #8 bars to be placed in one row, (b) determine the development lengths required if straight bars are used in the beam, and (c) determine the development lengths needed if 180° hooks are used in the support. (*Ans.* 3 #8, 74.5 in., 22.9 in.)



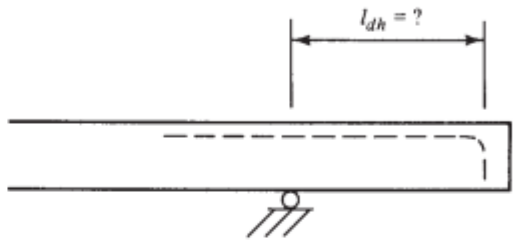
Problem 7.28 In the column shown, the lower column bars are #8 and the upper ones are #7. The bars are enclosed by ties spaced 12 in. on center. If $f_y = 60{,}000$ psi and $f'_c = 4000$ psi, what is the minimum lap splice length needed? Normal-weight concrete is to be used for the 12-in. × 12-in. column.



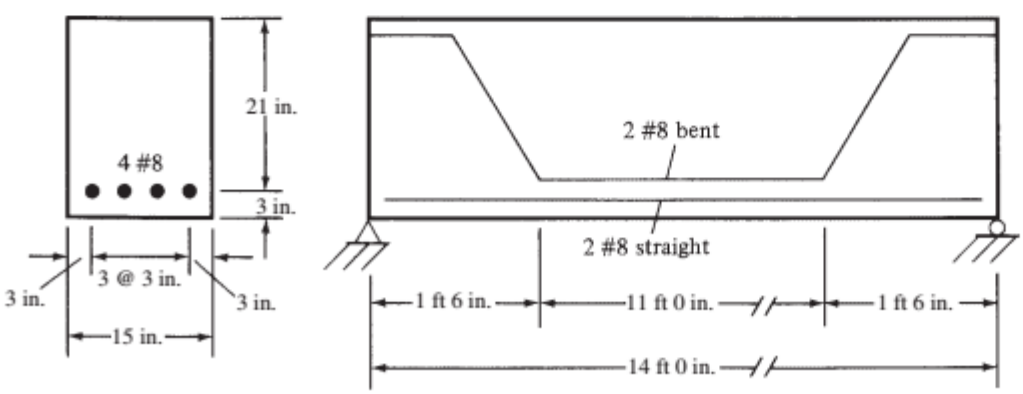
Problem 7.29 Calculations show that 2.24 in.^{2} of top or negative moment steel is required for the beam shown. Three #8 bars have been selected. Are the 4 ft. 6 in. embedment lengths shown satisfactory if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi? Bars are spaced 3 in. o.c. with 3-in. side and top cover measured from c.g. of bars. Use $K_{tr} = 0$. (*Ans.* No; $\ell_d = 58.8$ in. > 4 ft 6 in., not adequate)



Problem 7.30 Calculations show that 4.90 in.^{2} of top or negative steel is required for the beam shown. If four uncoated #10 bars have been selected, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi, determine the minimum development length needed for the standard 90° hooks shown. Assume that the bars have 3-in. side and top cover measured from c.g. of bars and are used in normal-weight concrete. The bars are not enclosed by ties or stirrups spaced at $3d_b$ or less.



Problem 7.31 If $f_y = 75{,}000$ psi, $f'_c = 3000$ psi, $w_D = 1.5$ k/ft, and $w_L = 5$ k/ft, are the development lengths of the straight bars satisfactory? Assume that the bars extend 6 in. beyond the centerline of the reactions and that $K_{tr} = 0$. $A_{s\,\text{required}} = 3.05$ in.^{2}. The bars are uncoated and the concrete is normal weight. (*Ans.* $\ell_d = 66.5$ in., embedment length is adequate)



Compression Splices

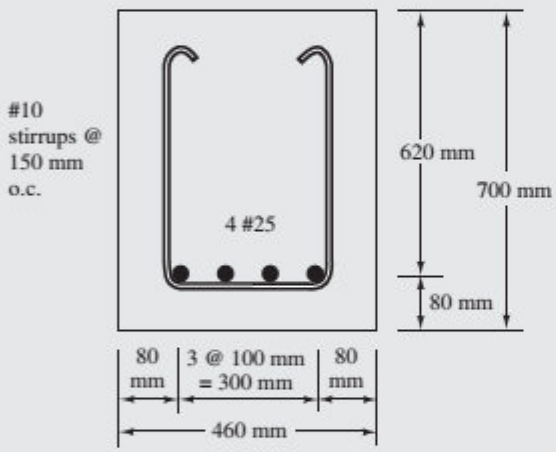
Problem 7.32 Determine the compression lap splices needed for a 14-in. × 14-in. reinforced concrete column with ties (whose effective area exceeds 0.0015 hs, as described in Section 10.7.5.2.1 of the ACI Code) for the cases to follow. There are eight #8 longitudinal bars equally spaced around the column.

(a) $f'_c = 4000$ psi and $f_y = 60{,}000$ psi
(b) $f'_c = 2000$ psi and $f_y = 50{,}000$ psi

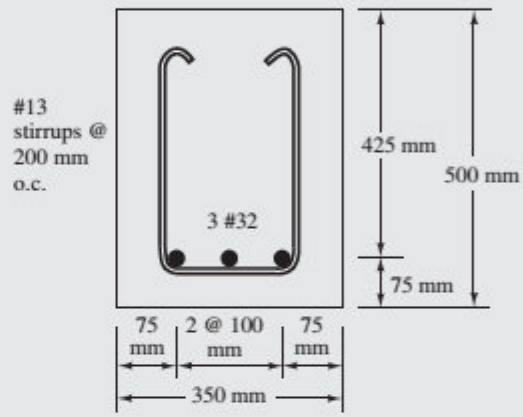
Problems in SI Units

For Problems 7.33 to 7.36, determine the tensile development lengths required using: (a) ACI Metric Equation 25.4.2.3a, assuming $K_{tr} = 0$, and (b) ACI Metric Equation 25.4.2.3a and the computed value of K_{tr}. Use $f_y = 420$ MPa and $f'_c = 28$ MPa.

Problem 7.33 (*Ans*. 931 mm, 777 mm)



Problem 7.34



Problem 7.35 Repeat Problem 7.33 if the longitudinal bars are #19. (*Ans*. 441 mm, 441 mm)

Problem 7.36 Repeat Problem 7.34 if the bars are epoxy coated.

Computer Problems

For Problems 7.37 and 7.38, use the Chapter 7 spreadsheet.

Problem 7.37 Repeat Problem 7.6. (*Ans*. 52.1 in., 44.0 in.)

Problem 7.38 Repeat Problem 7.9.

Problem 7.39 Repeat Problem 7.22.
(*Ans*. $\ell_{dt} = 13.6$ in. < 14 in. available ∴ OK)

CHAPTER 8 Shear and Diagonal Tension

8.1 Introduction

As repeatedly mentioned earlier in this book, the objective of today's reinforced concrete designer is to produce ductile members that provide warning of impending failure. To achieve this goal, the ACI Code provides design shear values that have larger safety factors against shear failures than do those provided for bending failures. The failures of reinforced concrete beams in shear are quite different from their failures in bending. Shear failures occur suddenly with little or no advance warning. Therefore, beams are designed to fail in bending under loads that are appreciably smaller than those that would cause shear failures. As a result, those members will undergo ductile failure. They may crack and sag a great deal if overloaded, but they will not fall apart, as they might if shear failures were possible.

8.2 Shear Stresses in Concrete Beams

Although no one has ever been able to accurately determine the resistance of concrete to pure shearing stress, the matter is not very important because pure shearing stress is probably never encountered in concrete structures. Furthermore, according to engineering mechanics, if pure shear is produced in a member, a principal tensile stress of equal magnitude will be produced on another plane. Because the tensile strength of concrete is less than its shearing strength, the concrete will fail in tension before its shearing strength is reached.

You have previously learned that in elastic homogeneous beams, where stresses are proportional to strains, two kinds of stresses occur (bending and shear), and they can be calculated with the following expressions:

$$f = \frac{Mc}{I}$$

$$v = \frac{VQ}{Ib}$$

An element of a beam not located at an extreme fiber or at the neutral axis is subject to both bending and shear stresses. These stresses combine into inclined compressive and tensile stresses, called *principal stresses*, which can be determined from the following expression:

$$f_p = \frac{f}{2} \pm \sqrt{\left(\frac{f}{2}\right)^2 + v^2}$$

The direction of the principal stresses can be determined using the following formula, in which α is the inclination of the stress to the beam's axis:

$$\tan 2\alpha = \frac{2v}{f}$$

Obviously, at different positions along the beam, the relative magnitudes of v and f change, and thus the directions of the principal stresses change. It can be seen from the preceding equation that at the neutral axis, the principal stresses will be located at a 45° angle with the horizontal.

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Iron grid.

You understand by this time that tension stresses in concrete are a serious matter. Diagonal principal tensile stresses, called *diagonal tension*, occur at different places and angles in concrete beams, and they must be carefully considered. If they reach certain values, additional reinforcement, called *web reinforcement*, must be supplied.

The discussion presented up to this point relating to diagonal tension applies rather well to plain concrete beams. If, however, reinforced concrete beams are being considered, the situation is quite different because the longitudinal bending tension stresses are resisted quite satisfactorily by the longitudinal reinforcement. These bars, however, do not provide significant resistance to the diagonal tension stresses.

8.3 Lightweight Concrete

In the 2008 ACI 318 Code, the effect of lightweight aggregate concrete on shear strength was modified by the introduction of the term λ (see Section 1.12). This term was added to most equations containing $\sqrt{f'_c}$. The resulting combined term, $\lambda\sqrt{f'_c}$, appears throughout this chapter as well as in Chapter 7 on development length and Chapter 15 on torsion. If normal-weight concrete is used, then λ is simply taken as 1. This unified approach to the effects of lightweight aggregate on the strength and other properties of concrete is a logical and simplifying improvement that was introduced in the 2008 ACI Code.

8.4 Shear Strength of Concrete

A great deal of research has been done on the subject of shear and diagonal tension for nonhomogeneous reinforced concrete beams, and many theories have been developed. Despite all this work and all the resulting theories, no one has been able to provide a clear explanation of the failure mechanism involved. As a result, design procedures are based primarily on test data.

If V_u is divided by the effective beam area, $b_w d$, the result is what is called an *average shearing stress*. This stress is not equal to the diagonal tension stress but merely serves as an indicator of its magnitude. Should this indicator exceed a certain value, shear or web reinforcing is considered necessary. In the ACI Code, the basic shear equations are presented in terms of shear forces, not shear stresses. In other words, the average shear stresses described in this paragraph are multiplied by the effective beam areas to obtain total shear forces.

For this discussion, V_n is considered to be the nominal or theoretical shear strength of a member. This strength is provided by the concrete and by the shear reinforcement.

$$V_n = V_c + V_s \qquad \text{(ACI Equation 22.5.1.1)}$$

The design shear strength of a member, ϕV_n, is equal to ϕV_c plus ϕV_s, which must at least equal the factored shear force to be taken, V_u

$$V_u = \phi V_c + \phi V_s$$

The shear strength provided by the concrete, V_c, is considered to equal an average shear stress strength (normally $2\lambda\sqrt{f'_c}$) times the effective cross-sectional area of the member, $b_w d$, where b_w is the width of a rectangular beam or of the web of a T beam or an I beam.

$$V_c = 2\lambda\sqrt{f'_c}\, b_w d \qquad \text{(ACI Table 22.5.5.1)}$$

$V_u d/M_u$ cannot be taken greater than one.

Or in SI units with f'_c in MPa

$$V_c = 0.17\lambda\sqrt{f'_c}\, b_w d$$

Beam tests have shown some interesting facts about the occurrence of cracks at different average shear stress values. For instance, where large moments occur even though appropriate longitudinal steel has been selected, extensive flexural cracks will be evident. As a result, the uncracked area of the beam cross section will be greatly reduced, and the nominal shear strength, V_c, can be as low as $1.9\lambda\sqrt{f'_c}\, b_w d$. In regions where the moment is small, however, the cross section will be either uncracked or slightly cracked, and a large portion of the cross section is available to resist shear. For such a case, tests show that a V_c of about $3.5\lambda\sqrt{f'_c}\, b_w d$ can be developed before shear failure occurs.[1]

Based on this information, the code (Equation 22.5.5.1) suggests that, conservatively, V_c (the shear force that the concrete can resist without web reinforcing) can go as high as $2\lambda\sqrt{f'_c}\, b_w d$. As an alternative, the following shear force (from Section 22.5.5.1 of the code) may be used, which takes into account the effects of the longitudinal reinforcing and the moment and shear magnitudes. This value must be calculated separately for each point being considered in the beam.

$$V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u}\right) b_w d \le 3.5\lambda\sqrt{f'_c}\, b_w d \qquad \text{(ACI Table 22.5.5.1)}$$

In SI units

$$V_c = \left(0.16\lambda\sqrt{f'_c} + 17\rho_w \frac{V_u d}{M_u}\right) b_w d \le 0.29\lambda\sqrt{f'_c}\, b_w d$$

[1] ACI-ASCE Committee 326, 1962, "Shear and Diagonal Tension," part 2, *Journal ACI*, 59, p. 277.

In these expressions, $\rho_w = A_s/b_w d$ and M_u are the factored moment occurring simultaneously with V_u, the factored shear at the section considered. The quantity $V_u d/M_u$ cannot be taken to be greater than unity in computing V_c by means of the above expressions.

From these expressions, it can be seen that V_c increases as the amount of reinforcing (represented by ρ_w) is increased. As the amount of steel is increased, the length and width of cracks will be reduced. If the cracks are kept narrower, more concrete is left to resist shear, and there will be closer contact between the concrete on opposite sides of the cracks. Hence there will be more resistance to shear by friction (called *aggregate interlock*) on the two sides of cracks.

Although this more complicated expression for V_c can easily be used for computer designs, it is quite tedious to apply when handheld calculators are used. The reason is that the values of ρ_w, V_u, and M_u are constantly changing as we move along the span, requiring the computation of V_c at numerous positions. As a result, the alternate value $2\lambda\sqrt{f'_c}b_w d$ is normally used. If the same member is to be constructed many times, the use of the more complex expression may be justified.

8.5 Shear Cracking of Reinforced Concrete Beams

Inclined cracks can develop in the webs of reinforced concrete beams, either as extensions of flexural cracks or occasionally as independent cracks. The first of these two types is the *flexure–shear crack*, an example of which is shown in Figure 8.1. These are the ordinary types of shear cracks found in both prestressed and nonprestressed beams. For them to occur, the moment must be larger than the cracking moment, and the shear must be rather large. The cracks run at angles of about 45° with the beam axis and probably start at the top of a flexure crack. The approximately vertical flexure cracks shown are not dangerous unless a critical combination of shear stress and flexure stress occurs at the top of one of the flexure cracks.

Occasionally, an inclined crack will develop independently in a beam, even though no flexure cracks are in that locality. Such cracks, which are called *web–shear cracks*, will sometimes occur in the webs of prestressed sections, particularly those with large flanges and thin webs. They also sometimes occur near the points of inflection of continuous beams or near simple supports. At such locations, small moments and high shear often occur. These types of cracks will form near the mid-depth of sections and will move on a diagonal path to the tension surface. Web–shear cracks are illustrated in Figure 8.2.

As a crack moves up to the neutral axis, the result will be a reduced amount of concrete left to resist shear—meaning that shear stresses will increase on the concrete above the crack.

It should be remembered that at the neutral axis, the bending stresses are zero, and the shear stresses are at their maximum values. The shear stresses will therefore determine what happens to the crack there.

After a crack has developed, the member will fail unless the cracked concrete section can resist the applied forces. If web reinforcing is not present, the items that are available to transfer the shear are as follows: (1) the shear resistance of the uncracked section above the crack (estimated to be 20% to 40% of the total resistance); (2) the aggregate interlock, that is, the friction developed due to the

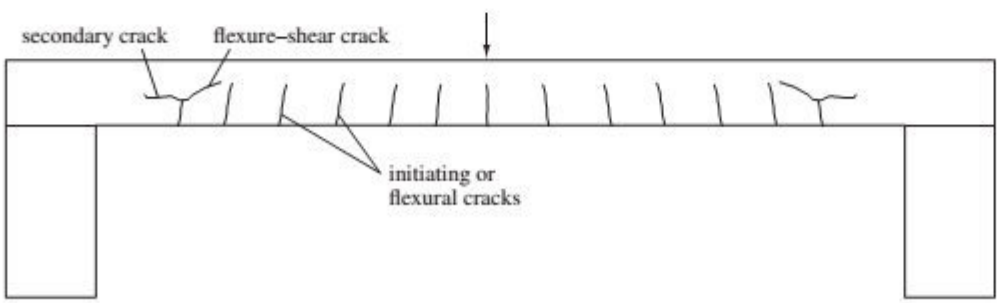

FIGURE 8.1 Flexure–shear crack.

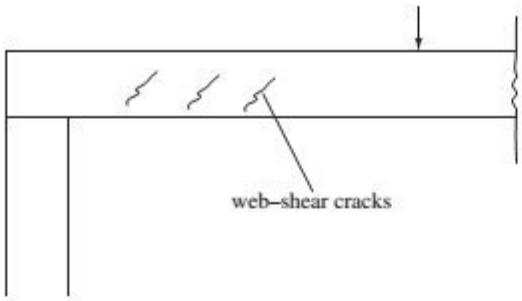

FIGURE 8.2 Web–shear cracks.

interlocking of the aggregate on the concrete surfaces on opposite sides of the crack (estimated to be 33% to 50% of the total); (3) the resistance of the longitudinal reinforcing to a frictional force, often called *dowel action* (estimated to be 15% to 25%); and (4) a tied-arch type of behavior that exists in rather deep beams produced by the longitudinal bars acting as the tie and by the uncracked concrete above and to the sides of the crack acting as the arch above.[2]

8.6 Web Reinforcement

When the factored shear, V_u, is high, it shows that large cracks are going to occur unless some type of additional reinforcing is provided. This reinforcing usually takes the form of stirrups that enclose the longitudinal reinforcing along the faces of the beam. The most common stirrups are ⊔ shaped, but they can be ⊔⊓⊔ shaped or perhaps have only a single vertical prong, as shown in Figure 8.3(c). Multiple stirrups such as the ones shown in Figure 8.3(e) are considered to inhibit splitting in the plane of the longitudinal bars. As a consequence, they are generally more desirable for wide beams than the ones shown in Figure 8.3(d). Sometimes it is rather convenient to use lap spliced stirrups, such as the ones shown in Figure 8.3(g). These stirrups, which are described in ACI Code, Section 25.7.1.7, are occasionally useful for deep members, particularly those with gradually varying depths. *However, they are considered to be unsatisfactory in seismic areas.*

Bars called *hangers* (usually with about the same diameter as that of the stirrups) are placed on the compression sides of beams to support the stirrups, as illustrated in Figure 8.3(a) to (j). The stirrups are passed around the tensile steel and, to meet anchorage requirements, they are run as far into the compression side of the beam as practical and hooked around the hangers. Bending of the stirrups around the hangers reduces the bearing stresses under the hooks. If these bearing stresses are too high, the concrete will crush and the stirrups will tear out. When a significant amount of torsion is present in a member, it will be necessary to use closed stirrups as shown in parts (f) through (j) of Figure 8.3 and as discussed in Chapter 15.

The width of diagonal cracks is directly related to the strain in the stirrups. Consequently, the ACI Code, Section 20.2.2.4, does not permit the design yield stress of the stirrups to exceed 60 ksi. This requirement limits the width of cracks that can develop. Such a result is important from the standpoint of both appearance and aggregate interlock. When the width of cracks is limited, it enables more aggregate interlock to develop. A further advantage of a limited yield stress is that the anchorage requirements at the top of the stirrups are not quite as stringent as they would be for stirrups with greater yield strengths.

The 60,000-psi limitation does not apply to deformed welded wire fabric because recent research has shown that the use of higher-strength wires has been quite satisfactory. Tests have shown that the width of inclined shear cracks at service load conditions is less for high-strength wire fabric than for those occurring in beams reinforced with deformed Grade 60 stirrups. The maximum stress permitted for deformed welded wire fabric is 80,000 psi ACI Code, Section 20.2.2.4.

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[2] Taylor, H. P. J., 1974, "The Fundamental Behavior of Reinforced Concrete Beams in Bending and Shear," *Shear in Reinforced Concrete*, Vol. 1, SP-42 (Detroit: American Concrete Institute), pp. 43–47.

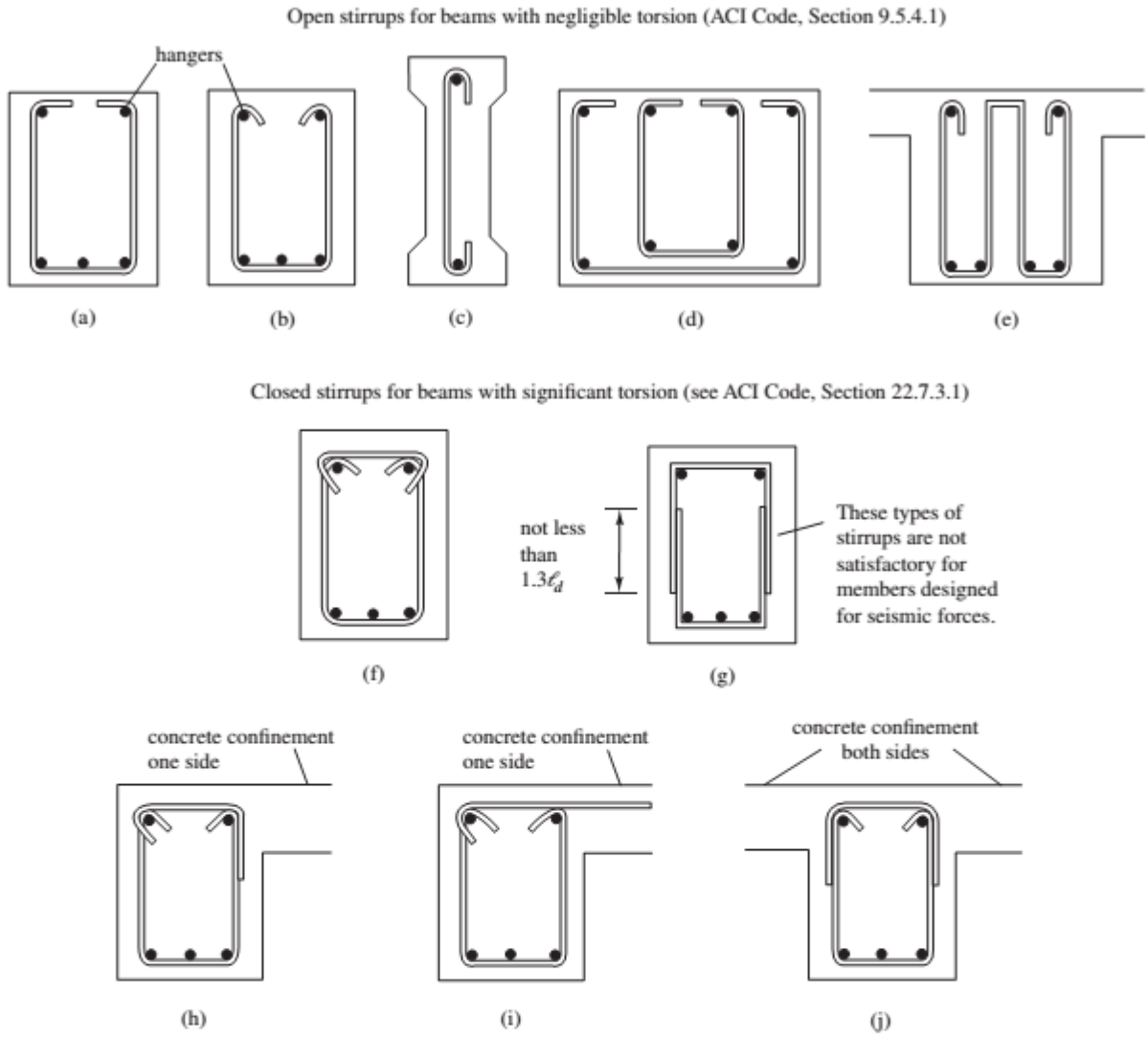

FIGURE 8.3 Types of stirrups.

In SI units, the maximum design yield stress values that may be used are 420 MPa for regular shear reinforcing and 550 MPa for welded deformed wire fabric.

8.7 Behavior of Beams with Web Reinforcement

The actual behavior of beams with web reinforcement is not really understood, although several theories have been presented through the years. One theory, which has been widely used for 100 years, is the so-called truss analogy, wherein a reinforced concrete beam with shear reinforcing is said to behave much like a statically determinate parallel chord truss with pinned joints. The flexural compression concrete is thought of as the top chord of the truss, whereas the tensile reinforcing is said to be the bottom chord. The truss web is made up of stirrups acting as vertical tension members and pieces of concrete between the approximately 45° diagonal tension cracks acting as diagonal compression

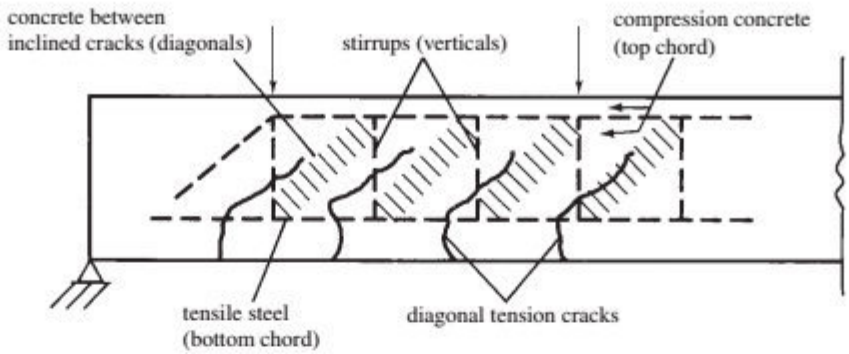

FIGURE 8.4 Truss analogy.

members.[3,4] The shear reinforcing used is similar in its action to the web members of a truss. For this reason, the term *web reinforcement* is used when referring to shear reinforcing. A "truss" of the type described here is shown in Figure 8.4.

Although the truss analogy has been used for many years to describe the behavior of reinforced concrete beams with web reinforcing, it does not accurately describe the manner in which shear forces are transmitted. For example, the web reinforcing does increase the shearing strength of a beam, but it has little to do with shear transfer in a beam before inclined cracks form.

The code requires web reinforcement for all major beams. In Sections 7.6.3.1 (for one-way slabs), 9.6.3.1 (for beams), and 10.6.2.1 (for columns), a minimum area of web reinforcing is required for all concrete flexural members except (a) footings and solid slabs; (b) certain hollow-core units; (c) concrete floor joists; (d) shallow beams with h not more than 10 in.; (e) beams integral with slabs with h less than 24 in. and h not greater than the larger of two-and-a-half times their flange thicknesses or one-half their web widths; or (f) beams constructed with steel fiber–reinforced, normal-weight concrete with f'_c not exceeding 6000 psi, h not greater than 24 in., and V_u not greater than $2\phi\sqrt{f'_c}b_w d$. Various tests have shown that shear failures do not occur before bending failures in shallow members. Shear forces are distributed across these wide sections. For joists, the redistribution is via the slabs to adjacent joists. Hooked or crimped steel fibers in dosages ≥100 lb per cubic yard exhibit higher shear strengths in laboratory tests. However, use of such fibers is not recommended when the concrete is exposed to chlorides, such as deicing salts.

Inclined or diagonal stirrups lined up approximately with the principal stress directions are more efficient in carrying the shears and preventing or delaying the formation of diagonal cracks. Such stirrups, however, are not usually considered to be very practical in the United States because of the high labor costs required for positioning them. Actually, they can be rather practical for precast concrete beams where the bars and stirrups are preassembled into cages before being used and where the same beams are duplicated many times.

Bent-up bars (usually at 45° angles) are another satisfactory type of web reinforcing (see Figure 8.5). Although bent-up bars are commonly used in flexural members in the United States, the average designer seldom considers the fact that they can resist diagonal tension. Two reasons for not counting their contribution to diagonal tension resistance are that there are only a few, if any, bent-up bars in a beam and that they may not be conveniently located for use as web reinforcement.

Diagonal cracks will occur in beams with shear reinforcing at almost the same loads at which they occur in beams of the same size without shear reinforcing. The shear reinforcing makes its presence known only after the cracks begin to form. At that time, beams must have sufficient shear reinforcing to resist the shear force not resisted by the concrete.

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[3] Ritter, W., 1899, "Die Bauweise Hennebique," *Schweizerische Bauzeitung*, Vol. 33, No. 7.

[4] Mörsch, E., 1912, *Der Eisenbetenbau, seine Theorie und Anwendung* (Stuttgart: Verlag Konrad Wittwer).

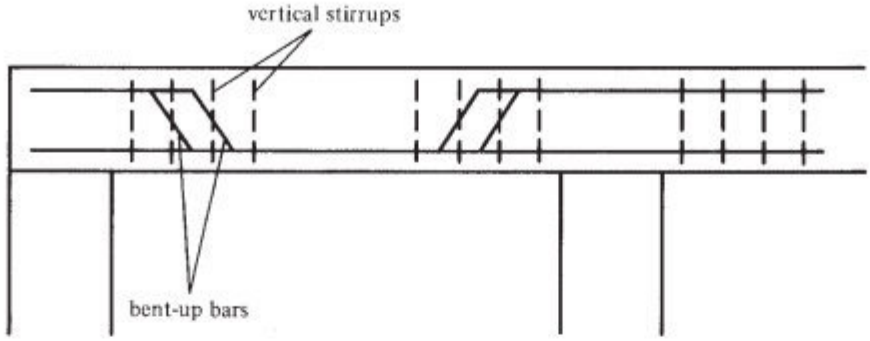

FIGURE 8.5 Bent-up bar web reinforcing.

After a shear crack has developed in a beam, only a little shear can be transferred across the crack unless web reinforcing is used to bridge the gap. When such reinforcing is present, it keeps the pieces of concrete on the two sides of the crack from separating. Several benefits result. These include:

1. The steel reinforcing passing across the cracks carries shear directly.
2. The reinforcing keeps the cracks from becoming larger, and this enables the concrete to transfer shear across the cracks by aggregate interlock.
3. The stirrups wrapped around the core of concrete act like hoops and thus increase the beam's strength and ductility. In a related fashion, the stirrups tie the longitudinal bars into the concrete core of the beam and restrain them from prying off the covering concrete.
4. The holding together of the concrete on the two sides of the cracks helps keep the cracks from moving into the compression zone of the beam. Remember that other than for deformed wire fabric, the yield stress of the web reinforcing is limited to 60 ksi to limit the width of the cracks.

8.8 Design for Shear

The maximum shear, V_u, in a beam must not exceed the design shear capacity of the beam cross section, ϕV_n, where ϕ is 0.75 and V_n is the nominal shear strength of the concrete and the shear reinforcing.

$$V_u \leq \phi V_n$$

The value of ϕV_n can be broken down into the design shear strength of the concrete, ϕV_c, plus the design shear strength of the shear reinforcing, ϕV_s. The value of ϕV_c is provided in the code for different situations, and thus we are able to compute the required value of ϕV_s for each situation:

$$V_u \leq \phi V_c + \phi V_s$$

For this derivation, an equal sign is used:

$$V_u = \phi V_c + \phi V_s$$

The purpose of stirrups is to minimize the size of diagonal tension cracks or to carry the diagonal tension stress from one side of the crack to the other. Very little tension is carried by the stirrups until after a crack begins to form. Before the inclined cracks begin to form, the strain in the stirrups is equal to the strain in the adjacent concrete. Because this concrete cracks at very low diagonal tensile stresses, the stresses in the stirrups at that time are very small, perhaps only 3 ksi to 6 ksi. You can see that these stirrups do not prevent inclined cracks and that they really aren't a significant factor until the cracks begin to develop.

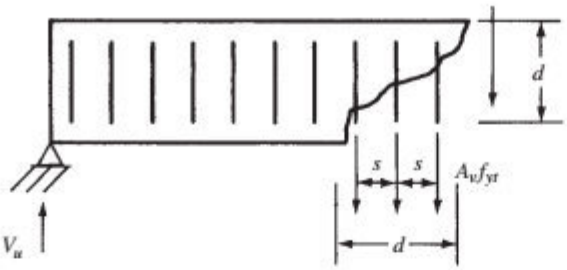

FIGURE 8.6 Beam with diagonal crack and vertical stirrups.

Tests made on reinforced concrete beams show that they will not fail by the widening of the diagonal tension cracks until the stirrups going across the cracks have been stressed to their yield stresses. For the derivation to follow, it is assumed that a diagonal tension crack has developed and has run up into the compression zone but not all the way to the top, as shown in Figure 8.6. It is further assumed that the stirrups crossing the crack have yielded.

The nominal shear strength of the stirrups, V_s, crossing the crack can be calculated from the following expression, where n is the number of stirrups crossing the crack and A_v is the cross-sectional area each stirrup has crossing the crack. If a ⊔ stirrup is used, A_v equals two times the cross-sectional area of the stirrup bar. If it is a ⊔⊓⊔ stirrup, A_v equals four times the cross-sectional area of the stirrup bar. The term f_{yt} is the specified yield strength of transverse reinforcement, or stirrups in this case.

$$V_s = A_v f_{yt} n$$

If it is conservatively assumed that the horizontal projection of the crack equals the effective depth, d, of the section (thus a 45° crack), the number of stirrups crossing the crack can be determined from the expression to follow, in which s is the center-to-center spacing of the stirrups:

$$n = \frac{d}{s}$$

Then

$$V_s = A_v f_{yt} \frac{d}{s} \qquad \text{(ACI Equation 22.5.10.5.3)}$$

From this expression, the required spacing of vertical stirrups is

$$s = \frac{A_v f_{yt} d}{V_s}$$

and the value of V_s can be determined as follows:

$$V_u = \phi V_c + \phi V_s$$

$$V_s = \frac{V_u - \phi V_c}{\phi}$$

Going through a similar derivation, the following expression can be determined for the required area for inclined stirrups, in which α is the angle between the stirrups and the longitudinal axis of the member. Inclined stirrups should be placed so they form an angle of at least 45° with respect to the longitudinal bars, and they must be securely tied in place.

$$V_s = \frac{A_v f_{yt} (\sin\alpha + \cos\alpha) d}{s} \qquad \text{(ACI Equation 22.5.10.5.4)}$$

And for a bent-up bar or a group of bent-up bars at the same distance from the support, we have

$$V_s = A_v f_{yt} \sin\alpha \leq 3\sqrt{f'_c} b_w d \qquad \text{(ACI Equation 22.5.10.6.2)}$$

8.9 ACI Code Requirements

This section presents a detailed list of the code requirements controlling the design of web reinforcing, even though some of these items have been previously mentioned in this chapter:

1. When the factored shear, V_u, exceeds one-half the shear design strength, ϕV_c, the code (Section 22.5.1.1) requires the use of web reinforcing. The value of V_c is normally taken as $2\lambda\sqrt{f'_c}b_w d$, but the code (Section 22.5.5.1) permits the use of the following less conservative value:

$$V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u}\right) b_w d \le 3.5\lambda\sqrt{f'_c} b_w d \qquad \text{(ACI Equation 22.5.5.1)}$$

As previously mentioned, M_u is the moment occurring simultaneously with V_u at the section in question. The value of $V_u d/M_u$ must not be taken as greater than 1.0 in calculating V_c, according to the code.

2. When shear reinforcing is required, the code states that the amount provided must fall between certain clearly specified lower and upper limits. If the amount of reinforcing is too low, it may yield or even snap immediately after the formation of an inclined crack. As soon as a diagonal crack develops, the tension previously carried by the concrete is transferred to the web reinforcing. To prevent the stirrups (or other web reinforcing) from snapping at that time, their area is limited to the minimum value provided at the end of the next paragraph.

ACI Code, Section 9.6.3.3, specifies a minimum amount of web reinforcing so as to provide an ultimate shear strength no less than $0.75\lambda\sqrt{f'_c}b_w s$. Using this provision of the code should prevent a sudden shear failure of the beam when inclined cracks occur. The shear strength calculated with this expression may not be less than $50b_w s$. If a $0.75\lambda\sqrt{f'_c}$ psi strength is available for a web width b_w and a length of beam s equal to the stirrup spacing, we will have

$$0.75\sqrt{f'_c}b_w s = A_v f_{yt}$$

$$A_{v\,\min} = \frac{0.75\sqrt{f'_c}b_w s}{f_{yt}} \qquad \text{(ACI Table 9.6.3.3)}$$

but not less than the value obtained with a 50-psi strength, $50b_w s/f_{yt}$.

If $\sqrt{f'_c}$ is greater than 4444 psi, the minimum value of A_v is controlled by the expression $0.75\sqrt{f'_c}b_w s/f_{yt}$. Should f'_c be less than 4444 psi, the minimum A_v value will be controlled by the $50b_w s/f_{yt}$ expression.

In SI units

$$A_{v\,\min} = 0.062\sqrt{f'_c}\frac{b_w s}{f_{yt}} \ge \frac{0.35 b_w s}{f_{yt}}$$

This expression from ACI Code, Section 9.6.3.3, provides the minimum area of web reinforcing, A_v, that is to be used as long as the factored torsional moment, T_u, does not exceed one-fourth of the cracking torque, T_{cr}. Such a torque will not cause an appreciable reduction in the flexural or shear strength of a member and may be neglected (ACI Code, Section 22.7.4.1). For non-prestressed members, this limiting value is

$$\phi\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)$$

In SI units

$$\phi 0.083\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)$$

In this expression, $\phi = 0.75$, A_{cp} is the area enclosed by the outside perimeter of the concrete cross section, and p_{cp} is the outside perimeter of the concrete cross section. The computation of T_u and T_{cr} for various situations is presented in Chapter 15.

Although you may feel that the use of such minimum shear reinforcing is not necessary, studies of earthquake damage in recent years have shown very large amounts of shear damage occurring in reinforced concrete structures, and it is felt that the use of this minimum value will greatly improve the resistance of such structures to seismic forces. Actually, many designers believe that the minimum area of web reinforcing should be used throughout beams, not just where V_u is greater than $\phi V_c/2$.

This requirement for a minimum amount of shear reinforcing may be waived if tests have been conducted showing that the required bending and shear strengths can be met without the shear reinforcing (ACI Code, Section 7.6.3.2 for one-way slabs and ACI Code, Section 9.6.3.2 for beams).

3. As previously described, stirrups cannot resist appreciable shear unless they are crossed by an inclined crack. Thus, to make sure that each 45° crack is intercepted by at least one stirrup, the maximum spacing of vertical stirrups permitted by the code (9.7.6.2.2) is the lesser of $d/2$ or 24 in. for non-prestressed members and $\frac{3}{4}h$ for prestressed members or 24 in. where h is the overall thickness of a member. Should, however, V_s exceed $4\sqrt{f'_c}b_w d$,[5] these maximum spacings are to be reduced by one-half (ACI Code, Section 9.7.6.2.2). These closer spacings will lead to narrower inclined cracks.

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Nuclear power plant construction.

[5] In SI, $V_s = 0.33\sqrt{f'_c}b_w d$.

Another advantage of limiting maximum spacing values for stirrups is that closely spaced stirrups will hold the longitudinal bars in the beam. They reduce the chance that the steel may tear or buckle through the concrete cover or possibly slip on the concrete.

Under no circumstances may V_s be allowed to exceed $8\sqrt{f'_c}b_w d$ (Code 22.5.1.2).[6] The shear strength of a beam cannot be increased indefinitely by adding more and more shear reinforcing, because the concrete will eventually disintegrate no matter how much shear reinforcing is added. The reader can understand the presence of an upper limit if he or she thinks for a little while about the concrete above the crack. The greater the shear in the member that is transferred by the shear reinforcing to the concrete above, the greater will be the chance of a combination shear and compression failure of that concrete.

4. Section 22.5.3.1 of the code states that the values of $\sqrt{f'_c}$ used for the design of web reinforcing may not exceed 100 psi[7] unless minimum web reinforcement in accordance with ACI Code, Section 9.6.3.3 or 9.6.4.2 is provided. Members meeting these requirements have sufficient postcrack capacities to prevent diagonal tension failures.

5. Section 25.7.1 of the code provides requirements about dimensions, development lengths, and so forth. For stirrups to develop their design strengths, they must be adequately anchored. Stirrups may be crossed by diagonal tension cracks at various points along their depths. Because these cracks may cross very close to the tension or compression edges of the members, the stirrups must be able to develop their yield strengths along the full extent of their lengths. It can then be seen why they should be bent around longitudinal bars of greater diameters than their own and extended beyond by adequate development lengths. Should there be compression reinforcing, the hooking of the stirrups around them will help prevent them from buckling.

Stirrups should be carried as close to the compression and tension faces of beams as the specified cover and longitudinal reinforcing will permit. The ends of stirrup legs should ideally have 135° or 180° hooks bent around longitudinal bars, with development lengths as specified in ACI Code, Section 25.7.1.3. Detailed information on stirrups follows:

(a) Stirrups with 90° bends and $6d_b$ extensions at their free ends may be used for #5 and smaller bars, as shown in Figure 8.7(a). Tests have shown that 90° bends with $6d_b$ extensions should not be used for #6 or larger bars (unless f_y is 40,000 psi or less) because they tend to pop out under high loads.

(b) If f_y is greater than 40,000 psi, #6, #7, and #8 bars with 90° bends may be used if the extensions are $12d_b$ [see Figure 8.7(b)]. The reason for this specification is that it is not possible to bend these higher-strength bars tightly around the longitudinal bars.

(c) Stirrups with 135° bends and $6d_b$ extensions may be used for #8 and smaller bars, as shown in Figure 8.7(c).

6. When a beam reaction causes compression in the end of a member in the same direction as the external shear, the shearing strength of that part of the member is increased. Tests of such reinforced concrete members have shown that, in general, as long as a gradually varying shear is present (as with a uniformly loaded member), the first crack will occur at a distance d from the face of the support. It is therefore permissible, according to the code (Section 9.4.3.2), to decrease somewhat the calculated shearing force for a distance d from the face of the support. This is done by using a V_u in that range equal to the calculated V_u at a distance d from the face of the support. Should a concentrated load be applied in this region, no such shear reduction is permitted. Such loads will be transmitted directly to the support above the 45° cracks, with the result that we are not permitted a reduction in the end shear for design purposes.

Should the reaction tend to produce tension in this zone, no shear stress reduction is permitted, because tests have shown that cracking may occur at the face of the support or even inside it. Figure 8.8

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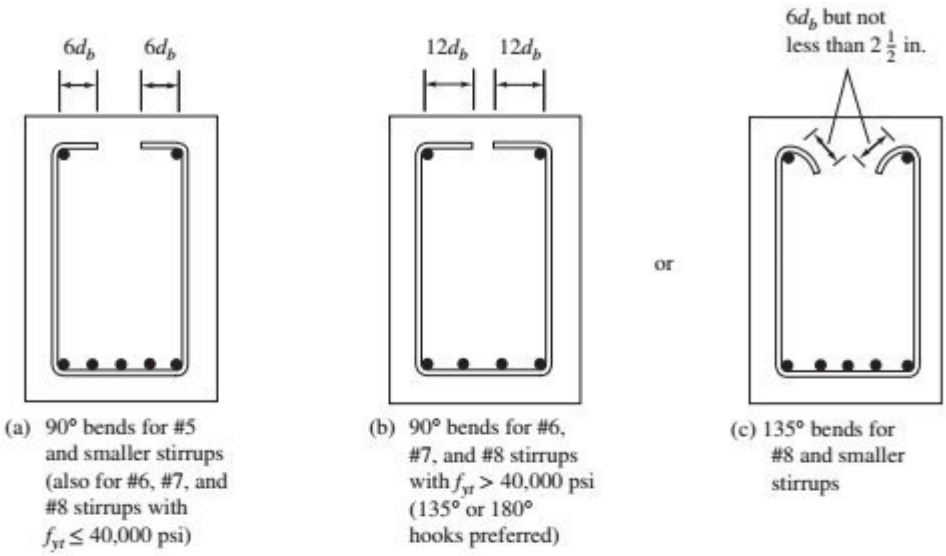
[6] It's $0.66\sqrt{f'_c}b_w d$ in SI units.
[7] It's 8.3 MPa in SI units.



(a) 90° bends for #5 and smaller stirrups (also for #6, #7, and #8 stirrups with $f_{yt} \leq 40{,}000$ psi)

(b) 90° bends for #6, #7, and #8 stirrups with $f_{yt} > 40{,}000$ psi (135° or 180° hooks preferred)

(c) 135° bends for #8 and smaller stirrups

Note: Fit stirrups as close to compression and tension surfaces as cover and other reinforcing permits.

FIGURE 8.7 Stirrup details.

shows two cases where the end shear reduction is not permitted. In the situation shown in Figure 8.8(a), the critical section will be at the face of the support. In Figure 8.8(b), an I-shaped section is shown, with the load applied to its tension flange. The loads have to be transferred across the inclined crack before they reach the support. Another crack problem like this one occurs in retaining wall footings and is discussed in Section 13.10 of this text.

7. Various tests of reinforced concrete beams of normal proportions with sufficient web reinforcing have shown that shearing forces have no significant effect on the flexural capacities of the beams. Experiments with deep beams, however, show that large shears will often keep those members from developing their full flexural capacities. As a result, the code requirements given in the preceding paragraphs are not applicable to beams whose clear spans divided by their effective depths are less than four or for regions of beams that are loaded with concentrated loads within a distance from the support equal to the member depth and that are loaded on one face and supported on the opposite face. Such a situation permits the development of compression struts between the loads and the supports. For such members as these, the code in its Appendix A provides an alternate method of design, which is referred to as "strut and tie" design. This method is briefly described in Appendix C of this text. Should the loads be applied through the sides or bottom of such members, their shear design should be handled as it is for ordinary beams. Members falling into this class include beams, short cantilevers, and corbels.

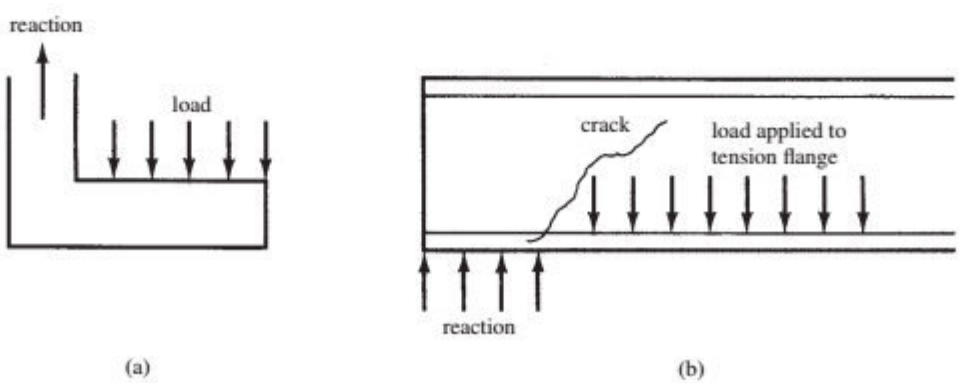


FIGURE 8.8 Two situations where end shear reduction is not permitted.

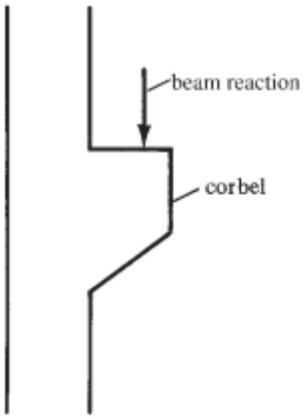


FIGURE 8.9 Corbel supporting beam reaction.

Corbels are brackets that project from the sides of columns and are used to support beams and girders, as shown in Figure 8.9. They are quite commonly used in precast construction. Special web reinforcing provisions are made for such members in Section 9.9 of the code and are considered in Section 8.12 of this chapter.

8. Section 9.8.1.5 of the ACI Code permits a shear of $1.1V_c$ for the ribs of joist construction, as where we have closely spaced T beams with tapered webs. For the 10% increase in V_c, the joist proportions must meet the provisions of ACI Code, Section 9.8. In ACI Code, Section 9.8, it is stated that the ribs must be no less than 4 in. wide, must have depths not more than three-and-a-half times the minimum width of the ribs, and may not have clear spacings between the ribs greater than 30 in.

8.10 Shear Design Example Problems

Example 8.1 illustrates the selection of a beam with a sufficiently large cross section so that no web reinforcing is required. The resulting beam is unusually large. It is normally considered much better practice to use appreciably smaller sections constructed with web reinforcing. The reader should also realize that it is good construction practice to use some stirrups in all reinforced concrete beams (even though they may not be required by shear) because they enable the workers to build for each beam a cage of steel that can be conveniently handled.

Example 8.1

Determine the minimum cross section required for a rectangular beam from a shear standpoint so that no web reinforcing is required by the ACI Code if $V_u = 38$ k and $f'_c = 4000$ psi. Use the conservative value of $V_c = 2\lambda\sqrt{f'_c}b_w d$.

SOLUTION

Shear strength provided by concrete is determined by the equation

$$\phi V_c = (0.75)\left[2\,(1.0)\,(\sqrt{4000}\text{ psi})b_w d\right] = 94.87 b_w d$$

But the ACI Code, Section 9.6.3.1, states that a minimum area of shear reinforcement is to be provided if V_u exceeds $\frac{1}{2}\phi V_c$

$$38{,}000\text{ lb} = \frac{1}{2}(94.87 b_w d)$$

$$b_w d = 801.1\text{ in.}^2$$

Use 24-in. × 36-in. beam ($d = 33.5$ in.)

The design of web reinforcing is illustrated by Examples 8.2 through 8.6. Maximum vertical stirrup spacings have been given previously, whereas no comment has been made about minimum spacings. Stirrups must be spaced far enough apart to permit the aggregate to pass through, and, in addition, they must be reasonably few in number so as to keep within reason the amount of labor involved in fabricating and placing them. Accordingly, minimum spacings of 3 in. or 4 in. are normally used. Usually #3 stirrups are assumed, and if the calculated design spacings are less than $d/4$, larger-diameter stirrups can be used. Another alternative is to use ⊔⊓⊔ stirrups instead of ⊔ stirrups. Different diameter stirrups should not be used in the same beam, or confusion will result.

As is illustrated in Examples 8.3, 8.5, and 8.6, it is quite convenient to draw the V_u diagram and carefully label it with values of such items as ϕV_c, $\phi V_c/2$, and V_u at a distance d from the face of the support and to show the dimensions involved.

Some designers place their first stirrup a distance of one-half of the end—calculated spacing requirement from the face. Others put the first stirrup 2 in. or 3 in. from the support.

From a practical viewpoint, stirrups are usually spaced with center-to-center dimensions that are multiples of 3 in. or 4 in. to simplify the fieldwork. Although this procedure may require an additional stirrup or two, total costs should be less because of reduced labor costs. A common field procedure is to place chalk marks at 2-ft intervals on the forms and to place the stirrups by eye in between those marks. This practice is combined with a somewhat violent placing of the heavy concrete in the forms, followed by vigorous vibration. These field practices should clearly show the student that it is foolish to specify odd theoretical stirrup spacings such as $4 \text{ @ } 6\frac{7}{16}$ in. and $6 \text{ @ } 5\frac{3}{8}$ in., because such careful positioning will not be achieved in the actual members. Thus, the designer will normally specify stirrup spacings in multiples of whole inches and perhaps in multiples of 3 in. or 4 in.

With available computer programs, it is easily possible to obtain theoretical arrangements of stirrups with which the least total amounts of shear reinforcing will be required. The use of such programs is certainly useful to the designer, but he or she needs to take the resulting values and revise them into simple economical patterns with simple spacing arrangements—as in multiples of 3 in., for example.

A summary of the steps required to design vertical stirrups is presented in Table 8.1. For each step, the applicable section number of the code is provided. The authors have found this to be a very useful table for students to refer to while designing stirrups.

TABLE 8.1 Summary of Steps Involved in Vertical Stirrup Design

Is Shear Reinforcing Necessary?	
1. Draw V_u diagram.	9.4.3.1
2. Calculate V_u at a distance d from the support (with certain exceptions).	9.4.3.2
3. Calculate $\phi V_c = 2\phi\lambda\sqrt{f'_c}b_w d$ (or use the alternate method).	22.5.5.1
4. Stirrups are needed if $V_u > \frac{1}{2}\phi V_c$ (with some exceptions for slabs, footings, shallow members, hollow-core units, steel fiber–reinforced beams, and joists).	9.6.3.1
Design of Stirrups	
1. Calculate theoretical stirrup spacing, $s = A_v f_{yt} d/V_s$ where $V_s = (V_u - \phi V_c)/\phi$.	22.5.10.5.3
2. Determine maximum spacing to provide minimum area of shear reinforcement, $s = A_v f_{yt}/0.75\sqrt{f'_c}b_w$ but not more than $A_v f_{yt}/50b_w$.	9.6.3.3
3. Compute maximum spacing: $d/2 \le 24$ in. if $V_s \le 4\sqrt{f'_c}b_w d$.	9.7.6.2.2
4. Compute maximum spacing: $d/4 \le 12$ in. if $V_s > 4\sqrt{f'_c}b_w d$.	9.7.6.2.2
5. V_s may not be $> 8\sqrt{f'_c}b_w d$.	22.5.1.2
6. Minimum practical spacing $\approx$ 3 in. or 4 in.	

Example 8.2

The beam shown in Figure 8.10 was selected using $f_y = 60{,}000$ psi and $f'_c = 3000$ psi, normal weight. Determine the theoretical spacing of #3 ⊔ stirrups for each of the following shears:

(a) $V_u = 12{,}000$ lb
(b) $V_u = 40{,}000$ lb
(c) $V_u = 60{,}000$ lb
(d) $V_u = 150{,}000$ lb

SOLUTION

(a) $V_u = 12{,}000$ lb (using $\lambda = 1.0$ for normal-weight concrete)

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}\,b_w d = (0.75)\,[2(1.0)\sqrt{3000}\text{ psi}]\,(14\text{ in.})\,(24\text{ in.}) = 27{,}605\text{ lb}$$

$$\frac{1}{2}\phi V_c = 13{,}803\text{ lb} > 12{,}000\text{ lb} \qquad \therefore \text{ Stirrups not required}$$

(b) $V_u = 40{,}000$ lb

Stirrups needed because $V_u > \frac{1}{2}\phi V_c$.

Theoretical spacing

$$\phi V_c + \phi V_s = V_u$$

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{40{,}000\text{ lb} - 27{,}605\text{ lb}}{0.75} = 16{,}527\text{ lb}$$

$$s = \frac{A_v f_{yt} d}{V_s} = \frac{(2)\,(0.11\text{ in.}^2)\,(60{,}000\text{ psi})\,(24\text{ in.})}{16{,}527\text{ lb}} = 19.17\text{ in.}$$

Maximum spacing to provide minimum A_v

$$s = \frac{A_v f_{yt}}{0.75\sqrt{f'_c}\,b_w} = \frac{(2)\,(0.11\text{ in.}^2)\,(60{,}000\text{ psi})}{(0.75\sqrt{3000}\text{ psi})\,(14\text{ in.})} = 22.95\text{ in.}$$

$$s = \frac{A_v f_{yt}}{50 b_w} = \frac{(2)\,(0.11\text{ in.}^2)\,(60{,}000\text{ psi})}{(50)\,(14\text{ in.})} = 18.86\text{ in.}$$

$$V_s = 16{,}527\text{ lb} < (4)\,(\sqrt{3000}\text{ psi})\,(14\text{ in.})\,(24\text{ in.}) = 73{,}614\text{ lb}$$

$$\therefore \text{ Maximum } s = \frac{d}{2} = 12\text{ in.} \leftarrow \text{controls} \qquad s = 12.0\text{ in.}$$

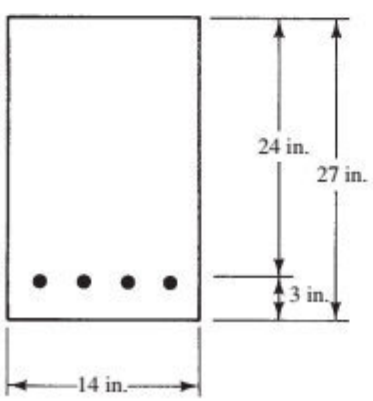


FIGURE 8.10 Beam cross section for Example 8.2.

(c) $V_u = 60{,}000$ lb

Theoretical spacing

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{60{,}000 \text{ lb} - 27{,}605 \text{ lb}}{0.75} = 43{,}193 \text{ lb}$$

$$s = \frac{A_v f_{yt} d}{V_s} = \frac{(2)\,(0.11 \text{ in.}^2)\,(60{,}000 \text{ psi})\,(24 \text{ in.})}{43{,}193 \text{ lb}} = 7.33 \text{ in.} \leftarrow \text{controls}$$

Maximum spacing to provide minimum A_v

$$s = \frac{A_v f_{yt}}{0.75\sqrt{f'_c} b_w} = \frac{(2)\,(0.11 \text{ in.}^2)\,(60{,}000 \text{ psi})}{(0.75\sqrt{3000} \text{ psi})\,(14 \text{ in.})} = 22.95 \text{ in.}$$

$$s = \frac{A_v f_{yt}}{50 b_w} = \frac{(2)\,(0.11 \text{ in.}^2)\,(60{,}000 \text{ psi})}{(50)\,(14 \text{ in.})} = 18.86 \text{ in.}$$

$$V_s = 43{,}193 \text{ lb} < (4)\,(\sqrt{3000} \text{ psi})\,(14 \text{ in.})\,(24 \text{ in.}) = 73{,}614 \text{ lb}$$

$$\therefore \text{Maximum } s = \frac{d}{2} = 12 \text{ in.} \qquad \underline{s = 7.33 \text{ in.}}$$

(d) $V_u = 150{,}000$ lb

$$V_s = \frac{150{,}000 \text{ lb} - 27{,}605 \text{ lb}}{0.75} = 163{,}193 \text{ lb}$$

$$163{,}193 \text{ lb} > (8)\,(\sqrt{3000} \text{ psi})\,(14 \text{ in.})\,(24 \text{ in.}) = 147{,}228 \text{ lb}$$

V_s may not be taken $> 8\sqrt{f'_c} b_w d$

$$\underline{\therefore \text{Need larger beam and/or one with larger } f'_c \text{ value}}$$

Example 8.3

Select #3 ⊔ stirrups for the beam shown in Figure 8.11, for which $w_D = 4$ k/ft and $w_L = 6$ k/ft. $f'_c = 4000$ psi, normal weight, and $f_{yt} = 60{,}000$ psi.

SOLUTION

V_u at the face of the left support = (7 ft)(1.2 × 4 klf + 1.6 × 6 klf) = 100.8 k = 100,800 lb

$$V_u \text{ at a distance } d \text{ from face of support} = \left(\frac{84 \text{ in.} - 22.5 \text{ in.}}{84 \text{ in.}}\right)(100{,}800 \text{ lb}) = 73{,}800 \text{ lb}$$

$$\phi V_c = \phi 2\lambda\sqrt{f'_c} b_w d = (0.75)\,[2(1.0)\sqrt{4000} \text{ psi}]\,(15 \text{ in.})\,(22.5 \text{ in.}) = 32{,}018 \text{ lb}$$

These values are shown in Figure 8.12.

$$V_u = \phi V_c + \phi V_s$$

$$\phi V_s = V_u - \phi V_c = 73{,}800 \text{ lb} - 32{,}018 \text{ lb} = 41{,}782 \text{ lb}$$

$$V_s = \frac{41{,}782 \text{ lb}}{0.75} = 55{,}709 \text{ lb}$$

Maximum spacing of stirrups = $d/2 = 11.25$ in., since V_s is $< 4\sqrt{f'_c}b_w d = 85{,}382$ lb. Maximum theoretical spacing at a distance d from the face of the left support is

$$s = \frac{A_v f_{yt} d}{V_s} = \frac{(2)\,(0.11\text{ in.}^2)\,(60{,}000\text{ psi})\,(22.5\text{ in.})}{(55{,}709\text{ lb})} = 5.33\text{ in.} \leftarrow \text{controls}$$

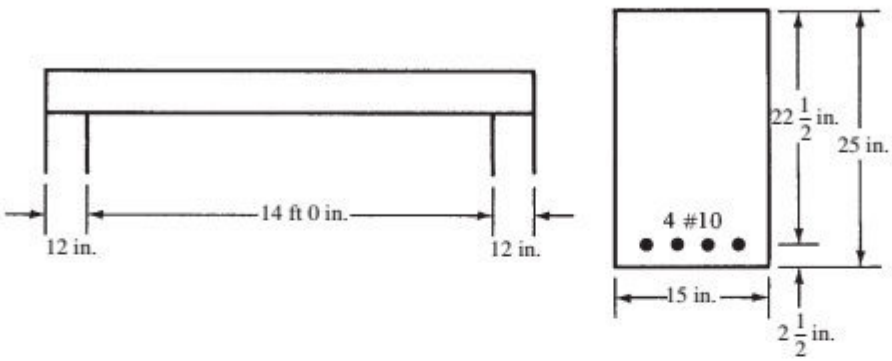


FIGURE 8.11 Given information for Example 8.3.

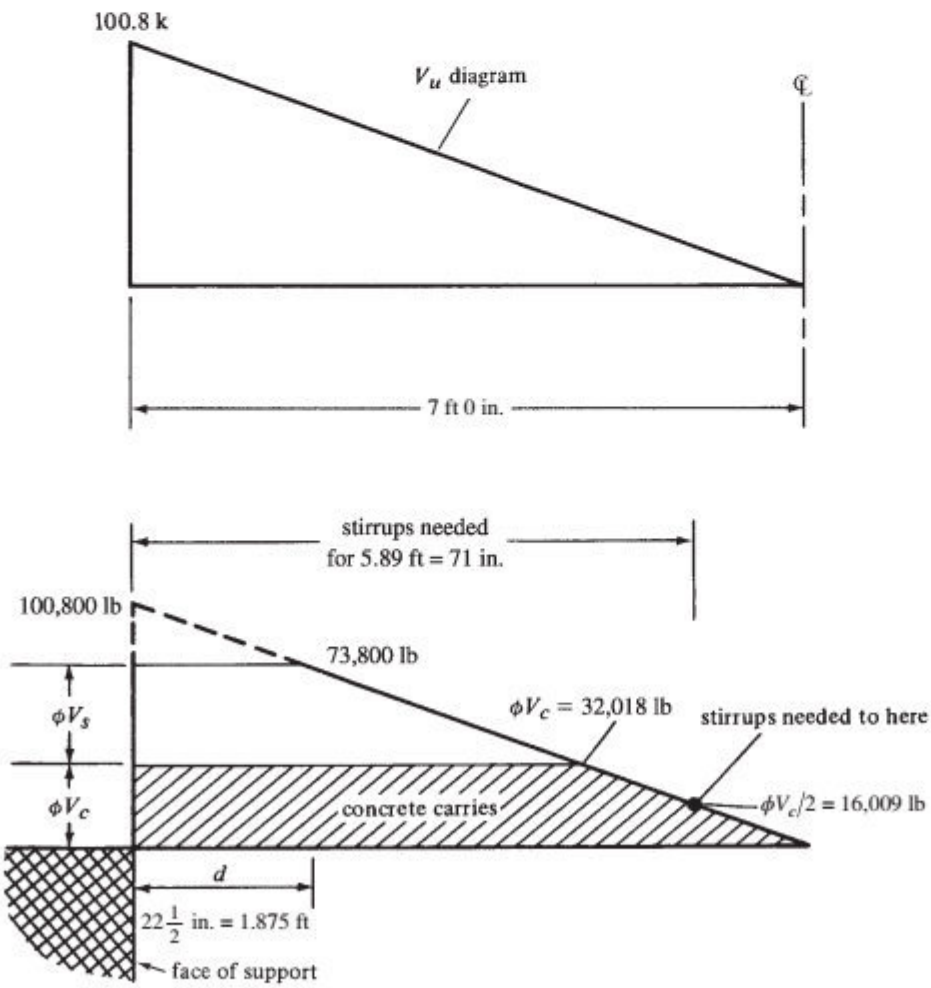


FIGURE 8.12 Shear diagram for Example 8.3.

Maximum spacing to provide minimum A_v of stirrups

$$s = \frac{A_v f_{yt}}{0.75\sqrt{f'_c} b_w} = \frac{(2)\ (0.11\ \text{in.}^2)\ (60{,}000\ \text{psi})}{(0.75\sqrt{4000}\ \text{psi})\ (15\ \text{in.})} = 18.55\ \text{in.}$$

$$s = \frac{A_v f_{yt}}{50 b_w} = \frac{(2)\ (0.11\ \text{in.}^2)\ (60{,}000\ \text{psi})}{(50)\ (15\ \text{in.})} = 17.6\ \text{in.}$$

At what location is $s = 9$ in. OK?

$$V_u = \phi V_c + \phi V_s = 32{,}018\ \text{lb} + 0.75\left[\frac{A_v f_y d}{s}\right]$$

$$= 32{,}018\ \text{lb} + \frac{(0.75)(2)(0.11\ \text{in.}^2)(60{,}000\ \text{psi})(22.5\ \text{in.})}{9\ \text{in.}}$$

$$= 56{,}768\ \text{lb}$$

$$V_u = 100{,}800\ \text{lb} - 14{,}400x = 56{,}768\ \text{lb},\ x = 3.058\ \text{ft} = 36.69\ \text{in.}$$

At what location can stirrups be terminated? Termination occurs when $V_u < 1/2\phi V_c$

$$V_u = 100{,}800\ \text{lb} - 14{,}400x = 16{,}009\ \text{lb},\ x = 5.888\ \text{ft} = 70.66\ \text{in.}$$

Results of similar calculations that relate the value of x to stirrup spacing, s, are shown in the table.

Distance from Face of Support (in.)	V_u (lb)	$V_s = \dfrac{V_u - \phi V_c}{\phi}$ (lb)	Theoretical $s = \dfrac{A_v f_{yt} d}{V_s}$ (in.)
0 to $d = 22.5$	73,800	55,709	5.33
26.38	69,143	49,500	6
30.80	63,842	42,432	7
34.11	59,863	37,126	8
40.82	51,819	26,401	$d/2 = 11.25$

Spacings selected

Distance	Cumulative Distance	
1 @ 2 in. = 2 in.	2 in.	
8 @ 5 in. = 40 in.	42 in. > 36.69 in	
4 @ 9 in. = 36 in.	78 in. > 70.66 in.	
73 in.		Symmetric about centerline

As previously mentioned, it is a good practice to space stirrups at multiples of 3 in. or 4 in. on center. As an illustration, it is quite reasonable to select for Example 8.3 the following spacings: 1 @ 2 in., 8 @ 5 in., and 4 @ 9 in. A total of 13 stirrups is required at each end of the beam. Many other options are possible, perhaps requiring fewer stirrups. For example, 1 @ 2 in., 5 @ 5 in., 2 @ 7 in. and 2 @ 11 in. also satisfy the shear requirements with 11 instead of 13 stirrups at each end. However, it is slightly more complicated than the first stirrup schedule. Some design offices might even use 1 @ 2 in. and 14 @ 5 in., just to keep things simple.

In Example 8.4, which follows, the value of V_c for the beam of Example 8.3 is computed by the alternate method of Section 22.5.5.1 of the code.

Example 8.4

Compute the value of V_c at a distance 3 ft from the face of the left support of the beam of Example 8.3 and Figure 8.11 by using ACI Table 22.5.5.1.

$$V_c = \left(1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_u}\right) b_w d \le 3.5\lambda\sqrt{f'_c} b_w d$$

SOLUTION

$$\lambda = 1.0 \text{ (normal-weight aggregate)}$$
$$w_u = (1.2)\ (4 \text{ klf}) + (1.6)(6 \text{ klf}) = 14.4 \text{ k/ft}$$

Measuring x from the center of the left support, the value of x corresponding to 3 ft from the face is 3.5 ft.

$$V_u = \frac{w_u l}{2} - w_u x = \frac{(14.4 \text{ k/ft})\ (15\text{ft})}{2} - (14.4 \text{ k/ft})(x)$$

$$= 57.6 \text{ k (at } x = 3.5 \text{ ft from center of the left support)}$$

$$M_u = \frac{w_u l x}{2} - \frac{w_u x^2}{2} = \frac{(14.4 \text{ k/ft})\ (15 \text{ ft})\ (3.5 \text{ ft})}{2} - \frac{(14.4 \text{ k/ft})(3.5 \text{ ft})^2}{2}$$

$$= (14.4 \text{ k/ft})\ (100.8 \text{ k}) - (3 \text{ ft})\ (1.5 \text{ ft})\ (14.4 \text{ k/ft}) = 289.8 \text{ ft-k}$$

$$\rho_w = \frac{5.06 \text{ in.}^2}{(15 \text{ in.})\ (22.5 \text{ in.})} = 0.0150$$

$$\frac{V_u d}{M_u} = \frac{(57.6 \text{ k})\ (22.5 \text{ in.})}{(12)\ (289.8 \text{ ft-k})} = 0.374 < 1.0$$

$$V_c = [1.9(1.0)\sqrt{4000} \text{ psi} + (2500)\ (0.0150)\ (0.374)]\ (15 \text{ in.})\ (22.5 \text{ in.})$$

$$= \underline{\underline{45{,}290 \text{ lb}}} < (3.5\sqrt{4000} \text{ psi})\ (15 \text{ in.})\ (22.5 \text{ in.}) = 74{,}709 \text{ lb}$$

For the uniformly loaded beams considered up to this point, it has been assumed that both dead and live loads extended from end to end of the spans. Although this practice will produce the maximum V_u at the ends of simple spans, it will not produce maximums at interior points. For such points, maximum shears will be obtained when the uniform live load is placed from the point in question to the most distant end support. For Example 8.5, shear is determined at the beam end (live load running for entire span) and then at the beam centerline (live load to one side only), and a straight-line relationship is assumed in between. Although the ACI does not specifically comment on the variable positioning of live load to produce maximum shears, it certainly is its intent for engineers to position loads so as to maximize design shear forces.

Example 8.5

Select #3 ⊔ stirrups for the beam of Example 8.3, assuming the live load is placed to produce maximum shear at beam end and centerline.

SOLUTION

Maximum V_u at left end = (7 ft) (1.2 × 4 klf + 1.6 × 6 klf) = 110.8 k = 100,800 lb.

For maximum V_u at centerline, the live load is placed as shown in Figure 8.13.

$$V_u \text{ at centerline} = 50{,}400 \text{ lb} - (7 \text{ ft})\ (1.2 \times 4 \text{ klf}) = 16.8 \text{ k} = 16{,}800 \text{ lb}$$

$$V_c = 2(1.0)\ (\sqrt{4000} \text{ psi})\ (15 \text{ in.})\ (22.5 \text{ in.}) = 42{,}691 \text{ lb}$$

V_u at a distance d from face of support = 78,300 lb as determined by proportions from Figure 8.14.

$$V_u = \phi V_c + \phi V_s$$

$$\phi V_s = V_u - \phi V_c = 78{,}300 \text{ lb} - (0.75)\ (42{,}691 \text{ lb}) = 46{,}282 \text{ lb at left end}$$

$$V_s = \frac{46{,}282 \text{ lb}}{0.75} = 61{,}709 \text{ lb}$$

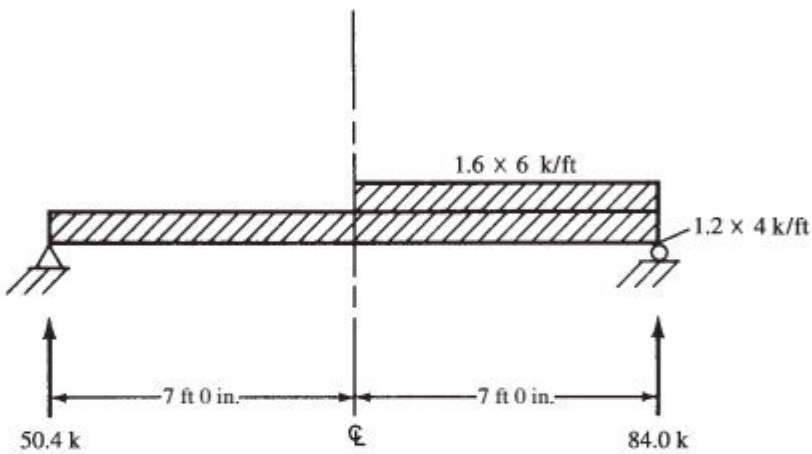


FIGURE 8.13 Load arrangement for maximum shear at beam midspan.

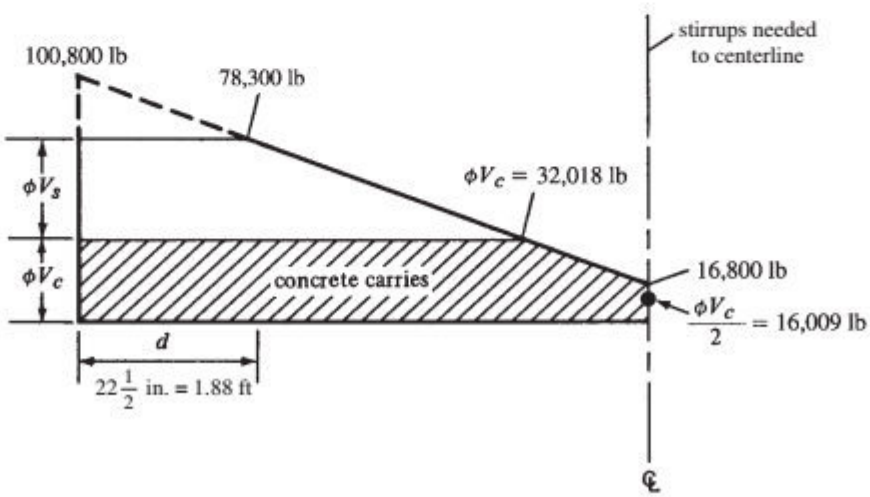


FIGURE 8.14 Shear diagram for Example 8.5.

At what location is $s = 9$ in. OK?

$$V_u = \phi V_c + \phi V_s = 32{,}018 + 0.75\left(\frac{A_v f_{yt} d}{s}\right) = 32{,}016 + \frac{(0.75)(2)(0.11\text{ in.}^2)(60{,}000\text{ psi})(22.5\text{ in.})}{9\text{ in.}}$$

$$= 56{,}768\text{ lb}$$

$$V_u = 100{,}800 - 12{,}000x = 56{,}768, x = 3.67\text{ ft} = 44.0\text{ in.}$$

Results of similar calculations that relate the value of x to stirrup spacing, s, are shown in the table.

The limiting spacings are the same as in Example 8.3. The theoretical spacings are given in the following table:

Distance from Face of Support (ft)	V_u (lb)	$V_s = \frac{V_u - \phi V_c}{\phi}$ (lb)	Theoretical Spacing Required $s = \frac{A_v f_{yt} d}{V_s}$ (in.)
0 to $d = 1.875$	78,300	61,709	4.81
2	76,800	59,709	4.97
2.638	69,143	49,500	6
3.67	56,768	33,000	9
5	40,800	11,709	> Maximum 11.25

One possible arrangement (#4 stirrups might be better)

1 @ 2 in. = 2 in.
8 @ 4 in. = 32 in.
2 @ 6 in. = 12 in.
5 @ 9 in. = 45 in.
91 in. > 84 in.

Symmetrical about centerline

Example 8.6

Select spacings for #3 ⊔ stirrups for a T beam with $b_w = 10$ in. and $d = 20$ in. for the V_u diagram shown in Figure 8.15, with $f_y = 60{,}000$ psi and $f'_c = 3000$ psi, normal-weight concrete.

SOLUTION

(with reference to Figure 8.16)

V_u at a distance d from face of support

$$= 44{,}000\text{ lb} + \left(\frac{72\text{ in.} - 20\text{ in.}}{72\text{ in.}}\right)(68{,}000\text{ lb} - 44{,}000\text{ lb}) = 61{,}333\text{ lb}$$

$\lambda = 1.0$ for normal-weight concrete

$$\phi V_c = (0.75)\,[2(1.0)\sqrt{3000}\text{ psi}]\,(10\text{ in.})\,(20\text{ in.}) = 16{,}432\text{ lb}$$

$$\frac{\phi V_c}{2} = \frac{16{,}432\text{ lb}}{2} = 8216\text{ lb}$$

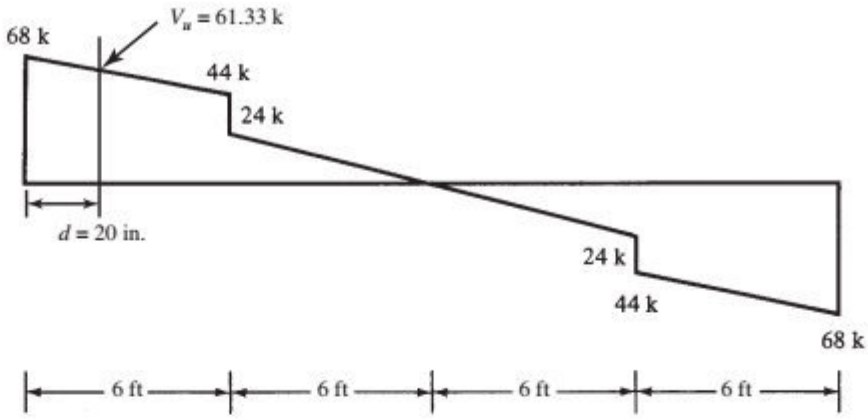


FIGURE 8.15 Shear diagram for Example 8.6.

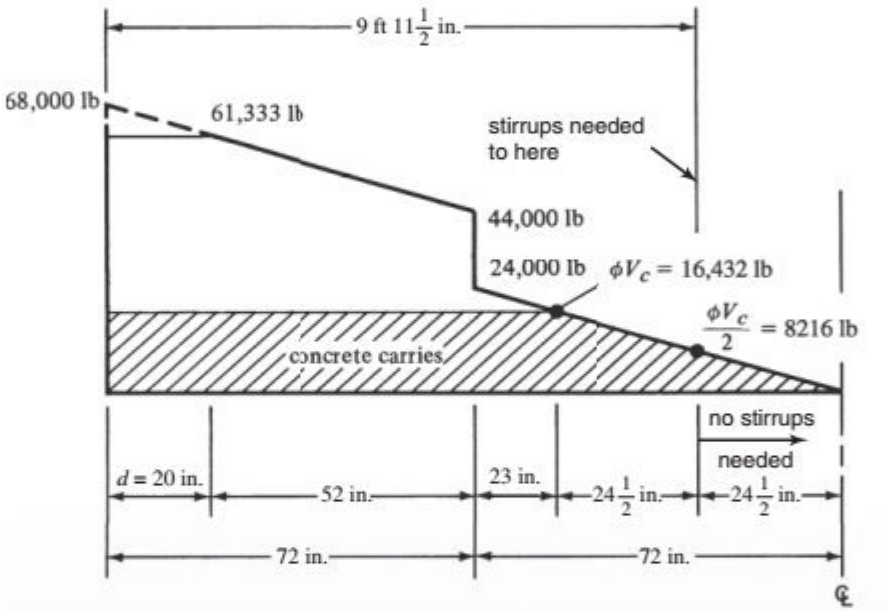


FIGURE 8.16 More detailed shear diagram for Example 8.6.

$$\text{Stirrups are needed for a distance} = 72 \text{ in.} + \left(\frac{24{,}000 \text{ lb} - 8216 \text{ lb}}{24{,}000 \text{ lb}}\right)(72 \text{ in.}) = 119.5 \text{ in.}$$

V_s at left end $(V_u - \phi V_c)/\phi = (61{,}333 \text{ lb} - 16{,}432 \text{ lb})/0.75 = 59{,}868$ lb, which is larger than $4\sqrt{f'_c}b_w d = (4\sqrt{3000} \text{ psi})(10 \text{ in.})(20 \text{ in.}) = 43{,}818$ lb but less than $8\sqrt{f'_c}b_w d$. Therefore, the maximum spacing of stirrups in that range is $d/4 = 5$ in.

Maximum spacing of stirrups $= d/4 = 20/4 = 5$ in. when $V_s > 4\sqrt{f'_c}b_w d = (4\sqrt{3000} \text{ psi})(10 \text{ in.})(20 \text{ in.}) = 43{,}818$ lb. By proportions from the V_s column in the table, V_s falls to 43,818 lb at approximately 4.66 ft, or 56 in., from the left end of the beam.

Maximum spacing permitted to provide minimum A_v of stirrups is the smaller of the two following values of s.

$$s = \frac{A_v f_{yt}}{0.75\sqrt{f'_c}b_w} = \frac{(2)(0.11 \text{ in.}^2)(60{,}000 \text{ psi})}{(0.75\sqrt{3000} \text{ psi})(10 \text{ in.})} = 32.13 \text{ in.}$$

$$s = \frac{A_v f_{yt}}{50 b_w} = \frac{(2)(0.11 \text{ in.}^2)(60{,}000 \text{ psi})}{(50)(10 \text{ in.})} = 26.4 \text{ in.}$$

The theoretical spacings at various points in the beam are computed in the following table:

Distance from Face of Support (in.)	V_u (lb)	$V_s = \dfrac{V_u - \phi V_c}{\phi}$ (lb)	Theoretical Spacing Required $s = \dfrac{A_v f_{yt} d}{V_s}$ (in.)	Maximum Spacing (in.)
0 to $d = 20$	61,333	59,868	4.41	4.41
36	56,000	52,758	5.00	5.00
72–	44,000	36,757	7.18	5.00
72+	24,000	10,091	26.16	10.00

A summary of the results of the preceding calculations is shown in Figure 8.17, where the solid dark line represents the maximum stirrup spacings permitted by the code and the dashed line represents the calculated theoretical spacings required for $V_u - \phi V_c$.

From this information, the authors selected the following spacings:

1 @ 3 in. = 3 in.
17 @ 4 in. = 68 in.
5 @ 10 in. = 50 in.
121 in. Symmetrical about centerline

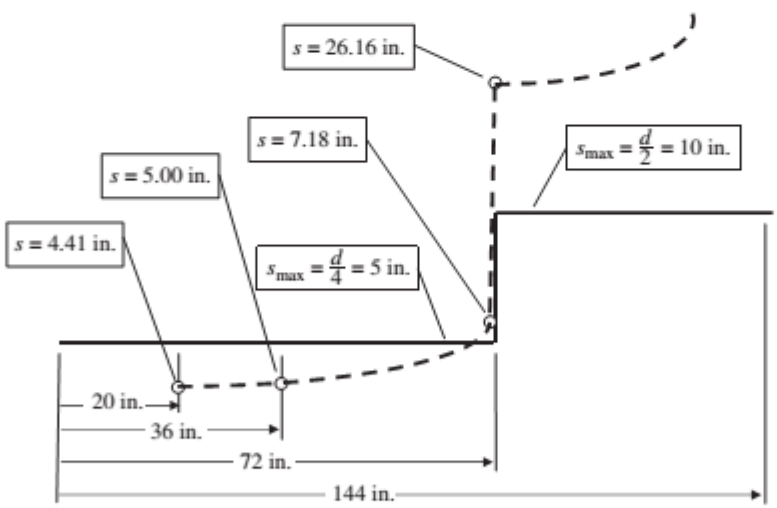


FIGURE 8.17 Detailed shear diagram for Example 8.6.

8.11 Economical Spacing of Stirrups

When stirrups are required in a reinforced concrete member, the code specifies maximum permissible spacings varying from $d/4$ to $d/2$. However, it is usually thought that stirrup spacings less than $d/4$ are rather uneconomical. Many designers use a maximum of three different spacings in a beam. These are

$d/4$, $d/3$, and $d/2$. It is easily possible to derive a value of ϕV_s for each size and style of stirrups for each of these spacings.[8]

Note that the number of stirrups is equal to d/s and that if we use spacings of $d/4$, $d/3$, and $d/2$ we can see that n equals 4, 3, or 2. Then the value of ϕV_s can be calculated for any particular spacing, size, and style of stirrup. For instance, for #3 ⊔ stirrups spaced at $d/2$ with $\phi = 0.75$ and $f_y = 60$ ksi,

$$\phi V_s = \frac{\phi A_v f_{yt} d}{s} = \frac{(0.75)\,(2 \times 0.11 \text{ in.}^2)\,(60 \text{ ksi})\,(d)}{d/2} = 19.8 \text{ k}$$

The values shown in Table 8.2 were computed in this way for 60-ksi stirrups.

For an example using this table, reference is made to the beam and V_u diagram of Example 8.3, which was shown in Figure 8.12 where 60-ksi #3 ⊔ stirrups were selected for a beam with a d of $22\frac{1}{2}$ in. For our closest spacing, $d/4$, we can calculate $\phi V_c + 39.6 \text{ k} = 32.018 \text{ k} + 39.6 \text{ k} = 71.6 \text{ k}$. Similar calculations are made for $d/3$ and $d/2$ spacings, and we obtain, respectively, 61.7 k and 51.8 k. The shear diagram is repeated in Figure 8.18, and the preceding values are located on the diagram by proportions or by scaling.

TABLE 8.2 Values for 60-ksi Stirrups

s	ϕV_s for #3 ⊔ Stirrups (k)	ϕV_s for #4 ⊔ Stirrups (k)
$d/2$	19.8	36
$d/3$	29.7	54
$d/4$	39.6	72

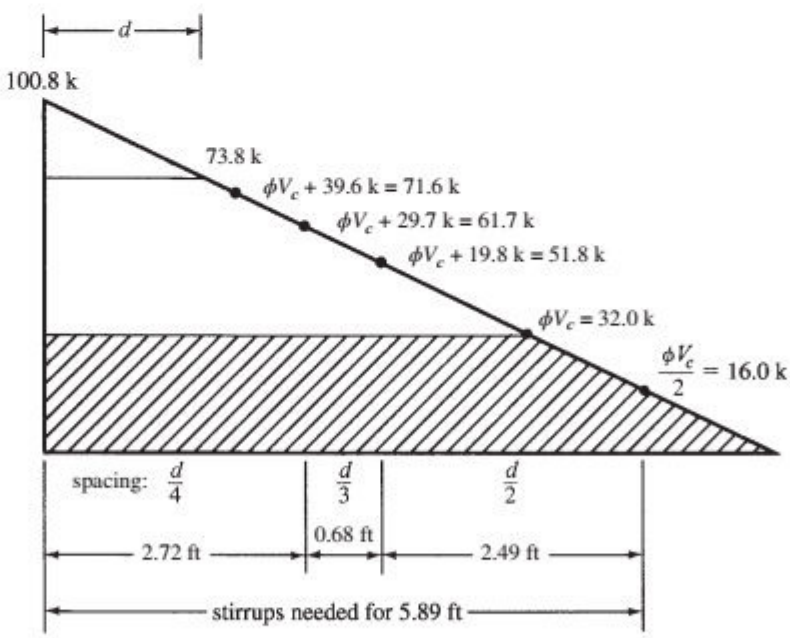


FIGURE 8.18 Application of Table 8.2 to Example 8.3.

[8] Neville, B. B., ed., 1984, *Simplified Design Reinforced Concrete Buildings of Moderate Size and Height* (Skokie, IL: Portland Cement Association), pp. 3–12 to 3–16.

From this information, we can see that we can use $d/4$ for the first 2.72 ft, $d/3$ for the next 0.68 ft, and $d/2$ for the remaining 2.49 ft. Then the spacings are smoothed (preferably to multiples of 3 in.). Also, for this particular beam, we would probably use the $d/4$ spacing on through the 0.68-ft section and then use $d/2$ the rest of the required distance.

8.12 Shear Friction and Corbels

If a crack occurs in a reinforced concrete member (whether caused by shear, flexure, shrinkage, etc.) and if the concrete pieces on opposite sides of the crack are prevented from moving apart, there will be a great deal of resistance to slipping along the crack due to the rough and irregular concrete surfaces. If reinforcement is provided across the crack to prevent relative displacement along the crack, shear will be resisted by friction between the faces, by resistance to shearing off of protruding portions of the concrete, and by dowel action of the reinforcement crossing the crack. The transfer of shear under these circumstances is called *shear friction*.

Shear friction failures are most likely to occur in short, deep members subject to high shears and small bending moments. These are the situations where the most nearly vertical cracks will occur. If moment and shear are both large, diagonal tension cracks will occur at rather large angles from the vertical. This situation has been discussed in Sections 8.1 through 8.11.

A short cantilever member having a ratio of clear span to depth (a/d) of 1.0 or less is often called a *bracket* or *corbel*. One such member is shown in Figure 8.19. The shear friction concept provides a convenient method for designing for cases where diagonal tension design is not applicable. The most common locations where shear friction design is used are for brackets, corbels, and precast connections, but it may also be applied to the interfaces between concretes cast at different times, to the interfaces between concrete and steel sections, and so on.

When brackets or corbels or short, overhanging ends or precast connections support heavy concentrated loads, they are subject to possible shear friction failures. The dashed lines in Figure 8.19 show the probable locations of these failures. It will be noted that for the end-bearing situations, the cracks tend to occur at angles of about 20° from the direction of the force application.

Space is not taken in this chapter to provide an example of shear friction design, but a few general remarks are presented. (In Section 12.13 of this text, a numerical shear friction example is presented

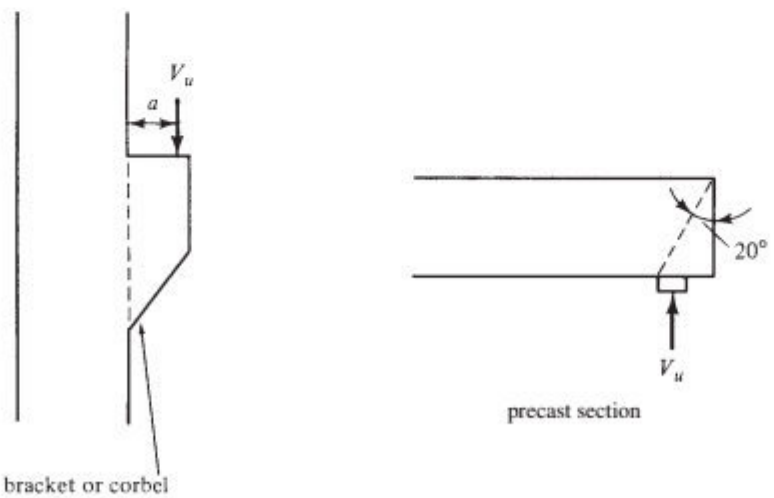


FIGURE 8.19 Possible shear friction failures.

in relation to the transfer of horizontal forces at the base of a column to a footing.) It is first assumed that a crack has occurred, as shown by the dashed lines of Figure 8.19. As slip begins to occur along the cracked surface, the roughness of the concrete surfaces tends to cause the opposing faces of the concrete to separate.

As the concrete separates, it is resisted by the tensile reinforcement (A_{vf}) provided across the crack. It is assumed that this steel is stretched until it yields. (An opening of the crack of 0.01 in. will probably be sufficient to develop the yield strength of the bars.) The clamping force developed in the bars, $A_{vf}f_y$, will provide a frictional resistance equal to $A_{vf}f_y\mu$, where μ is the coefficient of friction (values of which are provided for different situations in Section 22.9.4.2 of the code).

Then the design shear strength of the member must at least equal the shear, to be taken as

$$\phi V_n = V_u = \phi A_{vf} f_y \mu$$

The value of f_y used in this equation cannot exceed 60 ksi, and the shear friction reinforcement across or perpendicular to the shear crack may be obtained by

$$A_{vf} = \frac{V_u}{\phi f_y \mu}$$

This reinforcement should be appropriately placed along the shear plane. If there is no calculated bending moment, the bars will be uniformly spaced. If there is a calculated moment, it will be necessary to distribute the bars in the flexural tension area of the shear plane. The bars must be anchored sufficiently on both sides of the crack to develop their yield strength by means of embedment, hooks, headed bars, or other methods. Because space is often limited in these situations, it is often necessary to weld the bars to special devices, such as crossbars or steel angles. The bars should be anchored in confined concrete (i.e., column ties or external concrete or other reinforcement shall be used).

When beams are supported on brackets or corbels, there may be a problem with shrinkage and expansion of the beams, producing horizontal forces on the bracket or corbel. When such forces are present, the bearing plate under the concentrated load should be welded down to the tensile steel. Based on various tests, the ACI Code (Section 16.5.3.5) says that the horizontal force used must be at least equal to $0.2V_u$ unless special provisions are made to avoid tensile forces.

The presence of direct tension across a cracked surface obviously reduces the shear-transfer strength. Thus direct compression will increase its strength. As a result, Section 22.9.4.5 of the code permits the use of a permanent compressive load to increase the shear friction clamping force. A typical corbel design and its reinforcing are shown in Figure 8.20.

Enough concrete area must also be provided, and Section 22.9.4.4 of the code gives the upper limits on the shear force, V_n, transferred across a shear-friction failure surface based on concrete strength and contact area. For normal-weight concrete placed monolithically or placed against intentionally roughened concrete, V_n cannot exceed the smaller of

$$\begin{aligned} V_n &\le 0.2f'_cA_c \\ &\le (480 + 0.08f'_c)A_c \\ &\le 1600A_c \end{aligned}$$

For all other cases,

$$\begin{aligned} V_n &\le 0.2f'_cA_c \\ &\le 800A_c \end{aligned}$$

where A_c is the concrete contact area along the shear-friction failure surface. Units for these equations are: V_n(lb), f'_c(lb/in.^{2}), and A_c (in.^{2}).

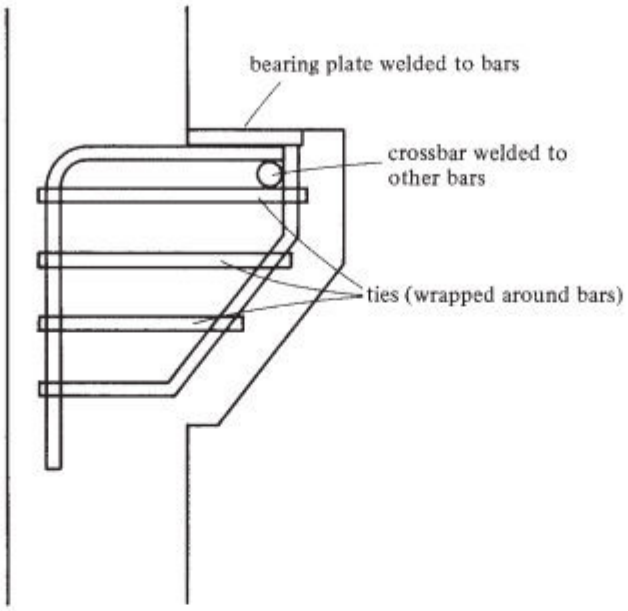


FIGURE 8.20 Example of corbel.

8.13 Shear Strength of Members Subjected to Axial Forces

Reinforced concrete members subjected to shear forces can at the same time be loaded with axial compression or axial tension forces due to wind, earthquake, gravity loads applied to horizontal or inclined members, shrinkage in restrained members, and so on. These forces can affect the shear design of our members. Compressive loads tend to prevent cracks from developing. As a result, they provide members with larger compressive areas and thus greater shear strengths. Tensile forces exaggerate cracks and reduce shear resistances because they will decrease compression areas.

When we have appreciable axial compression, the following equation can be used to compute the shear-carrying capacity of a concrete member:

$$V_c = 2\left(1 + \frac{N_u}{2000A_g}\right)\lambda\sqrt{f'_c}\,b_w d \qquad \text{(ACI Equation 22.5.6.1)}$$

For a member subjected to a significant axial tensile force, the shear capacity of the concrete may be determined from the following expression:

$$V_c = 2\left(1 + \frac{N_u}{500A_g}\right)\lambda\sqrt{f'_c}\,b_w d \qquad \text{(ACI Equation 22.5.7.1)}$$

In this expression, N_u, the axial load, is negative if the load is tensile. You might note that if the computed value of N_u/A_g for use in this equation is 500 psi or more, the concrete will have lost its capacity to carry shear. (The value of V_c used need not be taken as less than zero.)

The SI values for ACI Equations 22.5.6.1 and 22.5.7.1 are, respectively,

$$V_c = 0.17\left(1 + \frac{N_u}{14A_g}\right)(\lambda\sqrt{f'_c})b_w d$$

$$V_c = 0.17\left(1 + \frac{0.29N_u}{A_g}\right)(\lambda\sqrt{f'_c})b_w d$$

Instead of using ACI Equation 22.5.6.1 to compute the shear capacity of sections subject to axial compressive loads, ACI Table 22.5.6.1 may be used. In this equation, a revised moment, M_m, may be substituted for M_u at the section in question, and $V_u d/M_u$ is not limited to 1.0 as it normally is. Equation 22.5.6.1a is not applicable if M_m is negative.

$$\left(1.9\lambda\sqrt{f'_c} + 2500\rho_w \frac{V_u d}{M_m}\right) b_w d \le 3.5\lambda\sqrt{f'_c} b_w d \qquad \text{(ACI Equation 22.5.6.1a)}$$

$$M_m = M_u - N_u\left(\frac{4h-d}{8}\right)$$

$$V_c \text{ may not be} > 3.5\lambda\sqrt{f'_c} b_w d\sqrt{1+\frac{N_u}{500A_g}} \qquad \text{(ACI Equation 22.5.6.1b)}$$

In SI units, ACI Equation 22.5.6.1a is

$$V_c = \left(0.16\lambda\sqrt{f'_c} + 17\rho_w \frac{V_u d}{M_u}\right) b_w d \le 0.29\lambda\sqrt{f'_c} b_w d$$

Equation 11-7 is

$$V_c = 0.29\lambda\sqrt{f'_c} b_w d\sqrt{1+\frac{0.3N_u}{A_g}}$$

Example 8.7, which follows, illustrates the computation of the shear strength of an axially loaded concrete member.

Example 8.7

For the concrete section shown in Figure 8.21 for which f'_c is 3000 psi, normal weight ($\lambda = 1.0$),

(a) Determine V_c if no axial load is present using ACI Equation 22.5.5.1.

(b) Compute V_c using ACI Equation 22.5.6.1 if the member is subjected to an axial compression load of 12,000 lb.

(c) Repeat part (b) using revised ACI Equation 22.5.6.1a. At the section in question, assume $M_u = 30$ ft-k and $V_u = 40$ k. Use M_m in place of M_u.

(d) Compute V_c if the 12,000-lb load is tensile.

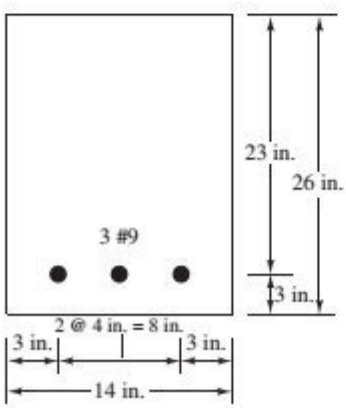


FIGURE 8.21 Beam cross section for Example 8.7.

SOLUTION

(a) $V_c = 2(1.0)\sqrt{3000}\ \text{psi}(14\ \text{in.})\ (23\ \text{in.}) = \underline{\underline{35{,}273\ \text{lb}}}$

(b) $V_c = 2\left[1 + \dfrac{20{,}000\ \text{lb}}{(2000)\ (14\ \text{in.})(26\ \text{in.})}\right]\ [(1.0)\sqrt{3000}\ \text{psi}]\ (14\ \text{in.})\ (23\ \text{in.}) = \underline{\underline{36{,}242\ \text{lb}}}$

$< 3.5(1.0)\sqrt{3000}\text{psi}(14\ \text{in.})\ (23\ \text{in.}) = 61{,}728\ \text{lb}$

(c) $M_m = (12\ \text{in/ft})\ (30{,}000\ \text{ft-lb}) - 12{,}000\ \text{lb}\left(\dfrac{4 \times 26\ \text{in.} - 23\ \text{in.}}{8}\right) = 238{,}500\ \text{in-lb}$

$\dfrac{V_u d}{M_m} = \dfrac{(40\ \text{k})\ (23\ \text{in.})}{238.5\ \text{in-k}} = 3.857 > 1.00,$ however, it is not limited to 1.0

$$V_c = \left[1.9(1.0)\sqrt{3000}\ \text{psi} + (2500)\left(\frac{3.00\ \text{in.}^2}{(14\ \text{in.})\ (23\ \text{in.})}\right)(3.857)\right](14\ \text{in.})\ (23\ \text{in.})$$

$$= \underline{\underline{62{,}437\ \text{lb}}}$$

But not $> 3.5(1.0)\sqrt{3000}\ \text{psi}\ (14\ \text{in.})\ (23\ \text{in.})\sqrt{1 + \dfrac{12{,}000\ \text{lb}}{(500\ \text{psi})\ (14\ \text{in.})\ (26\ \text{in.})}}$

$= 63{,}731\ \text{lb}$ $\underline{\underline{\text{OK}}}$

(d) $V_c = 2\left[1 + \dfrac{-12{,}000\ \text{lb}}{(500)\ (14\ \text{in.})\ (26\ \text{in.})}\right]\ [(1.0)\sqrt{3000}\ \text{psi}(14\ \text{in.})\ (23\ \text{in.})]$

$= \underline{\underline{32{,}950\ \text{lb}}}$

8.14 Shear Design Provisions for Deep Beams

There are some special shear design provisions given in Section 9.9 of the code for deep flexural members with ℓ_n/d values equal to or less than four that are loaded on one face and supported on the other face, so that compression struts can develop between the loads and the supports. Such a member is shown in Figure 8.22(a). Some members falling into this class are short, deep, heavily loaded beams; wall slabs under vertical loads; shear walls; and perhaps floor slabs subjected to horizontal loads.

If the loads are applied through the sides or the bottom (as where beams are framing into its sides or bottom) of the member, as illustrated in Figure 8.22(b) and (c), the usual shear design provisions described earlier in this chapter are to be followed, regardless of whether the member is deep.

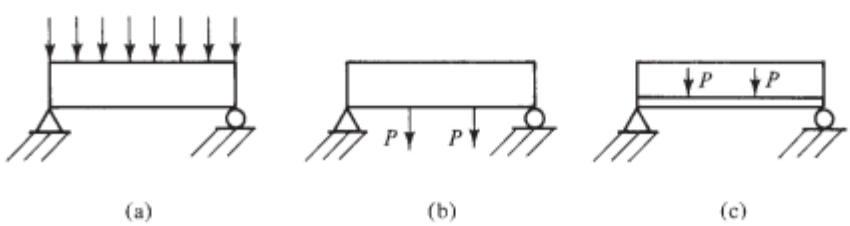


FIGURE 8.22 Deep beam configurations.

The angles at which inclined cracks develop in deep flexural members (measured from the vertical) are usually much smaller than 45°—on some occasions being very nearly vertical. As a result, web reinforcing when needed has to be more closely spaced than for beams of regular depths. Furthermore, the web reinforcing needed is in the form of both horizontal and vertical reinforcing. These almost vertical cracks indicate that the principal tensile forces are primarily horizontal, and thus horizontal reinforcing is particularly effective in resisting them.

The detailed provisions of the code relating to shear design for deep beams, together with the applicable ACI Section numbers, are as follows:

1. Deep beams are to be designed using the procedure described in Chapter 23 of the code (Appendix C in this textbook) or by using a nonlinear analysis (ACI Code, Section 9.9.1.2).
2. The nominal shear strength, V_n, for deep beams shall not exceed $10\sqrt{f'_c}b_w d$ (ACI Code, Section 9.9.2.1).
3. The area of shear reinforcing, A_v, perpendicular to the span must at least equal $0.0025\, b_w s$, and s may not be greater than $d/5$ or 12 in. (ACI Code, Section 9.9.3.1a). s is the spacing of the shear or torsion reinforcement measured in a direction parallel to the logitudinal reinforcement.
4. The area of shear reinforcement, A_{vh}, parallel to the span must not be less than $0.0025 b_w s_2$, and s_2 may not be greater than $d/5$ or 12 in. (ACI Code, Section 9.9.3.1b). s_2 is the spacing of the distributed longitudinal reinforcement.

You will note that more vertical than horizontal shear reinforcement is required because vertical reinforcement has been shown to be more effective than horizontal reinforcement. The subject of deep beams is continued in Appendix C of this textbook.

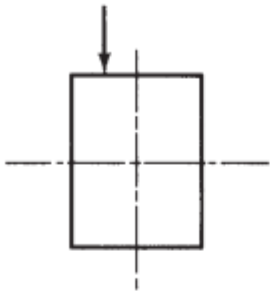
8.15 Introductory Comments on Torsion

Until recent years, the safety factors required by design codes for proportioning members for bending and shear were very large, and the resulting large members could almost always be depended upon to resist all but the very largest torsional moments. Today, however, with the smaller members selected using the strength design procedure, this is no longer true, and torsion needs to be considered much more frequently.

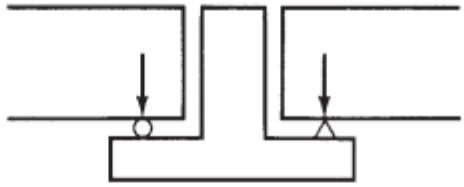
Torsion may be very significant for curved beams, spiral staircases, beams that have large loads applied laterally off center, and even spandrel beams running between exterior building columns. These latter beams support the edges of floor slabs, floor beams, curtain walls, and façades, and they are loaded laterally on one side. Several situations where torsion can be a problem are shown in Figure 8.23.

When plain concrete members are subjected to pure torsion, they will crack along 45° spiral lines when the resulting diagonal tension exceeds the design strength of the concrete. Although these diagonal tension stresses produced by twisting are very similar to those caused by shear, they will occur on all faces of a member. As a result, they add to the stresses caused by shear on one side of the beam and subtract from them on the other.

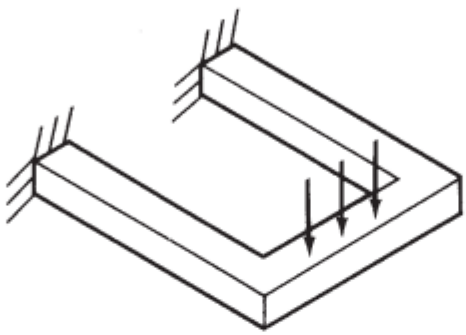
Reinforced concrete members subjected to large torsional forces may fail quite suddenly if they are not provided with torsional reinforcing. The addition of torsional reinforcing does not change the magnitude of the torsion that will cause diagonal cracks, but it does prevent the members from tearing apart. As a result, they will be able to resist substantial torsional moments without failure. Tests have shown that both longitudinal bars and closed stirrups or spirals are necessary to intercept the numerous diagonal tension cracks that occur on all surfaces of beams subject to appreciable torsional forces. There must be a longitudinal bar in each corner of the stirrups to resist the horizontal components of the diagonal tension caused by torsion. Chapter 15 of this text is completely devoted to torsion.



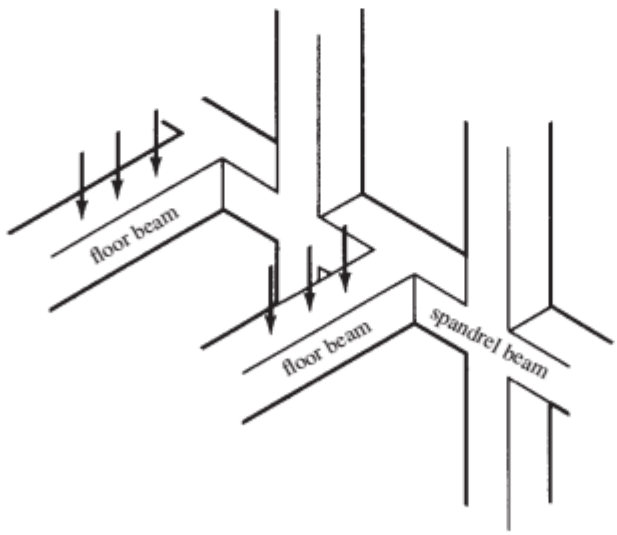
(a) Rectangular beam with off-center load



(b) Inverted T beam supporting beam reactions



(c) Balcony beams



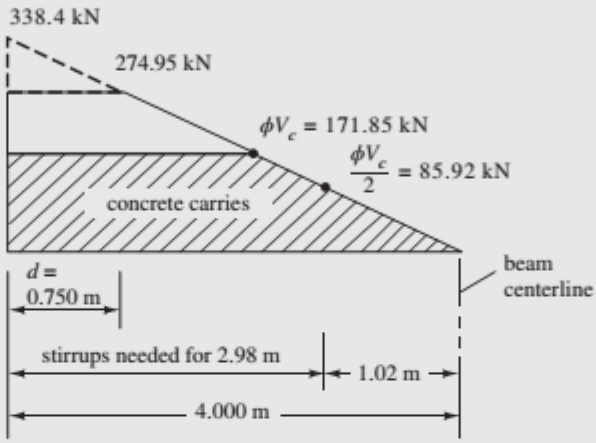
(d) Spandrel beam with torsion caused by floor beams

FIGURE 8.23 Some situations where torsion stresses may be significant.

8.16 SI Example

Example 8.8

Determine required spacing of #10 stirrups at the left end of the beam shown in Figure 8.24 if $f'_c = 21$ MPa, normal weight, and $f_y = 420$ MPa.



SOLUTION

$$V_u \text{ @ left end} = (4 \text{ m})(84.6 \text{ kN/m}) = 338.4 \text{ kN}$$

$$V_u \text{ @ a distance } d \text{ from left end}$$

$$= 338.4 \text{ kN} - \left(\frac{750 \text{ mm}}{1000 \text{ mm}}\right)(84.6 \text{ kN/m}) = 274.95 \text{ kN}$$

$$\phi V_c = (0.17\phi)\,(\lambda\sqrt{f'_c})\,b_w d = (0.17)(0.75)\,[(1.0)\,(\sqrt{21} \text{ MPa})]\,(400 \text{ mm})\,(750 \text{ mm})$$

$$= 175\,284 \text{ N} = 175.28 \text{ kN}$$

$$V_u = \phi V_c + \phi V_s$$

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{274.95 \text{ kN} - 175.28 \text{ kN}}{0.75} = 132.88 \text{ kN}$$

Assuming #10 Stirrups

$$\text{Theoretical } s = \frac{A_v f_{yt} d}{V_s} = \frac{(2)(71 \text{ mm}^2)(420 \text{ MPa})(750 \text{ mm})}{(132.88 \text{ kN})\,(10^3)} = 336 \text{ mm}$$

Maximum s to provide minimum A_v for stirrups is the lesser of (ACI Table 9.6.3.3)

$$s = A_v f_{yt}/0.062\sqrt{f'_c}b_w = (2 \times 71\ \text{mm}^2)(420\ \text{MPa})/(0.062)(\sqrt{21\ \text{MPa}})(400\ \text{mm}) = 525\ \text{mm}$$

and

$$s = \frac{A_v f_{yt}}{b_w} = \frac{(2 \times 71\ \text{mm}^2)\ (420\ \text{MPa})}{(0.35)400\ \text{mm}} = 426\ \text{mm}$$

$$\therefore \text{Maximum } s = \frac{d}{2} = \frac{700\ \text{mm}}{2} = 375\ \text{mm} \leftarrow \text{controls}$$

Use $s = 325$ mm.

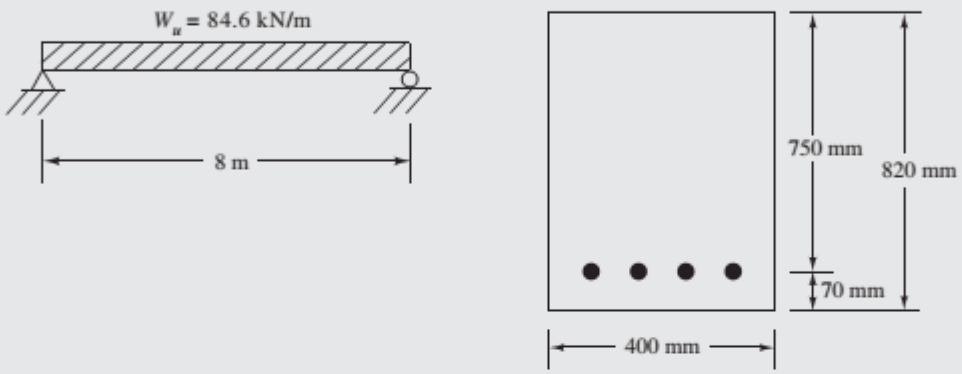


FIGURE 8.24 Given information for Example 8.8.

8.17 Computer Example

Example 8.9

Repeat Example 8.2(c) using the Excel spreadsheet provided for Chapter 8.

SOLUTION

Open the spreadsheet and enter values in the cells highlighted in shaded (only in the Excel spreadsheets, not the printed example). These include values for V_u, f'_c, λ, b_w, d, A_v, and f_{yt}. The required stirrup spacing s is shown in cell C19 ($s = 7.33$ in.). Use good judgment to enter an actual value for spacing in the cell (choose s). A value of choose $s = 7.00$ in. is shown. This value must not exceed the calculated value of s as well as the "Controlling s_{max}" listed a few cells. In the cell labeled "Check $\phi V_c + \phi V_s$" is the shear capacity of the section with the actual stirrup spacing you entered in "choose s." It will exceed the input value of V_u if the design is OK. In this case, the capacity is 61,548 lb, which exceeds V_u of 60,000 lb. Several warnings will appear if your "choose s =" value is too large.

Shear Design—Beams

	V_u =	60,000	lb				
	f'_c =	3000	psi				
	λ =	1					
	b_w =	14	in.				
	d =	24	in.				
	A_v =	0.22	in.^{2}				
	f_{yt} =	60,000	psi				
	ϕ =	0.75					
	V_c =	36,807	lb				
	ϕV_c =	27,605	lb	-			
	$\frac{1}{2}\phi V_c$ =	13,803	lb				
$V_s = (V_u - \phi V_c)/\phi$ =		43,193	lb				
				-			
Required ϕV_s =		32,395	lb				
	s =	7.33	in.				
choose s =		7.00	in.	-			
	s_{max} =	12	in.			-	
	$A_{v\ min}$ =	0.082	in.^{2}			-	
	s_{max} =	22.95	in.				
	s_{max} =	18.86	in.				
Controlling s_{max} =		12.00	in.				
Actual ϕV_s =		33,943	lb				
Check $\phi V_c - \phi V_s$ =		61,548	lb				

PROBLEMS

Problem 8.1 The ACI Code provides the following limiting shear values for members subject only to shear and flexure: $2\sqrt{f'_c}$, $4\sqrt{f'_c}$, and $8\sqrt{f'_c}$. What is the significance of each of these limits?

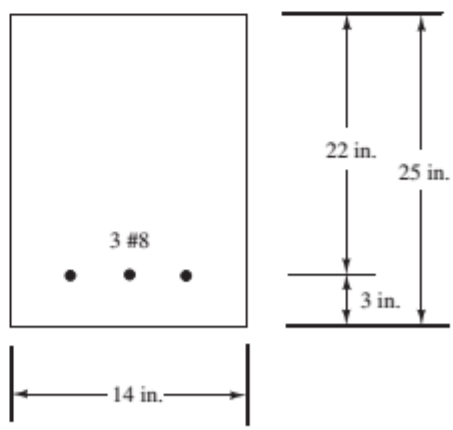
Problem 8.2 If the maximum shear force in a member occurs at a support, the code permits the designer to calculate the shear at a distance d from the face of the support in the presence of a certain condition. Describe the situation when this reduced shear may be used.

Problem 8.3 Why does the code limit the maximum design yield stress that may be used in the design calculations for shear reinforcing to 60,000 psi (not including welded wire fabric)?

Problem 8.4 What is shear friction and where is it most likely to be considered in reinforced concrete design?

Shear Analysis

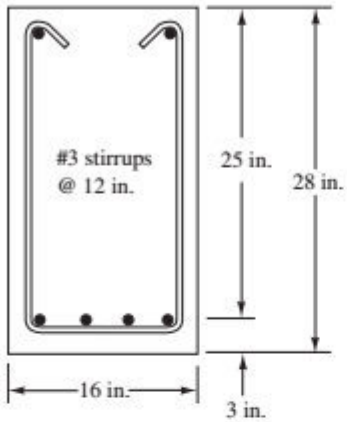
Problem 8.5 What is the design shear strength of the beam shown if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi? No shear reinforcing is provided. (*Ans.* theoretical $\phi V_c = 29{,}219$ lb, $\phi V_c/2 = 14{,}610$ lb controls)



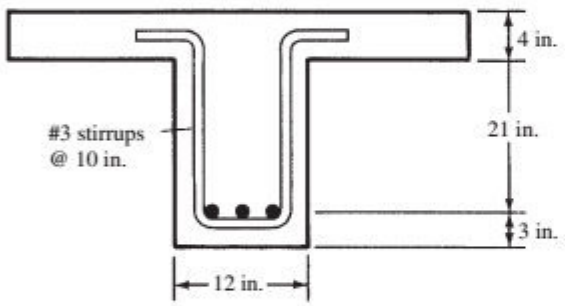
Problem 8.6 Repeat Problem 8.5 if the total depth of the beam is 32 in. and $f'_c = 3000$ psi.

For Problems 8.7 to 8.9, compute ϕV_n for the sections shown if f_{yt} of stirrups is 60 ksi and $f'_c = 4000$ psi.

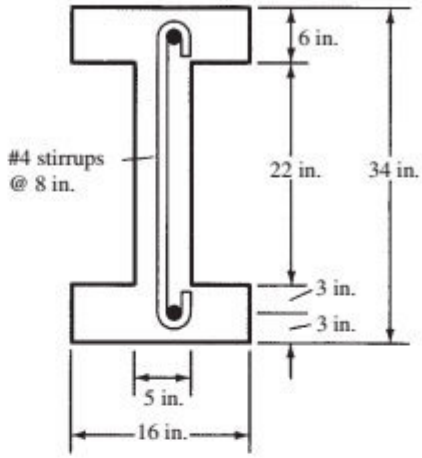
Problem 8.7 (*Ans.* 58,572 lb)



Problem 8.8



Problem 8.9 (*Ans.* 44,114 lb)

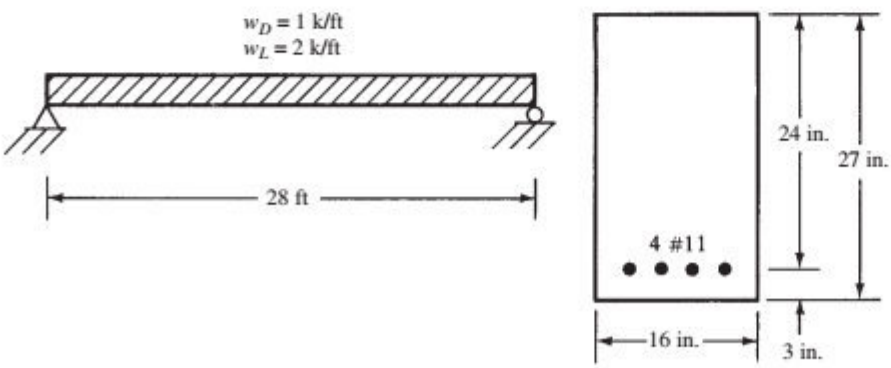


Shear Design

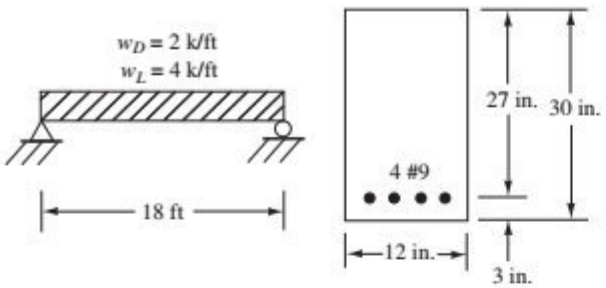
Problem 8.10 If $f'_c = 3000$ psi, $V_u = 60$ k, and $b_w = \frac{1}{2}d$, select a rectangular beam section if no web reinforcement is used. Use sand-lightweight concrete. b_w is an integer inch.

For Problems 8.11 to 8.19, for the beams and loads given, select stirrup spacings if $f'_c = 4000$ psi normal-weight concrete and $f_{yt} = 60{,}000$ psi. The dead loads shown include beam weights. Do not consider movement of live loads unless specifically requested. Assume #3 ⊔ stirrups unless given otherwise.

Problem 8.11 (*One ans.* 1 @ 6 in., 10 @ 12 in.)

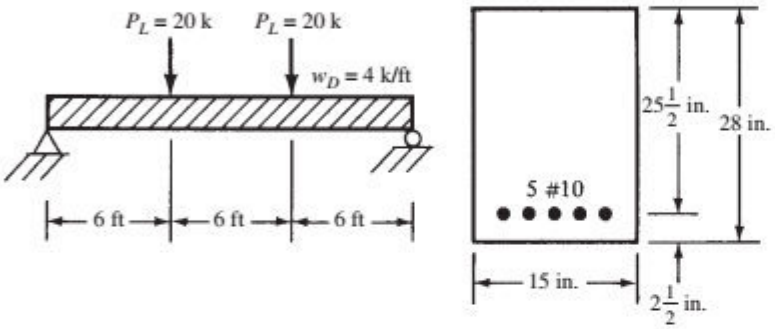


Problem 8.12

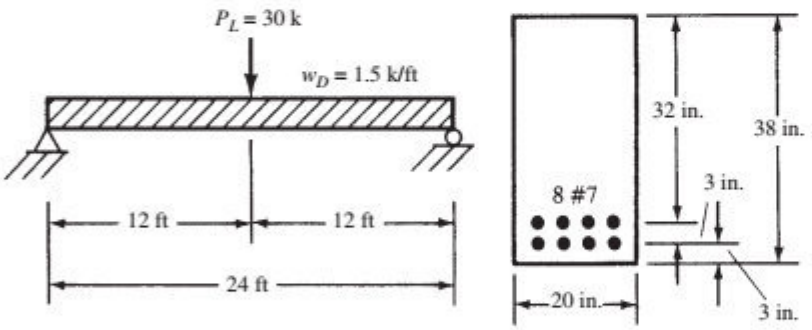


Problem 8.13 Repeat Problem 8.12 if live load positions are considered to cause maximum end and centerline shear. (*One ans.* 1 @ 4 in., 4 @ 8 in., 2 @ 10 in., 4 @ 13 in.)

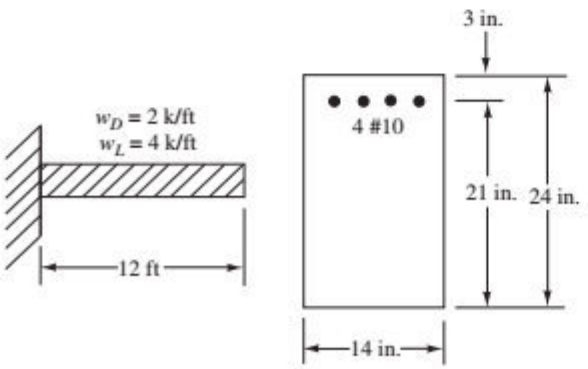
Problem 8.14



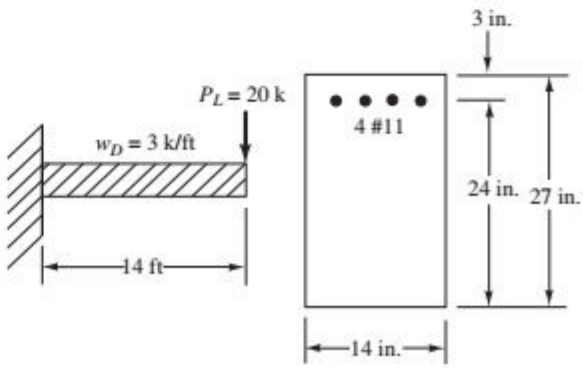
Problem 8.15 (*One ans.* 1 @ 6 in., 8 @ 12 in.)



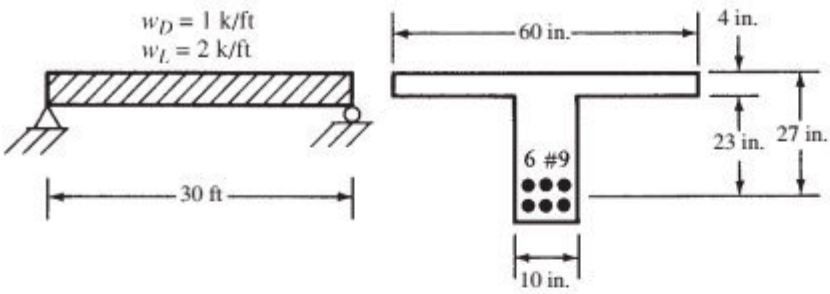
Problem 8.16 Use #4 ⊔ stirrups. (*One ans.* 1 @ 3 in., 8 @ 5 in., 2 @ 8 in., 5 @ 10 in.)



Problem 8.17 Use #4 ⊔ stirrups.



Problem 8.18



Problem 8.19 If the beam of Problem 8.14 has a factored axial compression load of 120 k in addition to other loads, calculate ϕV_c and redesign the stirrups. (*One ans.* 3 stirrups, 1 @ 4 in., 3 @ 10 in., 4 @ 12 in.)

Problem 8.20 Repeat Problem 8.19 if the axial load is tensile. Use #4 ⊔ stirrups.

For Problems 8.21 and 8.22, repeat the problems given using the Chapter 8 Excel spreadsheet.

Problem 8.21 If V_u = 56,400 lb at a particular section, determine the theoretical spacing of #3 ⊔ stirrups for the beam of Problem 8.11. (*Ans.* Theoretical s = 11.9 in., use maximum = 11 in.)

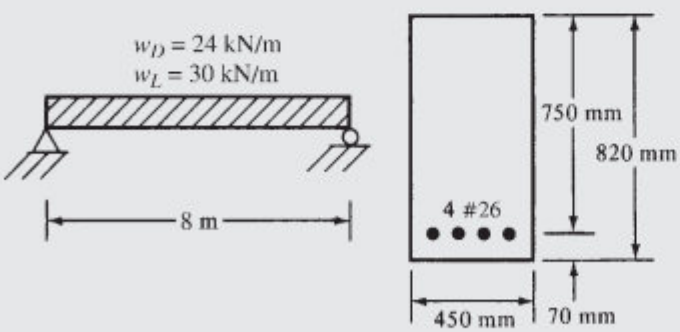
Problem 8.22 If V_u equals 79,600 lb at a particular section, determine the spacing of #4 ⊔ stirrups for the beam of Problem 8.12.

Problem 8.23 Prepare a flowchart for the design of stirrups for rectangular T or I beams.

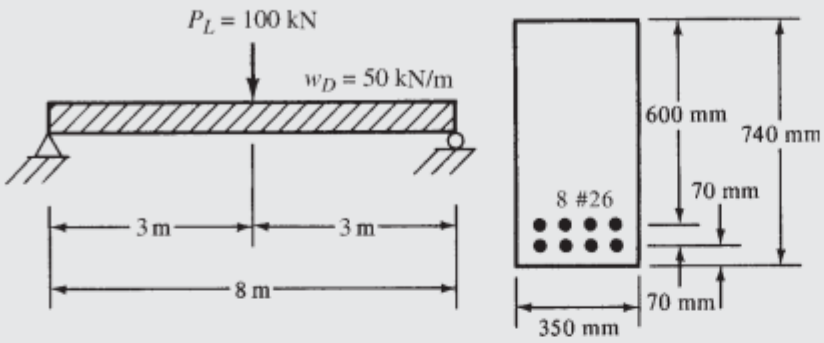
Problems in SI Units

For Problems 8.24 to 8.26, for the beams and loads given, select stirrup spacings if f'_c = 21 MPa and f_{yt} = 420 MPa. The dead loads shown include beam weights. Do not consider movement of live loads. Use #10 ⊔ stirrups.

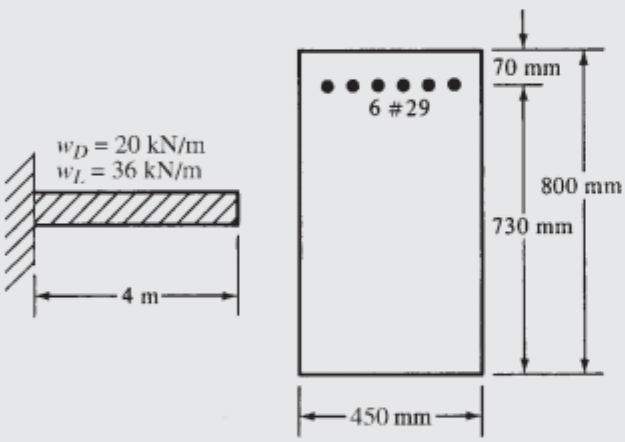
Problem 8.24



Problem 8.25 (*One ans.* #10 stirrups, 1 @ 100 mm, 13 @ 300 mm)



Problem 8.26



Introduction to Columns

CHAPTER 9

9.1 General

This chapter presents an introductory discussion of reinforced concrete columns, with particular emphasis on short, stocky columns subjected to small bending moments. Such columns are often said to be "axially loaded." Short, stocky columns with large bending moments are discussed in Chapter 10, while long or slender columns are considered in Chapter 11.

Concrete columns can be roughly divided into the following three categories:

Short compression blocks or pedestals—If the height of an upright compression member is less than three times its least lateral dimensions, it may be considered to be a pedestal. The ACI Code (Sections 2.3, 14.1.3d, and 14.5) states that a pedestal may be designed with unreinforced or plain concrete with a maximum design compressive stress equal to $0.60\phi f'_c$, where ϕ is 0.60. If bending stress is also present, its effect must be considered. Should the total load applied to the member be larger than the stress limits, it is necessary either to enlarge the cross-sectional area of the pedestal or to design it as a reinforced concrete column, as described in Section 9.9 of this chapter.

Short reinforced concrete columns—Should a reinforced concrete column fail due to initial material failure, it is classified as a short column. The load that it can support is controlled by the dimensions of the cross section and the strength of the materials of which it is constructed. We think of a short column as being a rather stocky member with little flexibility.

Long or slender reinforced concrete columns—As columns become more slender, bending deformations will increase, as will the resulting secondary moments. If these moments are of such magnitude as to significantly reduce the axial load capacities of columns, those columns are referred to as being *long* or *slender*.

When a column is subjected to *primary moments* (those moments caused by applied loads, joint rotations, etc.), the axis of the member will deflect laterally, with the result that additional moments equal to the column load times the lateral deflection will be applied to the column. These latter moments are called *secondary moments* or *P-Δ moments* and are illustrated in Figure 9.1.

A column that has large secondary moments is said to be a slender column, and it is necessary to size its cross section for the sum of both the primary and secondary moments. The ACI's intent is to permit columns to be designed as short columns if the secondary or $P\Delta$ effect does not reduce their strength by more than 5%. Effective slenderness ratios are described and evaluated in Chapter 11 and are used to classify columns as being short or slender. When the ratios are larger than certain values (depending on whether the columns are braced or unbraced laterally), they are classified as slender columns.

The effects of slenderness can be neglected in about 40% of all unbraced columns and about 90% of those braced against sidesway.[1] These percentages are probably decreasing year by year, however, due to the increasing use of slenderer columns designed by the strength method, using stronger materials and with a better understanding of column buckling behavior.

[1] Portland Cement Association, 2011, *Notes on ACI 318-11. Building Code Requirements for Structural Concrete* (Skokie, IL), p. 11–3.

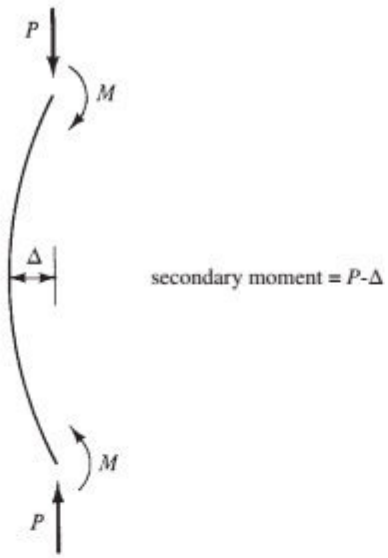


FIGURE 9.1 Secondary or P-Δ moment.

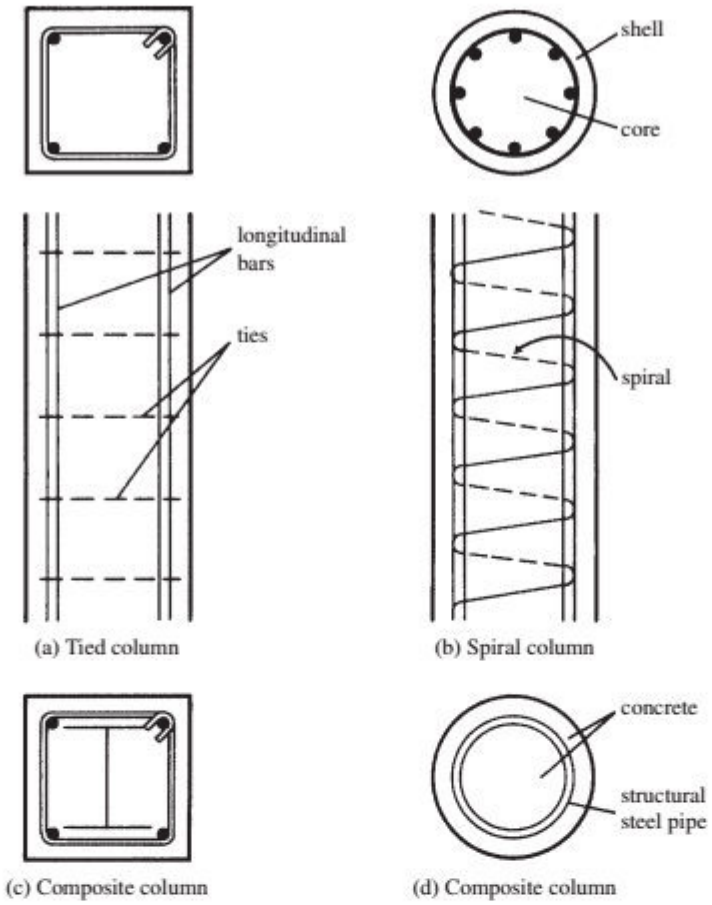
9.2 Types of Columns

A plain concrete column can support very little load, but its load-carrying capacity will be greatly increased if longitudinal bars are added. Further substantial strength increases may be made by providing lateral restraint for these longitudinal bars. Under compressive loads, columns tend not only to shorten lengthwise but also to expand laterally due to the Poisson effect. The capacity of such members can be greatly increased by providing lateral restraint in the form of closely spaced closed ties or helical spirals wrapped around the longitudinal reinforcement.

Reinforced concrete columns are referred to as *tied* or *spiral* columns, depending on the method used for laterally bracing or holding the bars in place. If the column has a series of closed ties, as shown in Figure 9.2(a), it is referred to as a *tied column*. These ties are effective in increasing the column strength. They prevent the longitudinal bars from being displaced during construction, and they resist the tendency of the same bars to buckle outward under load, which would cause the outer concrete cover to break or spall off. Tied columns are ordinarily square or rectangular, but they can be octagonal, round, L shaped, or so forth.

The square and rectangular shapes are commonly used because of the simplicity of constructing the forms. Sometimes, however, when they are used in open spaces, circular shapes are very attractive. The forms for round columns are often made from cardboard or plastic tubes, which are peeled off and discarded once the concrete has sufficiently hardened.

If a continuous helical spiral made from bars or heavy wire is wrapped around the longitudinal bars, as shown in Figure 9.2(b), the column is referred to as a *spiral column*. Spirals are even more effective than ties in increasing a column's strength. The closely spaced spirals do a better job of holding the longitudinal bars in place, and they also confine the concrete inside and greatly increase its resistance to axial compression. As the concrete inside the spiral tends to spread out laterally under the compressive load, the spiral that restrains it is put into hoop tension, and the column will not fail until the spiral yields or breaks, permitting the bursting of the concrete inside. Spiral columns are normally round, but they also can be made into rectangular, octagonal, or other shapes. For such columns, circular arrangements of the bars are still used. Spirals, though adding to the resilience of columns, appreciably increase costs. As a result, they are usually used only for large heavily loaded columns and for columns in seismic areas due to their considerable resistance to earthquake loadings. (In nonseismic zones, probably more than 9 out of 10 existing reinforced concrete columns are tied.) Spirals very effectively increase the ductility and toughness of columns, but they are much more expensive than ties.



(a) Tied column (b) Spiral column

(c) Composite column (d) Composite column

FIGURE 9.2 Types of columns.

Courtesy of EFCO Corp.

Column forms.

Composite columns, illustrated in Figure 9.2(c) and (d), are concrete columns that are reinforced longitudinally by structural steel shapes, which may or may not be surrounded by structural steel bars, or they may consist of structural steel tubing filled with concrete (commonly called *lally columns*).

9.3 Axial Load Capacity of Columns

In actual practice, there are no perfect axially loaded columns, but a discussion of such members provides an excellent starting point for explaining the theory involved in designing real columns with their eccentric loads. Several basic ideas can be explained for purely axially loaded columns, and the strengths obtained provide upper theoretical limits that can be clearly verified with actual tests.

It has been known for several decades that the stresses in the concrete and the reinforcing bars of a column supporting a long-term load cannot be calculated with any degree of accuracy. You might think that such stresses could be determined by multiplying the strains by the appropriate moduli of elasticity. But this idea does not work too well practically because the modulus of elasticity of the concrete is changing during loading due to creep and shrinkage. Thus, the parts of the load carried by the concrete and steel vary with the magnitude and duration of the loads. For instance, the larger the percentage of dead loads and the longer they are applied, the greater the creep in the concrete and the larger the percentage of load carried by the reinforcement.

Although stresses cannot be predicted in columns in the elastic range with any degree of accuracy, several decades of testing have shown that the ultimate strength of columns can be estimated very well. Furthermore, it has been shown that the proportions of live and dead loads, the length of loading, and other such factors have little effect on the ultimate strength. It does not even matter whether the concrete or the steel approaches its ultimate strength first. If one of the two materials is stressed close to its ultimate strength, its large deformations will cause the stress to increase quicker in the other material.

For these reasons, only the ultimate strength of columns is considered here. At failure, the theoretical ultimate strength or nominal strength of a short axially loaded column is quite accurately determined by the expression that follows, in which A_g is the gross concrete area and A_{st} is the total cross-sectional area of longitudinal reinforcement, including bars and steel shapes:

$$P_o = 0.85f'_c(A_g - A_{st}) + f_y A_{st} \qquad \text{(ACI Equation 22.4.2.2)}$$

9.4 Failure of Tied and Spiral Columns

Should a short, tied column be loaded until it fails, parts of the shell or covering concrete will spall off and, unless the ties are quite closely spaced, the longitudinal bars will buckle almost immediately, because their lateral support (the covering concrete) is gone. Such failures may often be quite sudden, and apparently they have occurred rather frequently in structures subjected to earthquake loadings.

When spiral columns are loaded to failure, the situation is quite different. The covering concrete or shell will spall off, but the core will continue to stand, and if the spiral is closely spaced, the core will be able to resist an appreciable amount of additional load beyond the load that causes spalling. The closely spaced loops of the spiral, together with the longitudinal bars, form a cage that very effectively confines the concrete. As a result, the spalling off of the shell of a spiral column provides a warning that failure is going to occur if the load is further increased.

American practice is to neglect any excess capacity after the shell spalls off, because it is felt that once the spalling occurs, the column will no longer be useful—at least from the viewpoint of the occupants of the building. For this reason, the spiral is designed so that it is just a little stronger than the shell that is assumed to spall off. The spalling gives a warning of impending failure, and then the

column will take a little more load before it fails. Designing the spiral so that it is just a little stronger than the shell does not increase the column's ultimate strength much, but it does result in a more gradual or ductile failure.

The strength of the shell is given by the following expression, where A_c is the area of the core, which is considered to have a diameter that extends from out to out of the spiral:

$$\text{Shell strength} = 0.85f'_c(A_g - A_c)$$

By considering the estimated hoop tension that is produced in spirals due to the lateral pressure from the core and by tests, it can be shown that spiral steel is at least twice as effective in increasing the ultimate column capacity as is longitudinal steel.[2,3] Therefore, the strength of the spiral can be computed approximately by the following expression, in which ρ_s is the percentage of spiral steel:

$$\text{Spiral strength} = 2\rho_s A_c f_{yt}$$

Equating these expressions and solving for the required percentage of spiral steel, we obtain

$$0.85f'_c(A_g - A_c) = 2\rho_s A_c f_{yt}$$

$$\rho_s = 0.425\frac{(A_g - A_c)f'_c}{A_c f_{yt}} = 0.425\left(\frac{A_g}{A_c} - 1\right)\frac{f'_c}{f_{yt}}$$

To make the spiral a little stronger than the spalled concrete, the code (25.7.3) specifies the minimum spiral percentage with the expression to follow, in which f_{yt} is the specified yield strength of the spiral reinforcement up to 100,000 psi.

$$\rho_s = 0.45\left(\frac{A_g}{A_c} - 1\right)\frac{f'_c}{f_{yt}} \qquad \text{(ACI Equation 25.7.3.3)}$$

Once the required percentage of spiral steel is determined, the spiral may be selected with the expression to follow, in which ρ_s is written in terms of the volume of the steel in one loop:

$$\rho_s = \frac{\text{volume of spiral in one loop}}{\text{volume of concrete core for a pitch } s}$$

$$= \frac{V_{\text{spiral}}}{V_{\text{core}}}$$

$$= \frac{a_s\pi(D_c - d_b)}{(\pi D_c^2/4)s} = \frac{4a_s(D_c - d_b)}{sD_c^2}$$

In this expression, a_s is the cross-sectional area of the spiral bar, D_c is the diameter of the core out to out of the spiral, and d_b is the diameter of the spiral bar (see Figure 9.3). The designer can assume a diameter for the spiral bar and solve for the pitch required. If the results do not seem reasonable, he or she can try another diameter. The pitch used must be within the limitations listed in the next section of this chapter. Actually, Table A.14 (see Appendix A), which is based on this expression, permits the designer to select spirals directly.

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[2] Park, A., and Paulay, T., 1975, *Reinforced Concrete Structures* (Hoboken, NJ: John Wiley & Sons), pp. 25, 119–121.

[3] Considere, A., 1902, "Compressive Resistance of Concrete Steel and Hooped Concrete, Part I," *Engineering Record*, December 20, pp. 581–583; "Part II," December 27, pp. 605–606.

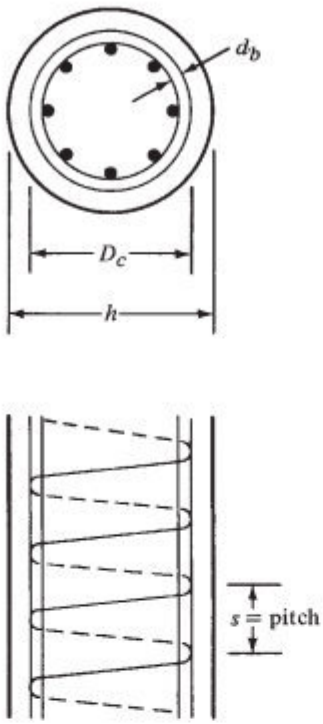
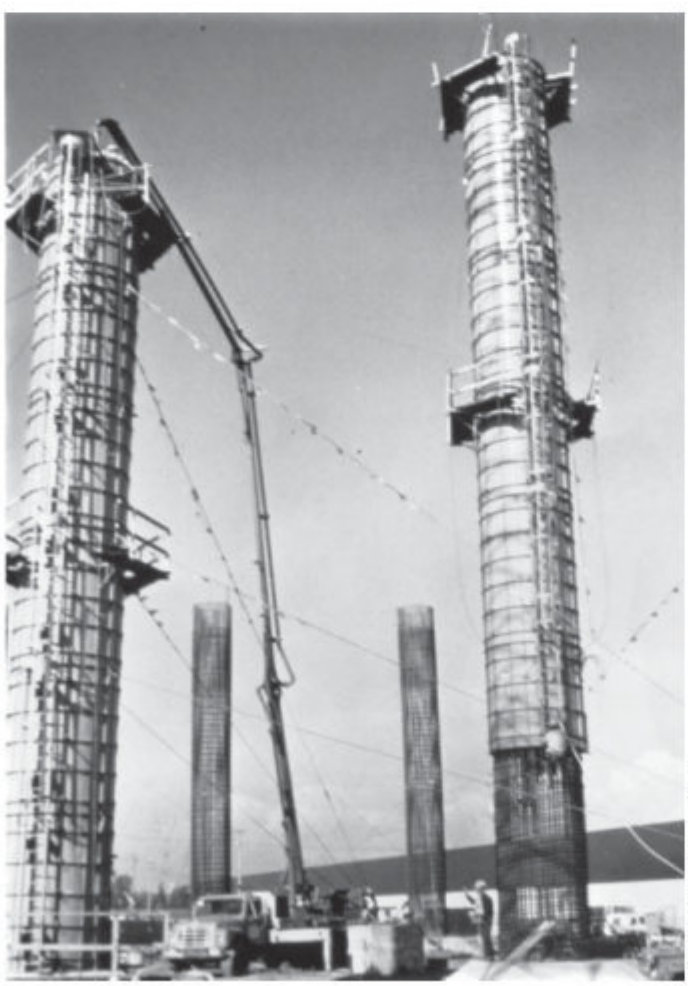
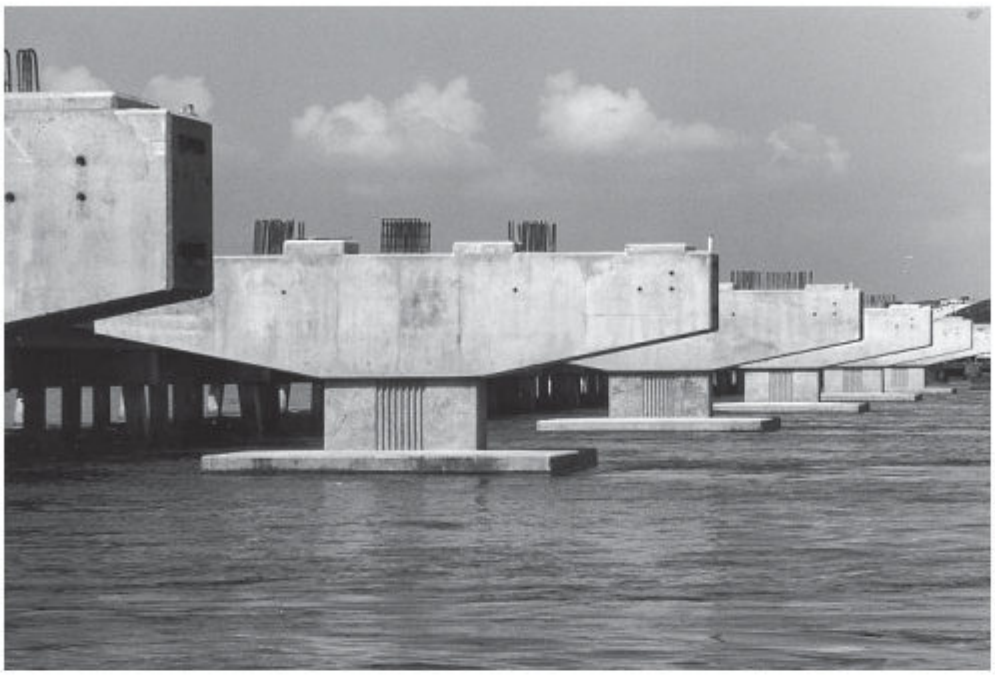


FIGURE 9.3 Drawing showing column spiral terms.



Courtesy of EFCO Corp.

Round spiral columns.



Courtesy of EFCO Corp.

Columns and hammerhead cap forms for the Gandy Bridge, Tampa, Florida.

9.5 Code Requirements for Cast-in-Place Columns

The ACI Code specifies quite a few limitations on the dimensions, reinforcement, lateral restraint, and other items pertaining to concrete columns. Some of the most important limitations are as follows.

1. The percentage of longitudinal reinforcement may not be less than 1% of the gross cross-sectional area of a column (ACI Code, Section 10.6.1.1). It is felt that if the amount of steel is less than 1%, there is a distinct possibility of a sudden nonductile failure, as might occur in a plain concrete column. The 1% minimum steel value will also lessen creep and shrinkage and provide some bending strength for the column. Actually, the code (Section 10.3.1.2) does permit the use of less than 1% steel if the column has been made larger than is necessary to carry the loads because of architectural or other reasons. In other words, a column can be designed with 1% longitudinal steel to support the factored load, and then more concrete can be added with no increase in reinforcement and no increase in calculated load-carrying capacity. In actual practice, the steel percentage for such members is kept to an absolute minimum of 0.005.

2. The maximum percentage of steel may not be greater than 8% of the gross cross-sectional area of the column (ACI Code, Section 10.6.1.1). This maximum value is given to prevent too much crowding of the bars. Practically, it is rather difficult to fit more than 4% or 5% steel into the forms and still get the concrete down into the forms and around the bars. When the percentage of steel is high, the chances of having honeycomb in the concrete is decidedly increased. If this happens, there can be a substantial reduction in the column's load-carrying capacity. Usually the percentage of reinforcement should not exceed 4% when the bars are to be lap spliced. It is to be remembered that if the percentage of steel is very high, the bars may be bundled.

3. The minimum numbers of longitudinal bars permissible for compression members (ACI Code, Section 10.7.3.1) are as follows: four for bars within rectangular or circular ties, three for bars within triangular-shaped ties, and six for bars enclosed within spirals or for columns of special moment frames enclosed by circular hoops. Should there be fewer than eight bars in a circular arrangement, the orientation of the bars will affect the moment strength of eccentrically loaded columns. This matter should be considered in design according to the ACI Commentary (R10.7.3.1).

4. The code does not directly provide a minimum column cross-sectional area, but it is obvious that minimum widths or diameters of about 8 in. to 10 in. are necessary to provide the necessary cover outside of ties or spirals and to provide the necessary clearance between longitudinal bars from one face of the column to the other. To use as little rentable floor space as possible, small columns are frequently desirable. In fact, thin columns may often be enclosed or "hidden" in walls.

5. When tied columns are used, the ties shall not be less than #3, provided that the longitudinal bars are #10 or smaller. The minimum size is #4 for longitudinal bars larger than #10 and for bundled bars. Deformed wire or welded wire fabric with an equivalent area may also be used (ACI Code, Section 25.7.2.2).

> In SI units, ties should not be less than #10 for longitudinal bars #32 or smaller and #13 for larger longitudinal bars and for bundled bars.

The center-to-center spacing of ties shall not be more than 16 times the diameter of the longitudinal bars, 48 times the diameter of the ties, or the least lateral dimension of the column. The ties must be arranged so that every corner and alternate longitudinal bar will have lateral support provided by the corner of a tie having an included angle not greater than 135°. No bars can be located a greater distance than 6 in. clear[4] on either side from such a laterally supported bar. These requirements are given by the ACI Code in Section 25.7.2.1. Figure 9.4 shows tie arrangements for several column cross sections. Some of the arrangements with interior ties, such as the ones shown in the bottom two rows of the figure, are rather expensive. Should longitudinal bars be arranged in a circle, round ties may be placed around them and the bars do not have to be individually tied or restrained otherwise (Section 25.7.2.4). The ACI also states (Section 10.7.6.1.5) that the requirements for lateral ties may be waived if tests and structural analysis show that the columns are sufficiently strong without them and that such construction is feasible.

There is little evidence available concerning the behavior of spliced bars and bundled bars. For this reason, Section R25.4.2.1 of the commentary states that it is advisable to provide ties when using very high compressive strength concrete.

Ties should not be placed more than one-half a spacing above the top of a footing or slab and not more than one-half a spacing below the lowest reinforcing in a slab or drop panel (to see a drop panel, refer to Figure 16.1 in Chapter 16). Where beams frame into a column from all four directions, the last tie may be below the lowest reinforcing in any of the beams.

6. The ACI Code (Section 25.7.3) states that spirals may not have diameters less than $\frac{3}{8}$ in.[5] and that the clear spacing between them may not be less than 1 in. or greater than 3 in.[6] Should splices be necessary in spirals, they are to be provided by lapping according to ACI Code, Section 25.7.3.6, but not less than 12 in.[7] Lap splice lengths are also given in ACI Code, Section 25.7.3.6, for plain uncoated bars and wires, for epoxy-coated deformed bars and wires, and so on. Special spacer bars may be used to hold the spirals in place and at the desired pitch until the concrete hardens. These spacers consist of vertical bars with small hooks. Spirals are supported by the spacers, not by the longitudinal bars. Section R26.6.2.2(b) of the ACI Commentary provides suggested numbers of spacers required for different-size columns.

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[4] 150 mm in SI.
[5] 10 mm in SI.
[6] 25 mm and 75 mm in SI.
[7] 300 mm in SI.

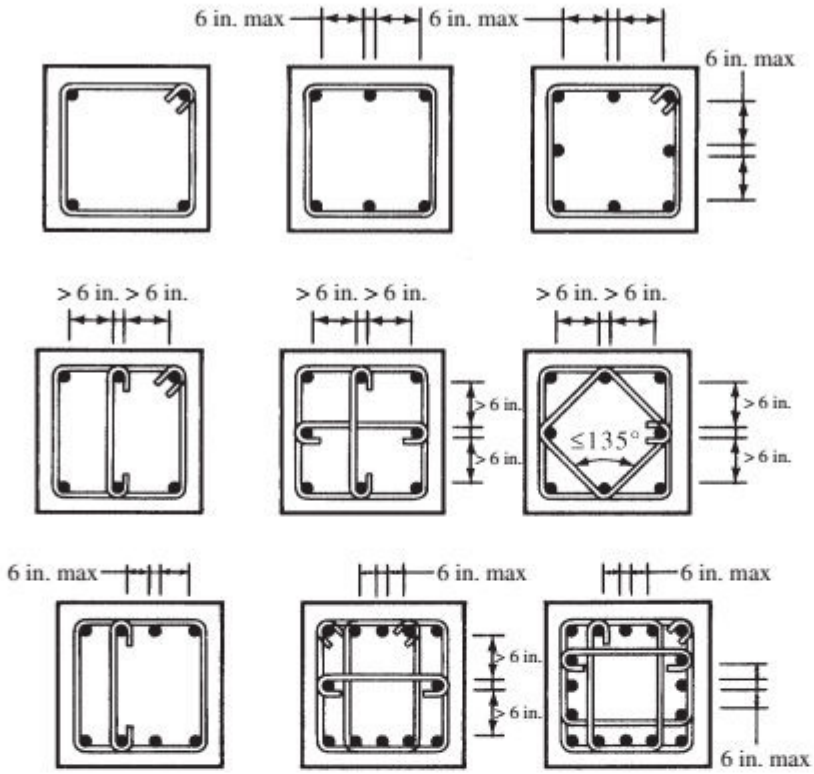

FIGURE 9.4 Typical tie arrangements.

7. The ACI 318 Code (Section 25.7.2.4) states that where longitudinal bars are located around the perimeter of a circle, a complete circular tie is permitted. The ends of the circular tie must overlap by not less than 6 in. and terminate with standard hooks that engage a longitudinal column bar. Overlaps at ends of adjacent circular ties shall be staggered around the perimeter enclosing the longitudinal bars. The code commentary for this provision warns that vertical splitting and loss of tie restraint are possible where the overlapped ends of adjacent circular ties are anchored at a single longitudinal bar. Adjacent circular ties should not engage the same longitudinal bar with end hook anchorages. While the transverse reinforcement in members with longitudinal bars located around the periphery of a circle can be either spirals or circular ties, spirals are usually more effective.

9.6 Safety Provisions for Columns

The values of ϕ to be used for columns as specified in Section 21.2.2 of the code are well below those used for flexure and shear (0.90 and 0.75, respectively). A value of 0.65 is specified for tied columns and 0.75 for spiral columns. A slightly larger ϕ is specified for spiral columns because of their greater toughness. In Section 10.10 of this text, a variable ϕ factor is discussed whereby it is increased under certain conditions.

The failure of a column is generally a more severe matter than is the failure of a beam, because a column generally supports a larger part of a structure than does a beam. In other words, if a column fails in a building, a larger part of the building will fall down than if a beam fails. This is particularly true for a lower-level column in a multistory building. As a result, lower ϕ values are desirable for columns.

There are other reasons for using lower ϕ values in columns. As an example, it is more difficult to do as good a job in placing the concrete for a column than it is for a beam. The reader can readily

see the difficulty of getting concrete down into narrow column forms and between the longitudinal and lateral reinforcing. As a result, the quality of the resulting concrete columns is probably not as good as that of beams and slabs.

The failure strength of a beam is normally dependent on the yield stress of the tensile steel—a property that is quite accurately controlled in the steel mills. The failure strength of a column is closely related to the concrete's ultimate strength, a value that is quite variable. The length factors also drastically affect the strength of columns and thus make the use of lower ϕ factors necessary.

It seems impossible for a column to be perfectly axially loaded. Even if loads could be perfectly centered at one time, they would not stay in place. Furthermore, columns may be initially crooked or have other flaws, with the result that lateral bending will occur. Wind and other lateral loads cause columns to bend, and the columns in rigid-frame buildings are subjected to moments when the frame is supporting gravity loads alone.

9.7 Design Formulas

In the pages that follow, the letter e is used to represent the eccentricity of the load. The reader may not understand this term because he or she has analyzed a structure and has computed an axial load, P_u, and a bending moment, M_u, but no specific eccentricity, e, for a particular column. The term e represents the distance the axial load, P_u, would have to be off center of the column to produce M_u. Thus

$$P_u e = M_u$$

or

$$e = \frac{M_u}{P_u}$$

Nonetheless, there are many situations where there are no calculated moments for the columns of a structure. For many years, the code specified that such columns had to be designed for certain minimum moments even though no calculated moments were present. This was accomplished by requiring designers to assume certain minimum eccentricities for their column loads. These minimum values were 1 in. or $0.05h$, whichever was larger, for spiral columns and 1 in. or $0.10h$ for tied columns. (The term h represents the outside diameter of round columns or the total depth of square or rectangular columns.) A moment equal to the axial load times the minimum eccentricity was used for design.

In today's code, minimum eccentricities are not specified, but the same objective is accomplished by requiring that theoretical axial load capacities (see Text Section 9.3) be multiplied by a factor which is equal to 0.85 for spiral columns and 0.80 for tied columns. Thus, as shown in Section 22.4.2 of the code, the axial load capacity of columns may not be greater than the following values:

For spiral columns ($\phi = 0.75$)

$$\phi P_n(\max) = 0.85\phi[0.85f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 22.4.2.1b[8])}$$

For tied columns ($\phi = 0.65$)

$$\phi P_n(\max) = 0.80\phi[0.85f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 22.4.2.1a[9])}$$

It is to be clearly understood that the preceding expressions are to be used only when the moment is quite small or when there is no calculated moment.

The equations presented here are applicable only for situations where the moment is sufficiently small so that e is less than $0.10h$ for tied columns or less than $0.05h$ for spiral columns. Short columns

[8] The equation shown includes the ϕ factor, which is not included in ACI Equation 22.4.2.1b.
[9] The equation shown includes the ϕ factor, which is not included in ACI Equation 22.4.2.1a.

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Destruction of old apartment buildings.

can be completely designed with these expressions as long as the e values are under the limits described. Should the e values be greater than the limiting values and/or should the columns be classified as long ones, it will be necessary to use the procedures described in the next two chapters.

9.8 Comments on Economical Column Design

Reinforcing bars are quite expensive, and thus the percentage of longitudinal reinforcement used in reinforced concrete columns is a major factor in their total costs. This means that under normal circumstances, a small percentage of steel should be used (perhaps in the range of 1.5% to 3%). This can be accomplished by using larger column sizes and/or higher-strength concretes. Furthermore, if the percentage of bars is kept in approximately this range, it will be found that there will be sufficient room for conveniently placing them in the columns.

Higher-strength concretes can be used more economically in columns than in beams. Under ordinary loads, only 30% to 40% of a beam cross section is in compression, while the remaining 60% to 70% is in tension and thus assumed to be cracked. This means that if a high-strength concrete is used for a beam, 60% to 70% of it is wasted. For the usual column, however, the situation is quite different because a much larger percentage of its cross section is in compression. As a result, it is quite economical to use high-strength concretes for columns. Although some designers have used concretes with ultimate strengths as high as 19,000 psi (as at Two Union Square in Seattle) for column design with apparent economy, the use of 5000-psi to 6000-psi columns is the normal rule when higher strengths are specified for columns.

Grade 60 reinforcing bars are generally used for best economy in the columns of most structures. However, Grade 75 bars may provide better economy in high-rise structures, particularly when they are used in combination with higher-strength concretes.

In general, tied columns are more economical than spiral columns, particularly if square or rectangular cross sections are to be used. Of course, spiral columns, high-strength concretes, and high percentages of steel save floor space.

As few different column sizes as possible should be used throughout a building. In this regard, it is completely uneconomical to vary a column size from floor to floor to satisfy the different loads it must

support. This means that the designer may select a column size for the top floor of a multistory building (using as small a percentage of steel as possible) and then continue to use that same size vertically for as many stories as possible, by increasing the steel percentage floor by floor as required. Furthermore, it is desirable to use the same column size as much as possible on each floor level. This consistency of sizes will provide appreciable savings in labor costs.

The usual practice for the columns of multistory reinforced concrete buildings is to use one-story-length vertical bars tied together in preassembled cages. This is the preferred procedure when the bars are #11[10] or smaller, where all the bars can be spliced at one location just above the floor line. For columns where staggered splice locations are required (as for larger-size bars), the number of splices can be reduced by using preassembled two-story cages of reinforcing.

Unless the least column dimensions or longitudinal bar diameters control tie spacings, the selection of the largest practical tie sizes will increase their spacings and reduce their number. This can result in some savings. Money can also be saved by avoiding interior ties, such as the ones shown in the bottom two rows of columns in Figure 9.4. With no interior ties, the concrete can be placed more easily and lower slumps used (thus lower-cost concrete).

In fairly short buildings, the floor slabs are often rather thin, and thus deflections may be a problem. As a result, rather short spans and thus close column spacings may be used. As buildings become taller, the floor slabs will probably be thicker to help provide lateral stability. For such buildings, slab deflections will not be as much of a problem, and the columns may be spaced farther apart.

Even though the columns in tall buildings may be spaced at fairly large intervals, they still will occupy expansive floor space. For this reason, designers try to place many of their columns on the building perimeters so they will not use up the valuable interior space. In addition, the omission of interior columns provides more flexibility for the users for placement of partitions and also makes large open spaces available.

9.9 Design of Axially Loaded Columns

As a brief introduction to columns, the design of three axially loaded short columns is presented in this section and the next. Moment and length effects are completely neglected. Examples 9.1 and 9.3 present the design of axially loaded square tied columns, while Example 9.2 illustrates the design of a similarly loaded round spiral column. Table A.15 in Appendix A provides several properties for circular columns that are particularly useful for designing round columns.

Example 9.1

Design a square tied column to support an axial dead load D of 130 k and an axial live load L of 180 k. Initially assume that 2% longitudinal steel is desired, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi.

SOLUTION

$$P_u = (1.2)\,(130\text{ k}) + (1.6)\,(180\text{ k}) = 444\text{ k}$$

Selecting Column Dimensions

$$\phi P_n = \phi 0.80[0.85f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 22.4.2.1a)}$$

$$444 = (0.65)(0.80)\,[(0.85)(4\text{ ksi})(A_g - 0.02A_g) + (60\text{ ksi})(0.02A_g)]$$

$$A_g = 188.40\text{ in.}^2 \qquad \text{Use 14 in. } \times \text{ 14 in. } (A_g = 196\text{ in.}^2)$$

[10] #36 in SI.

Selecting Longitudinal Bars

Substituting into column equation with known A_g and solving for A_{st}, we obtain from ACI Equation 22.4.2.1a,

$$444 = (0.65)(0.80)[(0.85)(4\text{ ksi})(196\text{ in.}^2 - A_{st}) + (60\text{ ksi})A_{st}]$$

$$A_{st} = 3.31\text{ in.}^2 \qquad \underline{\text{Use 6 \#7 bars (3.61 in.}^2\text{)}}$$

Design of Ties (Assuming #3 Bars)

Spacing:

(a) 48 in. $\times \frac{3}{8}$ in. = 18 in.

(b) 16 in. $\times \frac{7}{8}$ in. = 14 in. ←

(c) Least dim. = 14 in. ← Use #3 ties @ 14 in.

A sketch of the column cross section is shown in Figure 9.5.

Check Code Requirements

Following are the ACI Code limitations for columns. Space is not taken in future examples to show all of these essential checks, but they must be made.

(7.6.1) Longitudinal bar clear spacing = $\frac{9}{2}$ in. − $\frac{7}{8}$ in. = 3.625 in. > 1 in. and d_b of $\frac{7}{8}$ in. OK

(10.9.1) Steel percentage $0.01 < \rho = \dfrac{3.61}{(14\text{ in.})(14\text{ in.})} = 0.0184 < 0.08$ OK

(10.9.2) Number of bars = 6 > min. no. of 4 OK

(7.10.5.1) Minimum tie size = #3 for #7 bars OK

(7.10.5.2) Spacing of ties OK

(7.10.5.3) Arrangement of ties OK

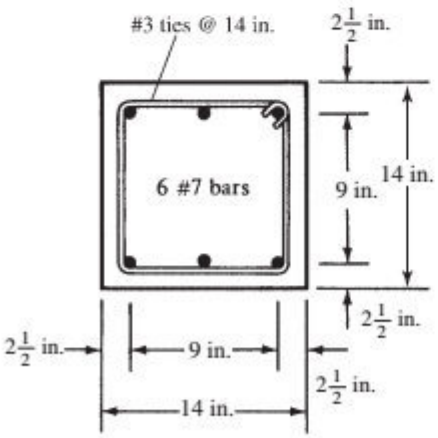


FIGURE 9.5 Final column cross section for Example 9.1.

Example 9.2

Design a round spiral column to support an axial dead load P_D of 240 k and an axial live load P_L of 300 k. Initially assume that approximately 2% longitudinal steel is desired, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi.

SOLUTION

$$P_u = (1.2)\,(240\text{ k}) + (1.6)\,(300\text{ k}) = 768\text{ k}$$

Selecting Column Dimensions and Bar Sizes

$$\phi P_n = \phi 0.85[0.85f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 22.4.2.1b)}$$

$$768\text{ k} = (0.75)(0.85)[(0.85)(4\text{ ksi})(A_g - 0.02A_g) + (60\text{ ksi})(0.02A_g)]$$

$$A_g = 266\text{ in.}^2 \qquad \text{Use 18-in. diameter column } (255\text{ in.}^2)$$

Using a column diameter with a gross area less than the calculated gross area ($255\text{ in.}^2 < 266\text{ in.}^2$) results in a higher percentage of steel than originally assumed.

$$768\text{ k} = (0.75)\,(0.85)\,[(0.85)\,(4\text{ ksi})\,(255\text{ in.}^2 - A_{st}) + (60\text{ ksi})A_{st}]$$

$$A_{st} = 5.97\text{ in.}^2 \qquad \text{Use 6 \#9 bars } (6.00\text{ in.}^2)$$

Check code requirements as in Example 9.1. A sketch of the column cross section is shown in Figure 9.6.

Design of Spiral

$$A_c = \frac{(\pi)(15\text{ in.})^2}{4} = 177\text{ in.}^2$$

$$\text{Minimum } \rho_s = (0.45)\left(\frac{A_g}{A_c} - 1\right)\frac{f'_c}{f_y} = (0.45)\left(\frac{255\text{ in.}^2}{177\text{ in.}^2} - 1\right)\left(\frac{4\text{ ksi}}{60\text{ ksi}}\right) = 0.0132$$

Assume a #3 spiral, $d_b = 0.375$ in. and $a_s = 0.11\text{ in.}^2$

$$\rho_s = \frac{4a_s(D_c - d_b)}{sD_c^2}$$

$$0.0132 = \frac{(4)(0.11\text{ in.}^2)(15\text{ in.} - 0.375\text{ in.})}{(s)(15\text{ in.})^2}$$

$$s = 2.17\text{ in.} \qquad \text{Say 2 in.}$$

(Checked with Appendix A, Table A.14.)

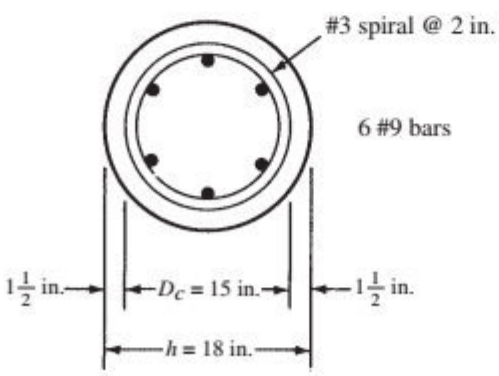


FIGURE 9.6 Final design for Example 9.2.

9.10 SI Example

Example 9.3

Design an axially loaded short square tied column for $P_u = 2600$ kN if $f'_c = 28$ MPa and $f_y = 350$ MPa. Initially assume $\rho = 0.02$.

SOLUTION

Selecting Column Dimensions

$$\phi P_n = \phi 0.80[0.85f'_c(A_g - A_{st}) + f_y A_{st}] \qquad \text{(ACI Equation 22.4.2.1a)}$$

$$2600 \text{ kN} = (0.65)(0.80)[(0.85)(28 \text{ MPa})(A_g - 0.02A_g) + (350 \text{ MPa})(0.02A_g)]$$

$$A_g = 164\,886 \text{ mm}^2$$

Use 400 mm × 400 mm ($A_g = 160\,000$ mm^2)

Selecting Longitudinal Bars

$$2600 \text{ kN} = (0.65)(0.80)[(0.85)(28 \text{ MPa})(160\,000 \text{ mm}^2 - A_{st}) + (350 \text{ MPa})A_{st}]$$

$$A_{st} = 3654 \text{ mm}^2$$

Use 6 #29 (3870 mm^2)

Design of Ties (Assuming #10 SI Ties)

(a) 16 mm × 28.7 mm = 459.2 mm
(b) 48 mm × 9.5 mm = 456 mm
(c) Least col. dim. = 400 mm ← Use #10 ties @ 400 mm

Check code requirements as in Example 9.1. A sketch of the column cross section is shown in Figure 9.7.

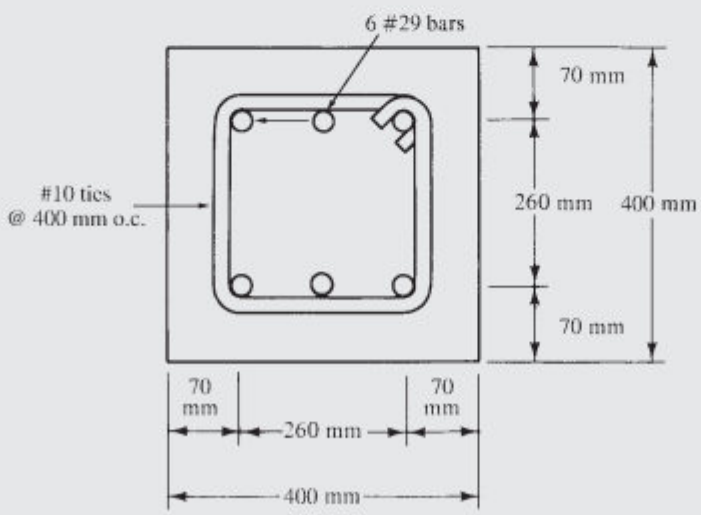


FIGURE 9.7 Final design for Example 9.3.

9.11 Computer Example

Example 9.4

Using the Chapter 9 Excel Spreadsheet, repeat Example 9.2.

SOLUTION

Open the Concentric Circular Column worksheet and enter the following:

$$P_u = 1.2P_D + 1.6P_L = 1.2(240\text{ k}) + 1.6(300\text{ k}) = 768\text{ k}$$

Enter material properties ($f'_c = 4000$ psi, $f_y = 60{,}000$ psi).
Enter a trial % of reinforcement of 2%
The required gross area is calculated ($A_g = 265.8$ in.^{2})
The corresponding theoretical diameter to produce A_g is then displayed, dia = 18.40 in.
Now select the diameter, D, that you want to use. If D is less than dia, then the actual steel percentage will be greater than the 2% assumed. Select $D = 18$ in.
The required area of steel is then calculated: $A_{st} = 6.00$ in.^{2}
Enter a number of bars and bar size, and compare the resulting A_{st} to that required.
Entering 6 #9 bars results in $A_{st} = 6.00$ in.^{2}, which is exactly equal to that theoretically required.
Note that $\rho_z = 0.0236 > 0.02$ as a result of using a diameter less than 18.4 in.

Circular Column Capacity		
P_u =	768	kips
f'_c =	4000	psi
f_y =	60,000	psi
trial %	2	%
A_g =	265.8	in.^{2}
dia =	18.40	in.
select D =	18	in.
A_{st} =	6.00	in.^{2}

	No. of bars	Bar size		
select bars	6	# 9	A_{st} =	6.00 in.^{2}
				bars selected have sufficient area

Bar dia.
1.128

$P_o = (0.85f'_c A_n + A_s f_y)$ = 1204.8 kips
$0.85\phi P_o$ = 768.1 kips <--- ACI 318-2014 Code Eq. 10.1
capacity is sufficient

ρ_z = 0.024
The reinforcement ratio is within the allowable limits of 0.01 and 0.08

PROBLEMS

Problem 9.1 Distinguish among tied, spiral, and composite columns.

Problem 9.2 What are primary and secondary moments?

Problem 9.3 Distinguish between long and short columns.

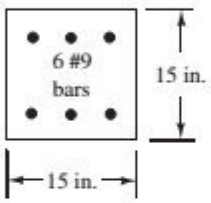
Problem 9.4 List several design practices that may help make the construction of reinforced concrete columns more economical.

Analysis of Axially Loaded Columns

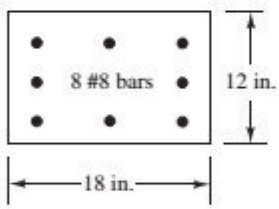
For Problems 9.5 to 9.8, compute the load-bearing capacity, ϕP_n, of the concentrically loaded short column. $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.

Problem 9.5 A 24-in. square column reinforced with 12 #10 bars. (*Ans*. 1465.4 k)

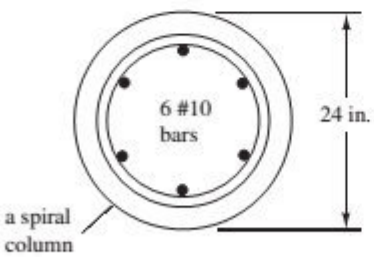
Problem 9.6



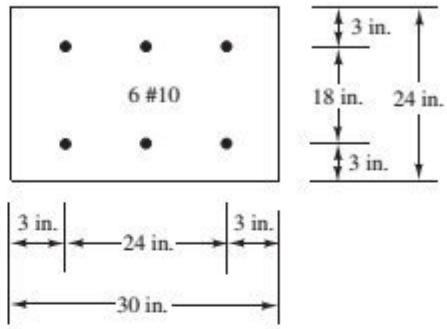
Problem 9.7 (*Ans*. 566.7 k)



Problem 9.8



Problem 9.9 Determine the load-bearing capacity of the concentrically loaded short column shown if $f_y = 60{,}000$ psi and $f'_c = 3000$ psi. (*Ans*. 1181.8 k)



Design of Axially Loaded Columns

For Problems 9.10 to 9.15, design columns for axial load only. Include the design of ties or spirals and a sketch of the cross sections selected, including bar arrangements. All columns are assumed to be short, and form sizes are available in 2-in. increments.

Problem 9.10 Square tied column: $P_D = 300$ k, $P_L = 500$ k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi. Initially assume $\rho_g = 2\%$.

Problem 9.11 Repeat Problem 9.10 if ρ_g is to be 3% initially. (*One ans*. 22-in. × 22-in. column with 10 #11 bars)

Problem 9.12 Round spiral column: $P_D = 300$ k, $P_L = 400$ k, $f'_c = 3500$ psi, and $f_y = 60{,}000$ psi. Initially assume $\rho_g = 4\%$.

Problem 9.13 Round spiral column: $P_D = 400$ k, $P_L = 250$ k, $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and p_g initially assumed = 2%. (*One ans*. 20-in. diameter column with 6 #9 bars)

Problem 9.14 Smallest possible square tied column: $P_D = 200$ k, $P_L = 300$ k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi.

Problem 9.15 Design a rectangular tied column with the long side equal to two times the length of the short side. $P_D = 700$ k, $P_L = 400$ k, $f'_c = 3000$ psi, and $f_y = 60{,}000$ psi. Initially assume that $p_g = 2\%$. (*One ans*. 20-in. × 40-in. column with 10 #11 bars)

Problems in SI Units

For Problems 9.16 to 9.18, design columns for axial load only for the conditions described. Include the design of ties or spirals and a sketch of the cross sections selected, including bar arrangements. All columns are assumed to be short and not exposed to the weather. Form sizes are in 50-mm increments.

Problem 9.16 Square tied column: $P_D = 600$ kN, $P_L = 800$ kN, $f'_c = 24$ MPa, and $f_y = 420$ MPa. Initially assume $\rho_g = 0.02$.

Problem 9.17 Smallest possible square tied column: $P_D = 700$ kN, $P_L = 300$ kN, $f'_c = 28$ MPa, and $f_y = 300$ MPa. (*One ans*. 250-mm × 250-mm column with 6 #29 bars)

Problem 9.18 Round spiral column: $P_D = 500$ kN, $P_L = 650$ kN, $f'_c = 35$ MPa, and $f_y = 420$ MPa. Initially assume $\rho_g = 0.03$.

For Problems 9.19 to 9.21, use the Chapters 9 and 10 Excel spreadsheets. Assume $d' = 2.5$ in. for each column.

Problem 9.19 Repeat Problem 9.6. (*Ans*. $\phi P_n = 574.4$ k)

Problem 9.20 Repeat Problem 9.10.

Problem 9.21 Repeat Problem 9.12. (*One ans*. 20-in.-diameter column with 8 #10 bars for which $\phi P_n = 964$ k > 960 k)

Design of Short Columns Subject to Axial Load and Bending

CHAPTER 10

10.1 Axial Load and Bending

All columns are subjected to some bending as well as axial forces, and they need to be proportioned to resist both. The so-called axial load formulas presented in Chapter 9 do take into account some moments, because they include the effect of small eccentricities with the 0.80 and 0.85 factors. These values are approximately equivalent to the assumption of actual eccentricities of $0.10h$ for tied columns and $0.05h$ for spiral columns.

Columns will bend under the action of moments, and those moments will tend to produce compression on one side of the columns and tension on the other. Depending on the relative magnitudes of the moments and axial loads, there are several ways in which the sections might fail. Figure 10.1 shows a column supporting a load, P_n. In the various parts of the figure, the load is placed at greater and greater eccentricities (thus producing larger and larger moments) until finally in part (f) the column is subject to such a large bending moment that the effect of the axial load is negligible. Each of the six cases shown is briefly discussed in the paragraphs to follow, where the letters (a) through (f) correspond to those same letters in the figure. The column is assumed to reach its ultimate capacity when the compressive concrete strain reaches 0.003.

(a) *Large axial load with negligible moment*—For this situation, failure will occur by the crushing of the concrete, with all reinforcing bars in the column having reached their yield stress in compression.

(b) *Large axial load and small moment such that the entire cross section is in compression*—When a column is subject to a small bending moment (i.e., when the eccentricity is small), the entire column will be in compression, but the compression will be higher on one side than on the other. The maximum compressive stress in the column will be $0.85f'_c$, and failure will occur by the crushing of the concrete with all the bars in compression.

(c) *Eccentricity larger than in case (b) such that tension begins to develop on one side of the column*—If the eccentricity is increased somewhat from the preceding case, tension will begin to develop on one side of the column, and the steel on that side will be in tension but less than the yield stress. On the other side, the steel will be in compression. Failure will occur as a result of the crushing of the concrete on the compression side.

(d) *A balanced loading condition*—As we continue to increase the eccentricity, a condition will be reached in which the reinforcing bars on the tension side will reach their yield stress at the same time that the concrete on the opposite side reaches its maximum compression, $0.85f'_c$. This situation is called the *balanced loading condition*.

(e) *Large moment with small axial load*—If the eccentricity is further increased, failure will be initiated by the yielding of the bars on the tensile side of the column prior to concrete crushing.

(f) *Large moment with no appreciable axial load*—For this condition, failure will occur as it does in a beam.

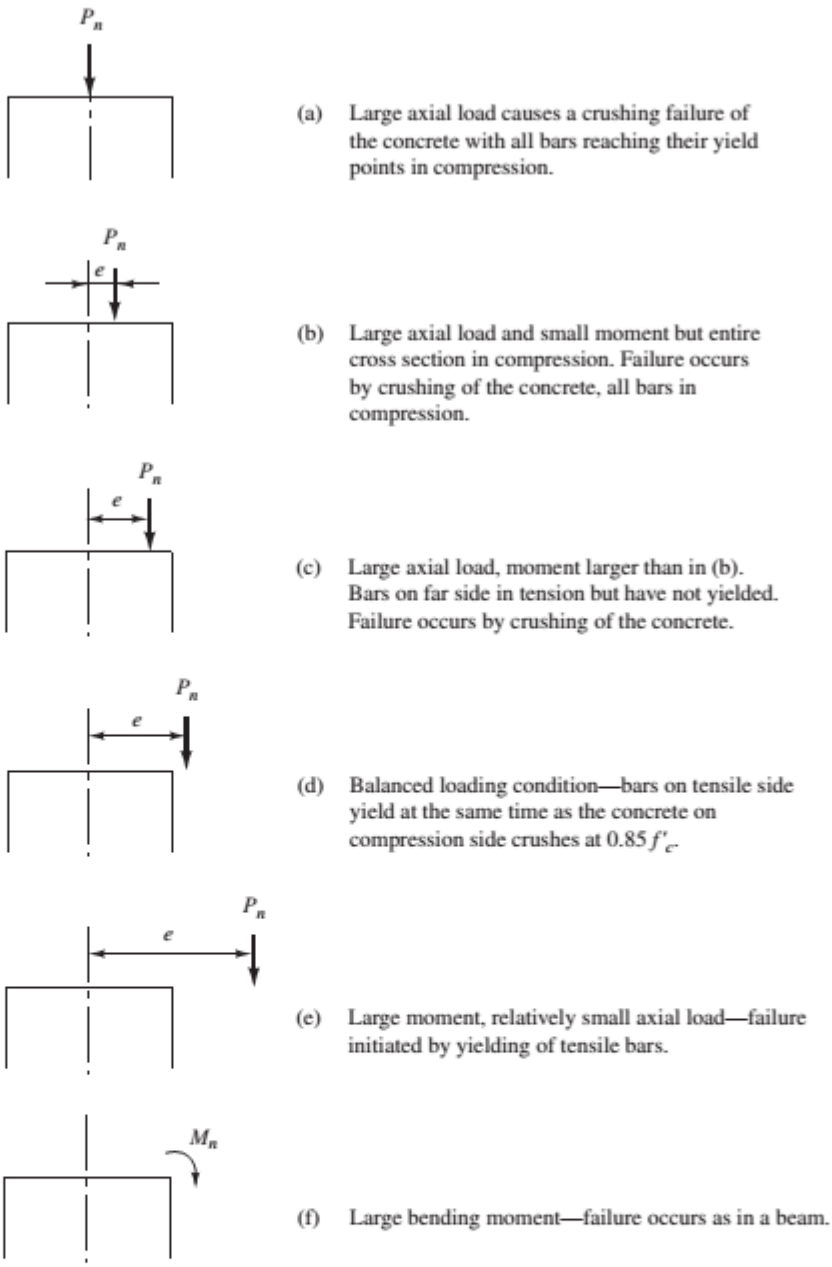

P_n
(a) Large axial load causes a crushing failure of the concrete with all bars reaching their yield points in compression.
P_n
e
(b) Large axial load and small moment but entire cross section in compression. Failure occurs by crushing of the concrete, all bars in compression.
P_n
e
(c) Large axial load, moment larger than in (b). Bars on far side in tension but have not yielded. Failure occurs by crushing of the concrete.
P_n
e
(d) Balanced loading condition—bars on tensile side yield at the same time as the concrete on compression side crushes at $0.85 f'_c$.
P_n
e
(e) Large moment, relatively small axial load—failure initiated by yielding of tensile bars.
M_n
(f) Large bending moment—failure occurs as in a beam.

FIGURE 10.1 Column subject to load with larger and larger eccentricities.

10.2 The Plastic Centroid

The eccentricity of a column load is the distance from the load to the *plastic centroid* of the column. The plastic centroid represents the location of the resultant force produced by the steel and the concrete. It is the point in the column cross section through which the resultant column load must pass to produce uniform strain at failure. For locating the plastic centroid, all concrete is assumed to be stressed in compression to $0.85f'_c$ and all steel to f_y in compression. For symmetrical sections, the plastic centroid coincides with the centroid of the column cross section, whereas for nonsymmetrical sections, it can be located by taking moments.

Courtesy of EFCO Corp.

Columns of variable heights between 25′ and 35′ will provide support for tank lids for the Eastside Treatment Plant in Sunnyvale, Texas.

Example 10.1 illustrates the calculations involved in locating the plastic centroid for a nonsymmetrical cross section. The nominal axial compressive strength, P_n, is determined by computing the total compressive forces in the concrete and the steel and adding them together. P_n is then assumed to act downward at the plastic centroid, at a distance $\overline{x}$ from one side of the column, and moments are taken on that side of the column of the upward compression forces acting at their centroids and the downward P_n.

Example 10.1

Determine the plastic centroid of the T-shaped column shown in Figure 10.2 if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

The plastic centroid falls on the x-axis, as shown in Figure 10.2, because of symmetry. The column is divided into two rectangles, the left one being 16 in. × 6 in. and the right one 8 in. × 8 in. C_1 is assumed to be the total compression in the left concrete rectangle, C_2 the total compression in the right rectangle, and C'_s the total compression in the reinforcing bars.

$$C_1 = (16 \text{ in.})\,(6 \text{ in.})\,(0.85)\,(4 \text{ ksi}) = 326.4 \text{ k}$$

$$C_2 = (8 \text{ in.})\,(8 \text{ in.})\,(0.85)\,(4 \text{ ksi}) = 217.6 \text{ k}$$

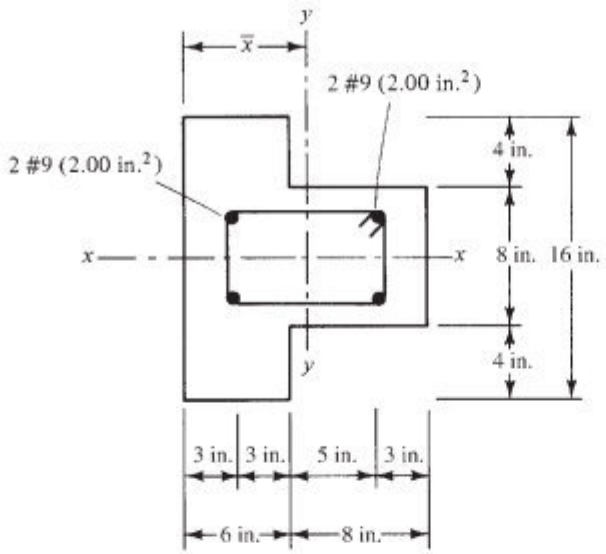


FIGURE 10.2 Column cross section for Example 10.1.

In computing C'_s, the concrete where the bars are located is subtracted; that is,

$$C'_s = (4.00 \text{ in.}^2)\,(60 \text{ ksi} - 0.85 \times 4 \text{ ksi}) = 226.4 \text{ k}$$

$$\text{Total compression} = P_n = 326.4 \text{ k} + 217.6 \text{ k} + 226.4 \text{ k} = 770.4 \text{ k}$$

Taking Moments about Left Edge of Column

$$-(326.4 \text{ k})\,(3 \text{ in.}) - (217.6 \text{ k})\,(10 \text{ in.}) - (226.4 \text{ k})\,(7 \text{ in.}) + (770.4 \text{ k})\,(\bar{x}) = 0$$

$$\underline{\bar{x} = 6.15 \text{ in.}}$$

10.3 Development of Interaction Diagrams

Should an axial compressive load be applied to a short concrete member, it will be subjected to a uniform strain or shortening, as shown in Figure 10.3(a). If a moment with zero axial load is applied to the same member, the result will be bending about the member's neutral axis such that strain is proportional to the distance from the neutral axis. This linear strain variation is shown in Figure 10.3(b). Should axial load and moment be applied at the same time, the resulting strain diagram will be a combination of two linear diagrams and will itself be linear, as illustrated in Figure 10.3(c). As a result of this linearity, we can assume certain numerical values of strain in one part of a column and determine strains at other locations by straight-line interpolation.

As the axial load applied to a column is changed, the moment that the column can resist will change. This section shows how an interaction curve of nominal axial load and moment values can be developed for a particular column.

Assuming that the concrete on the compression edge of the column will fail at a strain of 0.003, a strain can be assumed on the far edge of the column, and the values of P_n and M_n can be computed by statics. Holding the compression strain at 0.003 on the far edge, we can then assume a series of different strains on the other edge and calculate P_n and M_n for each.[1] Eventually a sufficient number of values will be obtained to plot an interaction curve such as the one shown in Figure 10.8. Example 10.2 illustrates the calculation of P_n and M_n for a column for one set of assumed strains.

[1] Leet, K., 1991, *Reinforced Concrete Design*, 2nd ed. (New York: McGraw-Hill), pp. 316–317.

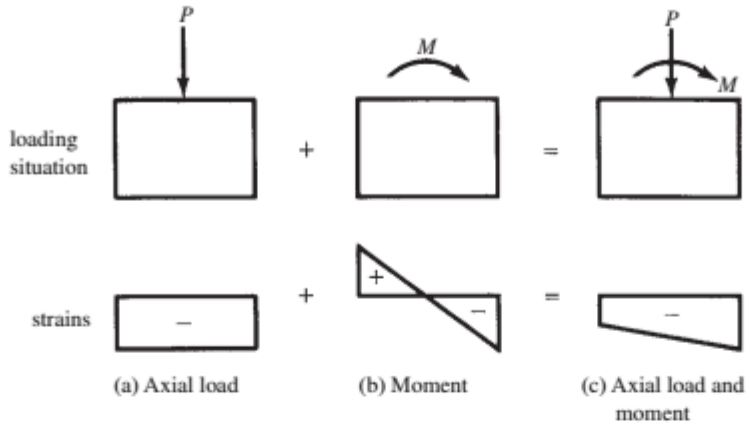


FIGURE 10.3 Column strains.

Example 10.2

It is assumed that the tied column of Figure 10.4 has a strain on its compression edge equal to –0.003 and has a tensile strain of +0.002 on its other edge. Determine the values of P_n and M_n that cause this strain distribution if $f_y = 60$ ksi and $f'_c = 4$ ksi. Approximate the tensile yield strain of the reinforcing steel as 0.002 (ACI Code, Section 21.2.2.1).

SOLUTION

Determine the values of c and of the steel strains ϵ'_s and ϵ_s by proportions with reference to the strain diagram shown in Figure 10.5.

$$c = \left(\frac{0.003}{0.003 + 0.002}\right)(24 \text{ in.}) = 14.40 \text{ in.}$$

$$\epsilon'_s = \left(\frac{11.90 \text{ in.}}{14.40 \text{ in.}}\right)(0.003) = 0.00248 > 0.00207 \qquad \therefore \text{ yields}$$

$$\epsilon_s = \left(\frac{7.10 \text{ in.}}{9.60 \text{ in.}}\right)(0.002) = 0.00148 \qquad \text{does not yield} \qquad \therefore \phi = 0.65 \text{ (Text Section 3.7)}$$

In the following calculations, C_c is the total compression in the concrete, C'_s is the total compression in the compression steel, and T_s is the total tension in the tensile steel. Each of these values is computed below.

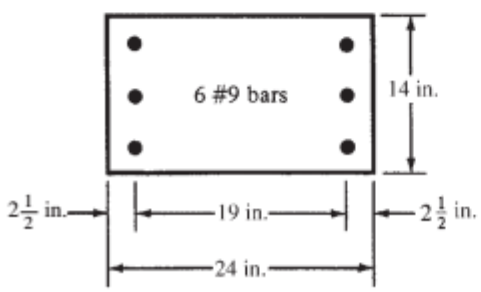


FIGURE 10.4 Column cross section for Example 10.2.

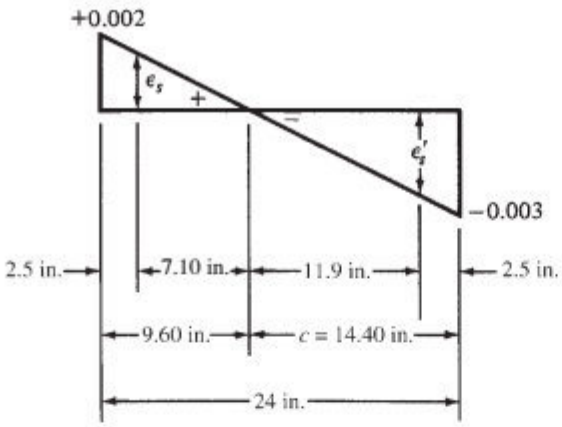


FIGURE 10.5 Strain diagram for Example 10.2.

The reader should note that C_s' is reduced by $0.85f_c'A_s'$ to account for concrete displaced by the steel in compression.

$$a = 0.85c = (0.85)\ (14.40\ \text{in.}) = 12.24\ \text{in.}$$

$$C_c = 0.85f_c'ab = (0.85)\ (12.24\ \text{in.})\ (14\ \text{in.})\ (4.0\ \text{ksi}) = -582.62\ \text{k}$$

$$C_s' = f_s'A_s' - 0.85A_s'f_c' = (60\ \text{ksi})\ (3.0\ \text{in.}^2) - (0.85)\ (3.0\ \text{in.}^2)\ (4.0\ \text{ksi}) = -169.8\ \text{k}$$

$$T_s = \epsilon_s E_s A_s = (0.00148)\ (29{,}000\ \text{ksi})\ (3.0\ \text{in.}^2) = +128.76\ \text{k}$$

By statics, P_n and M_n are determined with reference to Figure 10.6, where the values of C_c, C_s', and T_s are shown.

Summing forces vertically

$$-P_n + 169.8\ \text{k} + 582.62\ \text{k} - 128.76\ \text{k} = 0$$

$$P_n = 623.7\ \text{k}$$

$$\phi P_n = (0.65)\ (623.7\ \text{k}) = 405.4\ \text{k}$$

$\Sigma M = 0$ about Tensile Steel

$$(623.7\ \text{k})\ (9.50\ \text{in.}) + M_n - (582.62\ \text{k})\ (15.38\ \text{in.}) - (169.8\ \text{k})\ (19.00\ \text{in.}) = 0$$

$$M_n = 6261.3\ \text{in.-k} = 521.8\ \text{ft-k}$$

$$\phi M_n = (0.65)\ (6261.3\ \text{in.-k}) = 4069.8\ \text{in.-k} = 339.2\ \text{ft-k}$$

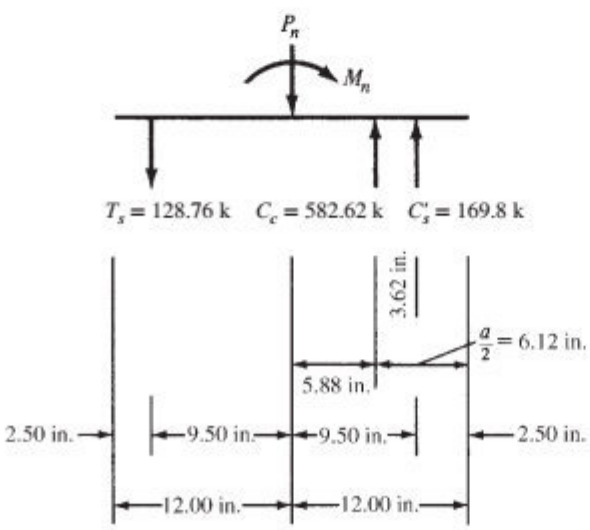


FIGURE 10.6 Internal column forces for Example 10.2.

In this manner, a series of P_n and M_n values is determined to correspond with a strain of −0.003 on the compression edge and varying strains on the far column edge. The resulting values are plotted on a curve, as shown in Figure 10.8.

A few remarks are made here concerning the extreme points on this curve. One end of the curve will correspond to the case where P_n is at its maximum value and M_n is zero. For this case, P_n is determined as in Chapter 9 for the axially loaded column of Example 10.2.

$$\begin{aligned} P_n &= 0.85f'_c(A_g - A_s) + A_s f_y \\ &= (0.85)\ (4.0\ \text{ksi})\ (14\ \text{in.} \times 24\ \text{in.} - 6.00\ \text{in.}^2) + (6.00\ \text{in.}^2)\ (60\ \text{ksi}) \\ &= 1482\ \text{k} \end{aligned}$$

On the other end of the curve, M_n is determined for the case where P_n is zero. This is the procedure used for a doubly reinforced member as previously described in Chapter 5. For the column of Example 10.2, M_n is equal to 297 ft-k.

A column reaches its ultimate capacity when the concrete reaches a compressive strain of 0.003. If the steel closest to the extreme tension side of the column reaches yield strain, or even more when the concrete reaches a strain of 0.003, the column is said to be tension controlled; otherwise, it is compression controlled. The transition point between these regions is the balance point. In Chapter 3, the term *balanced section* was used in referring to a section whose compression concrete strain reached 0.003 at the same time as the tensile steel reached its yield strain at f_y/E_s. In a beam, this situation theoretically occurred when the steel percentage equaled ρ_b. A column can undergo a balanced failure no matter how much steel it has if it has the right combination of moment and axial load.

For columns, the definition of balanced loading is the same as it was for beams—that is, a column has a strain of 0.003 on its compression side at the same time that its tensile steel on the other side has a strain of f_y/E_s. Although it is easily possible to prevent a balanced condition in beams by requiring that tensile steel strains be kept well above f_y/E_s, such is not the case for columns. Thus, for columns, it is not possible to prevent sudden compression failures or balanced failures. For every column, there is a balanced loading situation where an ultimate load, P_{bn}, placed at an eccentricity, e_b, will produce a moment, M_{bn}, at which time the balanced strains will be reached simultaneously.

Courtesy of EFCO Corp.

Washington Redskins Stadium.

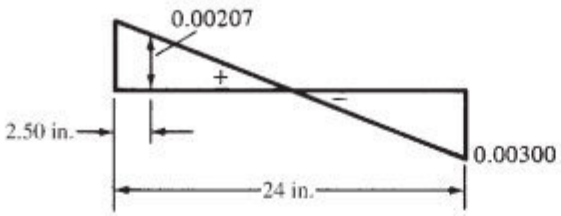


FIGURE 10.7 Strain diagram for balanced conditions.

At the balanced condition, we have a strain of −0.003 on the compression edge of the column and a strain of $f_y/29 \times 10^3$ ksi = 60 ksi/29×10^3 ksi = 0.00207 in the tensile steel. This information is shown in Figure 10.7. The same procedure used in Example 10.2 is used to find $P_n = 504.4$ k and $M_n = 559.7$ ft-k.

The curve for P_n and M_n for a particular column may be extended into the range where P_n becomes a tensile load. We can proceed in exactly the same manner as we did when P_n was compressive. A set of strains can be assumed, and the usual statics equations can be written and solved for P_n and M_n. Several different sets of strains were assumed for the column of Figure 10.4, and then the values of P_n and M_n were determined. The results were plotted at the bottom part of Figure 10.8 and were connected with the dashed line labeled "tensile loads."

Because axial tension and bending are not very common for reinforced concrete columns, the tensile load part of the curves is not shown in subsequent figures in this chapter. You will note that the largest tensile value of P_n will occur when the moment is zero. For that situation, all of the column steel has yielded, and all of the concrete has cracked. Thus, P_n will equal the total steel area, A_s, times the yield stress. For the column of Figure 10.4

$$P_n = A_s f_y = (6.0 \text{ in.}^2)\,(60 \text{ ksi}) = 360 \text{ k}$$

On some occasions, members subject to axial load and bending have unsymmetrical arrangements of reinforcing bars. Should this be the case, you must remember that eccentricity is correctly measured from the plastic centroid of the section.

In this chapter, P_n values were obtained only for rectangular tied columns. The same theory could be used for round columns, but the mathematics would be somewhat complicated because of the circular layout of the bars, and the calculations of distances would be rather tedious. Several approximate

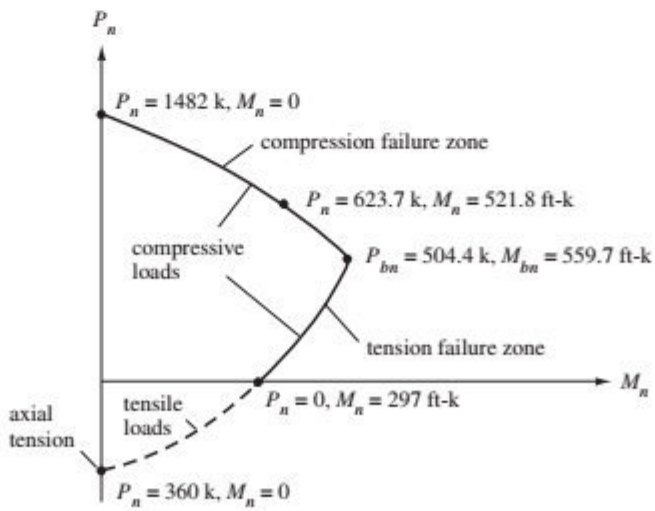
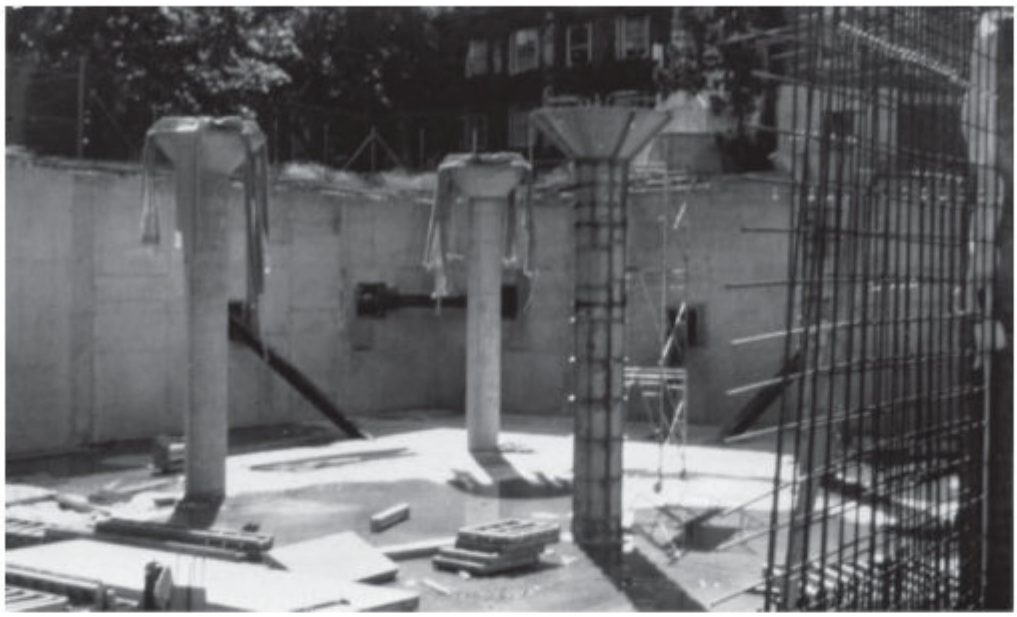

P_n
$P_n = 1482$ k, $M_n = 0$
compression failure zone
$P_n = 623.7$ k, $M_n = 521.8$ ft-k
compressive loads
$P_{bn} = 504.4$ k, $M_{bn} = 559.7$ ft-k
tension failure zone
M_n
$P_n = 0$, $M_n = 297$ ft-k
tensile loads
axial tension
$P_n = 360$ k, $M_n = 0$

FIGURE 10.8 Interaction curve for the column of Figure 10.4. Notice these are nominal values.



Courtesy of EFCO Corp.

Round columns.

methods have been developed that greatly simplify the mathematics. Perhaps the best known of these is the one proposed by Charles Whitney, in which equivalent rectangular columns are used to replace the circular ones.[2] This method gives results that correspond quite closely with test results.

In Whitney's method, the area of the equivalent column is made equal to the area of the actual circular column, and its depth in the direction of bending is 0.80 times the outside diameter of the real column. One-half the steel is assumed to be placed on one side of the equivalent column and one-half on the other. The distance between these two areas of steel is assumed to equal two-thirds of the diameter (D_s) of a circle passing through the center of the bars in the real column. These values are illustrated in Figure 10.9. Once the equivalent column is established, the calculations for P_n and M_n are made as for rectangular columns.

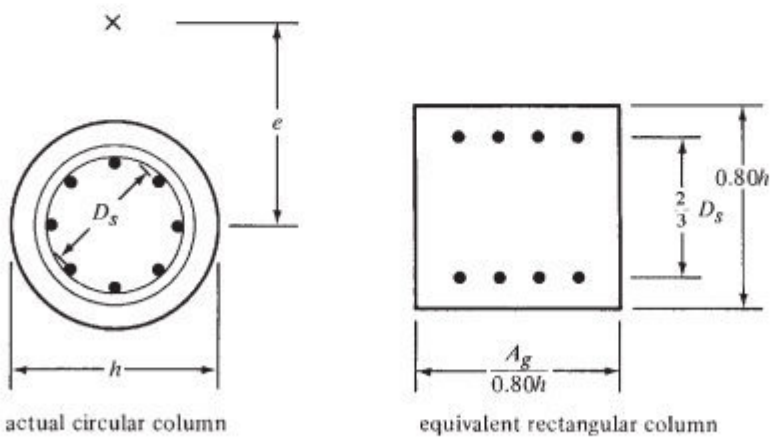


FIGURE 10.9 Replacing a circular column with an equivalent rectangular one.

[2] Whitney, Charles S., 1942, "Plastic Theory of Reinforced Concrete Design," *Transactions ASCE*, 107, pp. 251–326.

10.4 Use of Interaction Diagrams

We have seen that by statics, the values of P_n and M_n for a given column with a certain set of strains can easily be determined. Preparing an interaction curve with a hand calculator for just one column, however, is quite tedious. Imagine the work involved in a design situation where various sizes, concrete strengths, and steel percentages need to be considered. Consequently, designers resort almost completely to computer programs, computer-generated interaction diagrams, or tables for their column calculations. The remainder of this chapter is concerned primarily with computer-generated interaction diagrams such as the one in Figure 10.10. As we have seen, such a diagram is drawn for a column as the load changes from one of a pure axial nature through varying combinations of axial loads and moments and on to a pure bending situation.

Interaction diagrams are useful for studying the strengths of columns with varying proportions of loads and moments. Any combination of loading that falls inside the curve is satisfactory, whereas any combination falling outside the curve represents failure.

If a column is loaded to failure with an axial load only, the failure will occur at point A on the diagram (Figure 10.10). Moving out from point A on the curve, the axial load capacity decreases as the proportion of bending moment increases. At the very bottom of the curve, point C represents the bending strength of the member if it is subjected to moment only with no axial load present. In between the extreme points A and C, the column fails because of a combination of axial load and bending. Point B is called the *balanced point* and represents the balanced loading case, where theoretically a compression failure and tensile yielding occur simultaneously.

Refer to point D on the curve. The horizontal and vertical dashed lines to this point indicate a particular combination of axial load and moment at which the column will fail. Should a radial line be drawn from point 0 to the interaction curve at any point (as to D in this case), it will represent a constant eccentricity of load, that is, a constant ratio of moment to axial load.

You may be somewhat puzzled by the shape of the lower part of the curve from B to C, where bending predominates. From A to B on the curve, the moment capacity of a section increases as the axial load decreases, but just the opposite occurs from B to C. A little thought on this point, however,

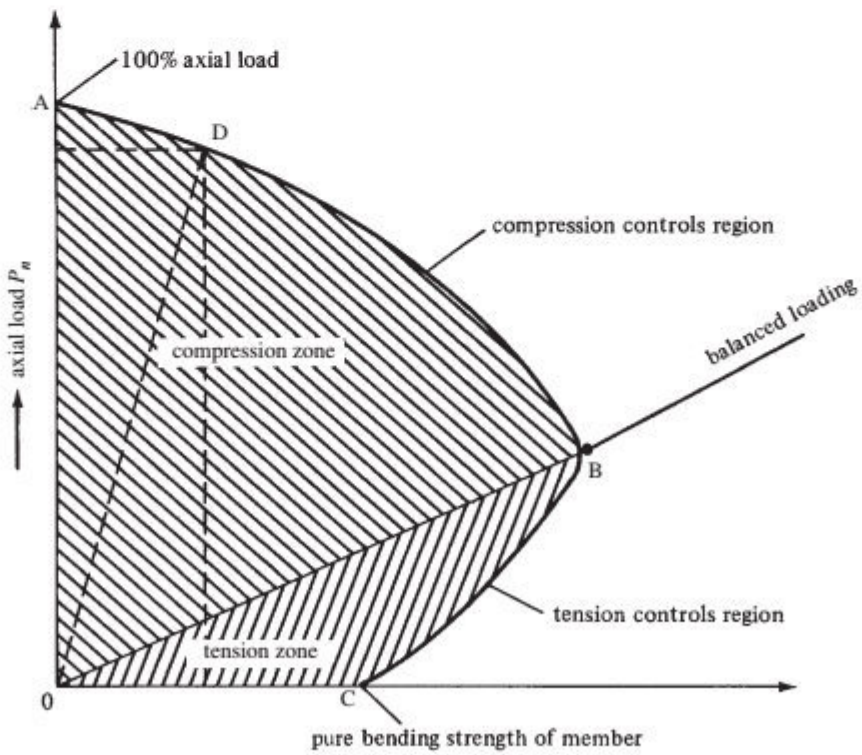


FIGURE 10.10 Column interaction diagram.

Reinforced concrete columns.

shows that the result is quite logical after all. The part of the curve from B to C represents the range of tensile failures. Any axial compressive load in that range tends to reduce the stresses in the tensile bars, with the result that a larger moment can be resisted.

In Figure 10.11, an interaction curve is drawn for the 14-in. × 24-in. column with 6 #9 bars considered in Section 10.3. If 8 #9 bars had been used in the same dimension column, another curve

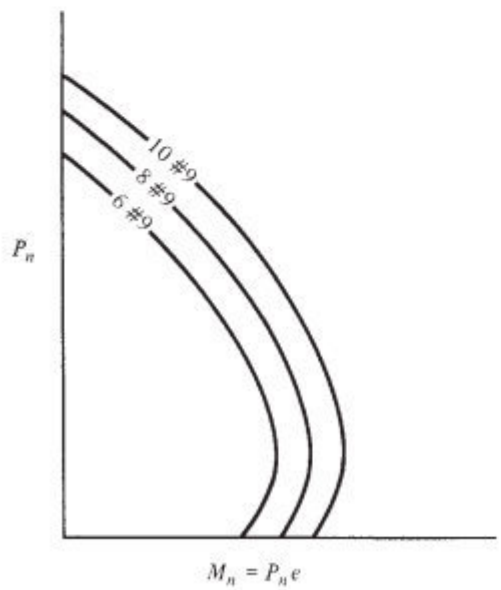


FIGURE 10.11 Interaction curves for a rectangular column with different sets of reinforcing bars.

could be generated as shown in the figure; if 10 #9 bars were used, still another curve would result. The shape of the new diagrams would be same as the curve for 6 #9 bars, but the values of P_n and M_n would be larger.

10.5 Code Modifications of Column Interaction Diagrams

If interaction curves for P_n and M_n values were prepared, they would be of the types shown in Figures 10.10 and 10.11. To use such curves to obtain design values, they would have to have three modifications made to them as specified in the code. These modifications are as follows:

(a) The ACI Code, Section 21.2.1a specifies strength reduction or ϕ factors (0.65 for tied columns and 0.75 for spiral columns) that must be multiplied by P_n values. If a P_n curve for a particular column were multiplied by ϕ, the result would be a curve something like the ones shown in Figure 10.12.

(b) The second modification also refers to ϕ factors. The code specifies values of 0.65 and 0.75 for tied and spiral columns, respectively. Should a column have quite a large moment and a very small axial load so that it falls on the lower part of the curve between points B and C (see Figure 10.10), the use of these small ϕ values may be a little unreasonable. For instance, for a member in pure bending (point C on the same curve), the specified ϕ is 0.90, but if the same member has a very small axial load added, ϕ would immediately fall to 0.65 or 0.75. Therefore, the ACI Code, Section 21.2.2 states that when members subject to axial load and bending have net tensile strains (ϵ_t) between the limits for compression-controlled and tensile-controlled sections, they fall in the transition zone for ϕ. In this zone, it is permissible to increase ϕ linearly from 0.65 or 0.75 to 0.90 as ϵ_t increases from the compression-controlled limit to 0.005. In this regard, the reader is again referred to Figure 3.5 in Chapter 3 where the transition zone and the variation of ϕ values are clearly shown. This topic is continued in Section 10.10 of the text.

(c) As described in Chapter 9, maximum permissible column loads were specified for columns no matter how small their e values. As a result, the upper part of each design interaction curve is shown as a horizontal line representing the appropriate value of

$$P_u = \phi P_{n\,\max} \text{ for tied columns} = 0.80\phi[0.85f'_c(A_g - A_{st}) + f_y A_{st}]$$

(ACI Equation 22.4.2.1b[3])

$$P_u = \phi P_{n\,\max} \text{ for spiral columns} = 0.85\phi[0.85f'_c(A_g - A_{st}) + f_y A_{st}]$$

(ACI Equation 22.4.2.1a)

These formulas were developed to be approximately equivalent to loads applied with eccentricities of $0.10h$ for tied columns and $0.05h$ for spiral columns.

Each of the three modifications described here is indicated on the design curve of Figure 10.13. In Figure 10.13, the solid curved line represents P_u and M_u, whereas the dashed curved line is P_n and M_n. The difference between the two curves is the ϕ factor. The two curves would have the same shape if the ϕ factor did not vary. Above the radial line labeled "balanced case," $\phi = 0.65$ (0.75 for spirals). Below the other radial line, labeled "strain of 0.005," $\phi = 0.9$. It varies between the two values in between, and the P_u versus M_u curve assumes a different shape.

[3] The equations shown include the ϕ factor, which is not included in ACI Equations 22.4.2.1a or 22.4.2.1b.

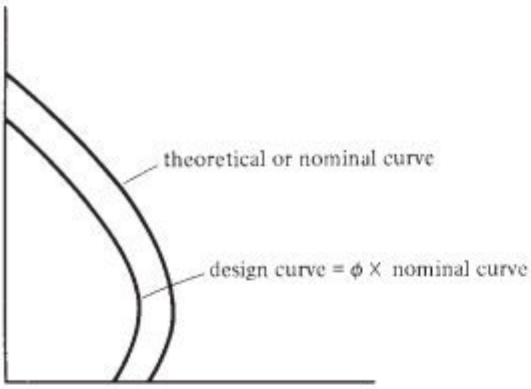


FIGURE 10.12 Curves for P_n and ϕP_n for a single column.

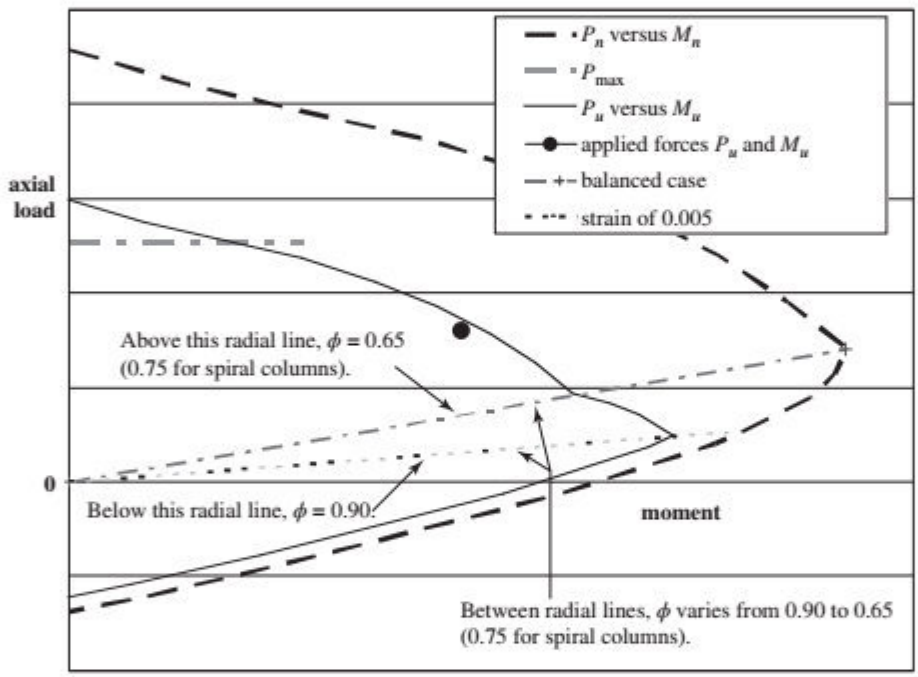

P_n versus M_n
P_{max}
P_u versus M_u
applied forces P_u and M_u
balanced case
strain of 0.005
axial load
Above this radial line, $\phi = 0.65$ (0.75 for spiral columns).
0
Below this radial line, $\phi = 0.90$.
moment
Between radial lines, ϕ varies from 0.90 to 0.65 (0.75 for spiral columns).

FIGURE 10.13 A column interaction curve adjusted for the three modifications described in text Section 10.5.

10.6 Design and Analysis of Eccentrically Loaded Columns Using Interaction Diagrams

If individual column interaction diagrams were prepared as described in the preceding sections, it would be necessary to have a diagram for each different column cross section, for each different set of concrete and steel grades, and for each different bar arrangement. The result would be an astronomical number of diagrams. The number can be tremendously reduced, however, if the diagrams are plotted with ordinates of $K_n = P_n/f'_c A_g$ (instead of P_n) and with abscissas of $R_n = P_n e/f'_c A_g h$ (instead of M_n). The resulting normalized interaction diagrams can be used for cross sections with widely

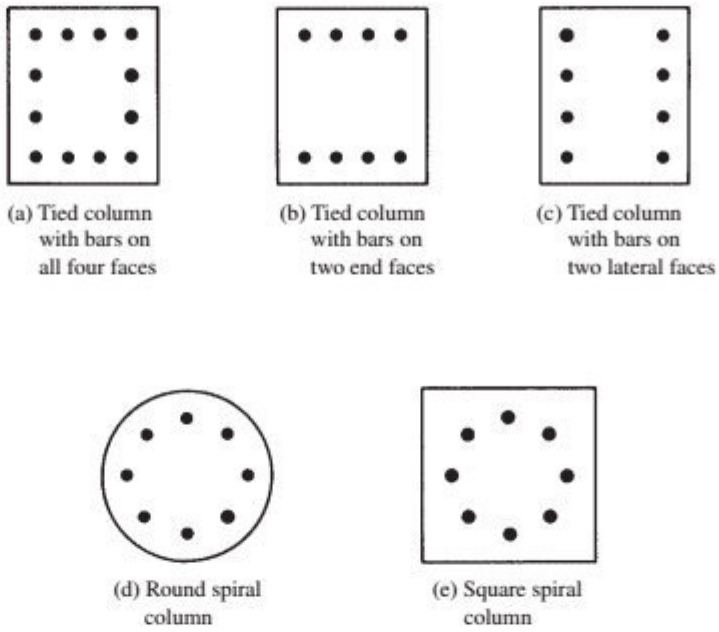

(a) Tied column with bars on all four faces
(b) Tied column with bars on two end faces
(c) Tied column with bars on two lateral faces
(d) Round spiral column
(e) Square spiral column

FIGURE 10.14 Column cross sections for normalized interaction curves in Appendix A, Graphs 2–13.

varying dimensions. The ACI has prepared normalized interaction curves in this manner for the different cross section and bar arrangement situations shown in Figure 10.14 and for different grades of steel and concrete.[4]

Two of the ACI diagrams are given in Figures 10.15 and 10.16, while Appendix A (Graphs 2 to 13) presents several other ones for the situations given in parts (a), (b), and (d) of Figure 10.14. *Notice that these ACI diagrams do not include the three modifications described in Section 10.5 of the text.*

The ACI column interaction diagrams are used in Examples 10.3 to 10.7 to design or analyze columns for different situations. In order to correctly use these diagrams, it is necessary to compute the value of γ (gamma), which is equal to the distance from the center of the bars on one side of the column to the center of the bars on the other side of the column divided by h, the depth of the column (both values being taken in the direction of bending). Usually the value of γ obtained falls in between a pair of curves, and interpolation of the curve readings will have to be made.

Caution

Be sure that the column picture at the upper right of the interaction curve being used agrees with the column being considered. In other words, are there bars on two faces of the column or on all four faces? If the wrong curves are selected, the answers may be quite incorrect.

Although several methods are available for selecting column sizes, a trial-and-error method is about as good as any. With this procedure, the designer estimates what he or she thinks is a reasonable

[4] American Concrete Institute, 1997, *Design Handbook* (Farmington Hills, MI: ACI). Publication SP-17 (97).

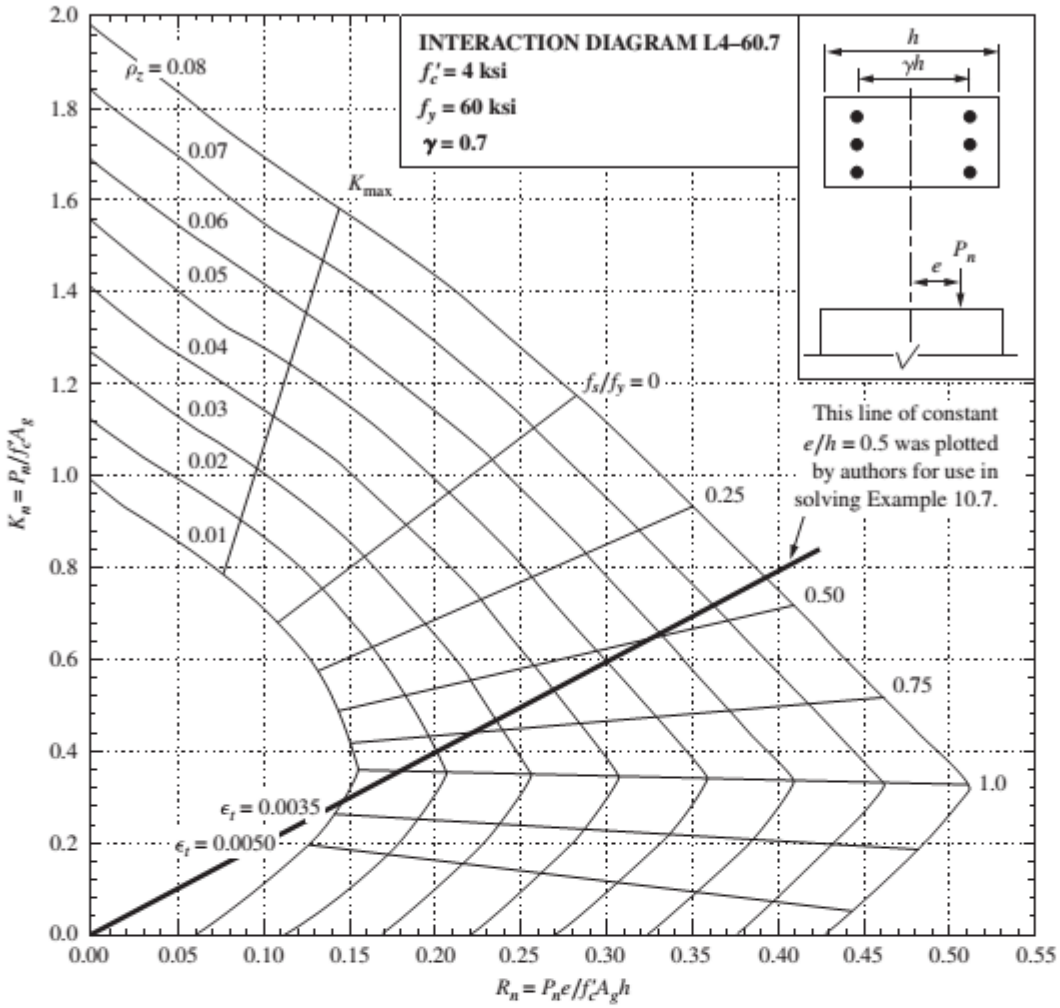


FIGURE 10.15 ACI rectangular column interaction diagrams when bars are placed on two faces only. (Permission of American Concrete Institute.)

column size and then determines the steel percentage required for that column size from the interaction diagram. If it is felt that the ρ_g determined is unreasonably large or small, another column size can be selected and the new required ρ_g selected from the diagrams, and so on. In this regard, the selection of columns for which ρ_g is greater than 4% or 5% results in congestion of the steel, particularly at splices, and consequent difficulties in getting the concrete down into the forms.

A slightly different approach is used in Example 10.4, where the average compression stress at ultimate load across the column cross section is assumed to equal some value—say, $0.5f'_c$ to $0.6f'_c$. This value is divided into P_n to determine the column area required. Cross-sectional dimensions are then selected, and the value of ρ_g is determined from the interaction curves. Again, if the percentage obtained seems unreasonable, the column size can be revised and a new steel percentage is obtained.

In Examples 10.3 to 10.5, reinforcing bars are selected for three columns. The values of $K_n = P_n/f'_cA_g$ and $R_n = P_ne/f'_cA_gh$ are computed. The position of those values is located on the appropriate graph, and ρ_g is determined and multiplied by the gross area of the column in question to determine the reinforcing area needed.

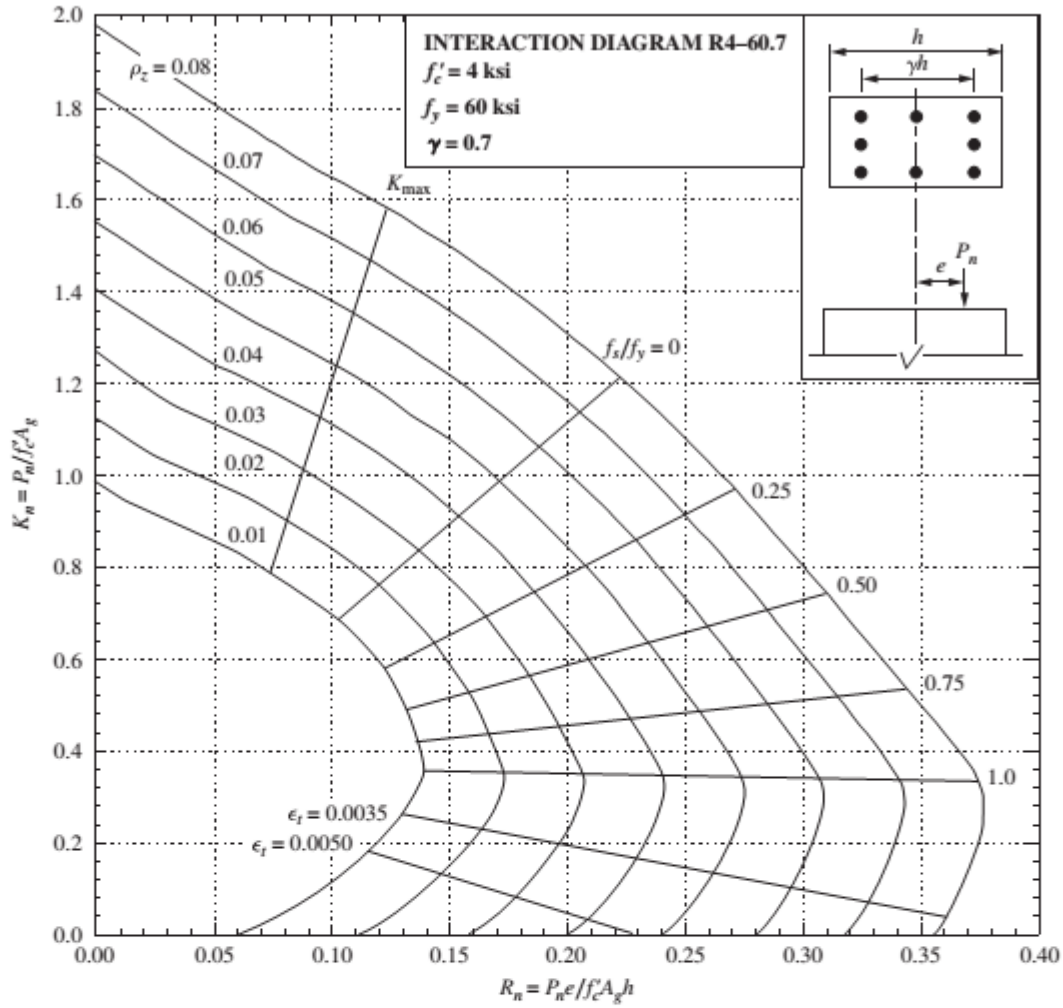


FIGURE 10.16 ACI rectangular column interaction diagram when bars are placed along all four faces. (Permission of American Concrete Institute.)

Example 10.3

The short 14-in. × 20-in. tied column of Figure 10.17 is to be used to support the following loads and moments: P_D = 125 k, P_L = 140 k, M_D = 75 ft-k, and M_L = 90 ft-k. If f'_c = 4000 psi and f_y = 60,000 psi, select reinforcing bars to be placed in its end faces only using appropriate ACI column interaction diagrams.

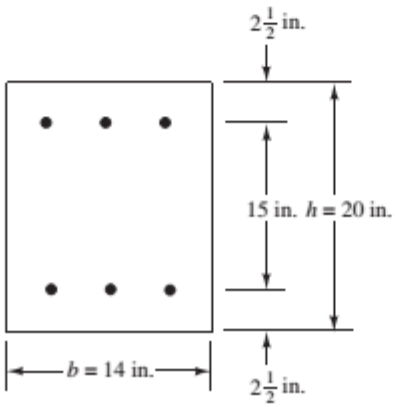


FIGURE 10.17 Column cross section for Example 10.3.

SOLUTION

$$P_u = (1.2)\,(125\text{ k}) + (1.6)\,(140\text{ k}) = 374\text{ k}$$

$$P_n = \frac{374\text{ k}}{0.65} = 575.4\text{ k}$$

$$M_u = (1.2)\,(75\text{ ft-k}) + (1.6)\,(90\text{ ft-k}) = 234\text{ ft-k}$$

$$M_n = \frac{234\text{ ft-k}}{0.65} = 360\text{ ft-k}$$

$$e = \frac{(12\text{ in./ft})\,(360\text{ ft-k})}{575.4\text{ k}} = 7.51\text{ in.}$$

$$\gamma = \frac{15\text{ in.}}{20\text{ in.}} = 0.75$$

Compute Values of K_n and R_n

$$K_n = \frac{P_n}{f'_c A_g} = \frac{575.4\text{ k}}{(4\text{ ksi})\,(14\text{ in.} \times 20\text{ in.})} = 0.513$$

$$R_n = \frac{P_n e}{f'_c A_g h} = \frac{(575.4\text{ k})\,(7.51\text{ in.})}{(4\text{ ksi})\,(14\text{ in.} \times 20\text{ in.})\,(20\text{ in.})} = 0.193$$

The value of γ falls between γ values for Graphs 3 and 4 of Appendix A. Therefore, interpolating between the two gives results as follows:

γ	0.70	0.75	0.80
ρ_g	0.0220	0.0202	0.0185

$$A_s = \rho_g bh = (0.0202)\,(14\text{ in.})\,(20\text{ in.}) = 5.66\text{ in.}^2$$

Use 6 #9 bars = 6.00 in.^{2}

Notes

(a) Note that $\phi = 0.65$ as initially assumed since the graphs used show f_s/f_y is < 1.0 and, thus, $\epsilon_t < 0.002$.

(b) Code requirements must be checked as in Example 9.1. (See Figure 10.25 to visualize this.)

Example 10.4

Design a short square column for the following conditions: $P_u = 600$ k, $M_u = 80$ ft-k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi. Place the bars uniformly around all four faces of the column.

SOLUTION

Assume the column will have an average compression stress = about $0.6f'_c = 2400$ psi = 2.4 ksi.

$$A_{g\text{ required}} = \frac{600\text{ k}}{2.4\text{ ksi}} = 250\text{ in.}^2$$

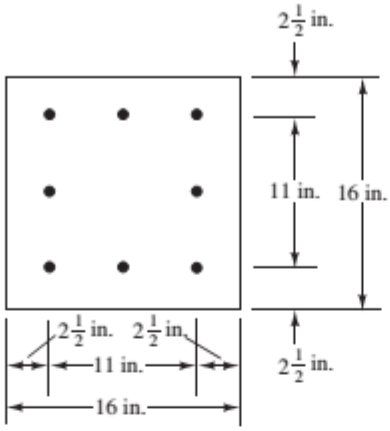


FIGURE 10.18 Column cross section for Example 10.4.

Try a 16-in. × 16-in. column ($A_g = 256$ in.^{2}) with the bar arrangement shown in Figure 10.18.

$$e = \frac{M_u}{P_u} = \frac{(12 \text{ in./ft})\,(80 \text{ ft-k})}{600 \text{ k}} = 1.60 \text{ in.}$$

$$P_n = \frac{P_u}{\phi} = \frac{600 \text{ k}}{0.65} = 923.1 \text{ k}$$

$$K_n = \frac{P_n}{f'_c A_g} = \frac{923.1 \text{ k}}{(4 \text{ ksi})\,(16 \text{ in.} \times 16 \text{ in.})} = 0.901$$

$$R_n = \frac{P_n e}{f'_c A_g h} = \frac{(923.1 \text{ k})\,(1.6 \text{ in.})}{(4 \text{ ksi})\,(16 \text{ in.} \times 16 \text{ in.})\,(16 \text{ in.})} = 0.0901$$

$$\gamma = \frac{11 \text{ in.}}{16 \text{ in.}} = 0.6875$$

Interpolating between values given in Graphs 6 and 7 of Appendix A, the results are as follows:

γ	0.600	0.6875	0.700
ρ_g	0.025	0.023	0.022

$$A_s = (0.023)\,(16 \text{ in.})\,(16 \text{ in.}) = 5.89 \text{ in.}^2$$

Use 8 #8 bars = 6.28 in.^{2}

Notes

(a) Note that $\phi = 0.65$ as initially assumed since the graphs used show $f_s/f_y < 1.0$ and, thus, $\epsilon_t < 0.002$.

(b) Code requirements must be checked as in Example 9.1. (See Figure 10.25.)

Example 10.5

Using the ACI column interaction graphs, select reinforcing bars for the short round spiral column shown in Figure 10.19 if $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, $P_u = 500$ k, and $M_u = 225$ ft-k.

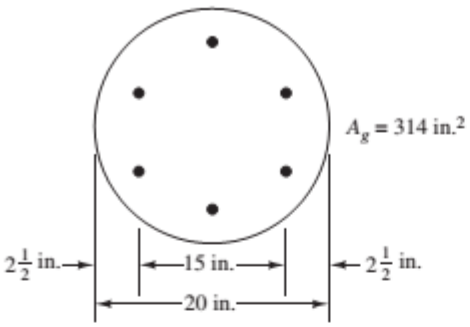


FIGURE 10.19 Column cross section for Example 10.5.

SOLUTION

$$e = \frac{(12 \text{ in./ft})\,(225 \text{ ft-k})}{500 \text{ k}} = 5.40 \text{ in.}$$

$$P_n = \frac{P_u}{\phi} = \frac{500 \text{ k}}{0.75} = 666.7 \text{ k}$$

$$K_n = \frac{P_n}{f'_c A_g} = \frac{666.7 \text{ k}}{(4 \text{ ksi})\,(314 \text{ in.}^2)} = 0.531$$

$$R_n = \frac{P_n e}{f'_c A_g h} = \frac{(666.7 \text{ k})\,(5.40 \text{ in.})}{(4 \text{ ksi})\,(314 \text{ in.}^2)\,(20 \text{ in.})} = 0.143$$

$$\gamma = \frac{15 \text{ in.}}{20 \text{ in.}} = 0.75$$

By interpolation between Graphs 11 and 12 of Appendix A, ρ_g is found to equal 0.0235 and $f_s/f_y < 1.0$.

$$\rho_g A_g = (0.0235)\,(314 \text{ in.}^2) = 7.38 \text{ in.}^2$$

Use 8 #9 bars = 8.00 in.^{2}

The same notes apply here as for Examples 10.3 and 10.4.

In Example 10.6, it is desired to select a 14-in. wide column with approximately 2% steel. This is done by trying different column depths and then determining the steel percentage required in each case.

Example 10.6

Design a 14-in.-wide rectangular short tied column with bars only in the two end faces for $P_u = 500$ k, $M_u = 250$ ft-k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi. Select a column with approximately 2% steel.

SOLUTION

$$e = \frac{M_u}{P_u} = \frac{(12 \text{ in./ft})\,(250 \text{ ft-k})}{500 \text{ k}} = 6.00 \text{ in.}$$

$$P_n = \frac{P_u}{\phi} = \frac{500 \text{ k}}{0.65} = 769.2 \text{ k}$$

Trying several column sizes and determining reinforcing requirements.

Trial sizes (in.)	14 × 20	14 × 22	14 × 24
$K_n = \dfrac{P_n}{f'_c A_g}$	0.687	0.624	0.572
$R_n = \dfrac{P_n e}{f'_c A_g h}$	0.206	0.170	0.143
$\gamma = \dfrac{h - (2 \times 2.50 \text{ in.})}{h}$	0.750	0.773	0.792
ρ_g by interpolation	0.0315	0.020	0.011

Use 14-in. × 22-in. column

$$A_s = (0.020)\ (14 \text{ in.} \times 22 \text{ in.}) = 6.16 \text{ in.}^2$$

Use 8 #8 bars = 6.28 in.^{2}

Same notes as for Examples 10.3 and 10.4.

One more illustration of the use of the ACI interaction is presented with Example 10.7. In this example, the nominal column load P_n at a given eccentricity that a column can support is determined.

With reference to the ACI interaction curves, the reader should carefully note that the value of R_n (which is $P_n e/f'_c A_g h$) for a particular column equals e/h times the value of K_n ($P_n/f'_c A_g$) for that column. This fact needs to be understood when the user desires to determine the nominal load that a column can support at a given eccentricity.

In Example 10.7, the nominal load that the short column of Figure 10.20 can support at an eccentricity of 10 in. with respect to the x-axis is determined. If we plot on the interaction diagram the intersection point of K_n and R_n for a particular column and draw a straight line from that point to the lower-left corner or origin of the figure, we will have a line with a constant e/h. For the column of Example 10.6, $e/h = 10$ in./20 in. $= 0.5$. Therefore, a line is plotted from the origin through a set of assumed values for K_n and R_n in the proportion of 10/20 to each other. In this case, K_n was set equal to 0.8 and $R_n = 0.5 \times 0.8 = 0.4$. Next, a line was drawn from that intersection point to the origin of the diagram, as shown in Figure 10.15. Finally, the intersection of this line with ρ_g (0.0316 in this example) was determined, and the value of K_n or R_n was read. This latter value enables us to compute P_n.

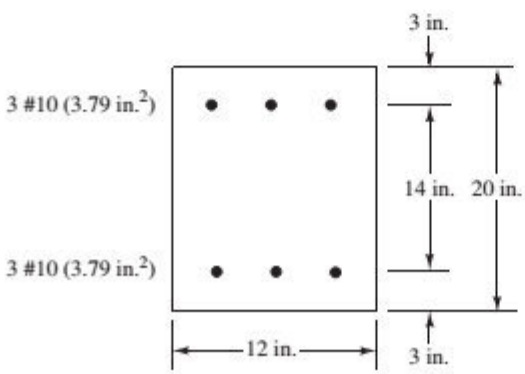


FIGURE 10.20 Column cross section for Example 10.6.

Example 10.7

Using the appropriate interaction curves, determine the value of P_n for the short tied column shown in Figure 10.20 if $e_x = 10$ in. Assume $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

$$\frac{e}{h} = \frac{10 \text{ in.}}{20 \text{ in.}} = 0.500$$

$$\rho_g = \frac{(2)\,(3.79 \text{ in.}^2)}{(12 \text{ in.})\,(20 \text{ in.})} = 0.0316$$

$$\gamma = \frac{14 \text{ in.}}{20 \text{ in.}} = 0.700$$

We plot a straight line through the origin and the intersection of assumed values of K_n and R_n (say, 0.8 and 0.4, respectively).

For ρ_g of 0.0316, we read the value of $R_n = 0.24$.

$$R_n = \frac{P_n e}{f'_c A_g h} = 0.24$$

$$P_n = \frac{(0.24)\,(4 \text{ ksi})\,(12 \text{ in.} \times 20 \text{ in.})\,(20 \text{ in.})}{10 \text{ in.}} = 460.8 \text{ k}$$

When the usual column is subjected to axial load and moment, it seems reasonable to assume initially that $\phi = 0.65$ for tied columns and 0.75 for spiral columns. It is to be remembered, however, that under certain conditions, these ϕ values may be increased, as discussed in detail in Section 10.10.

10.7 Shear in Columns

The shearing forces in interior columns in braced structures are usually quite small and normally do not control the design. However, the shearing forces in exterior columns can be large, even in a braced structure, particularly in columns bent in double curvature. Section 22.5.6.1 of the ACI Code provides the following equations for determining the shearing force that can be carried by the concrete for a member subjected simultaneously to axial compression and shearing forces.

$$V_c = 2\left(1 + \frac{N_u}{2000A_g}\right)\lambda\sqrt{f'_c}\,b_w d \qquad \text{(ACI Equation 22.5.6.1)}$$

In SI units this equation is:

$$V_c = 0.17\left(1 + \frac{N_u}{14A_g}\right)\left(\lambda\sqrt{f'_c}\right)b_w d$$

In these equations, N_u is the factored axial force acting simultaneously with the factored shearing force, V_u, that is applied to the member. The value of N_u/A_g is the average factored axial stress in the column and is expressed in units of psi. Should V_u be greater than $\phi V_c/2$, it will be necessary to calculate required tie spacing using the stirrup spacing procedures described in Chapter 8. The results will sometimes be closer tie spacing than required by the usual column rules discussed earlier in Section 9.5 of the text.

Sections 22.5.2.2 and 22.5.10.5.6 of the ACI Code specify the method for calculating the contribution of the concrete to the total shear strength of circular columns and for calculating the contribution of shear reinforcement for cases where circular hoops, ties, or spirals are present. According to the commentary of the code in Section 22.5.2.2, the entire cross section in circular columns is effective in resisting shearing forces. The shear area, $b_w d$, in ACI Equation 22.5.6.1 then would be equal to the gross area of the column. However, to provide for compatibility with other calculations requiring an effective depth, the ACI requires that, when applied to circular columns, the shear area in ACI Equation 22.5.6.1 be computed as an equivalent rectangular area in which

$$b_w = D$$
$$d = 0.8D$$

In these equations, D is the gross diameter of the column. If the constant modifying D in the effective depth equation were equal to $\pi/4$, which is equal to 0.7854, the effective rectangular area would be equal to the gross area of the circular column. Thus, the area of the column is overestimated by a little less than 2% when using the equivalent area prescribed by the ACI.

10.8 Biaxial Bending

Many columns are subjected to biaxial bending, that is, bending about both axes. Corner columns in buildings where beams and girders frame into the columns from both directions are the most common cases, but there are others, such as where columns are cast monolithically as part of frames in both directions or where columns are supporting heavy spandrel beams. Bridge piers are almost always subject to biaxial bending.

Circular columns have polar symmetry and, thus, the same ultimate capacity in all directions. The design process is the same, therefore, regardless of the directions of the moments. If there is bending about both the x- and y-axes, the biaxial moment can be computed by combining the two moments or their eccentricities as follows:

$$M_u = \sqrt{(M_{ux})^2 + (M_{uy})^2}$$

or

$$e = \sqrt{(e_x)^2 + (e_y)^2}$$

For shapes other than circular ones, it is necessary to consider the three-dimensional interaction effects. Whenever possible, it is desirable to make columns subject to biaxial bending circular in shape. Should it be necessary to use square or rectangular columns for such cases, the reinforcing bars should be placed uniformly around the perimeters.

You might quite logically think that you could determine P_n for a biaxially loaded column by using static equations, as was done in Example 10.2. Such a procedure will lead to the correct answer, but the mathematics involved is so complicated because of the shape of the compression side of the column that the method is not a practical one. Nevertheless, a few comments are made about this type of solution, and reference is made to Figure 10.21.

An assumed location is selected for the neutral axis, and the appropriate strain triangles are drawn, as shown in Figure 10.21. The usual equations are written with $C_c = 0.85f'_c$ times the shaded area A_c and with each bar having a force equal to its cross-sectional area times its stress. The solution of the equation yields the load that would establish that neutral axis—but the designer usually starts with certain loads and eccentricities and does not know the neutral axis location. Furthermore, the neutral axis is probably not even perpendicular to the resultant $e = \sqrt{(e_x)^2 + (e_y)^2}$.

For column shapes other than circular ones, it is desirable to consider three-dimensional interaction curves such as the one shown in Figure 10.22. In this figure, the curve labeled M_{nxo} represents

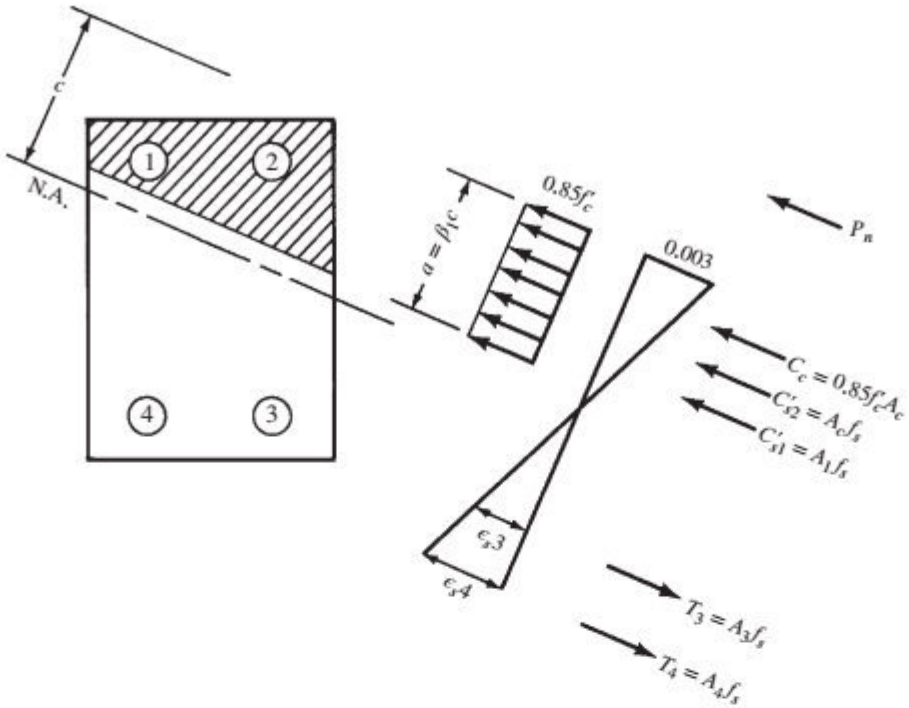


FIGURE 10.21 Column cross section with skewed neutral axis from biaxial bending.

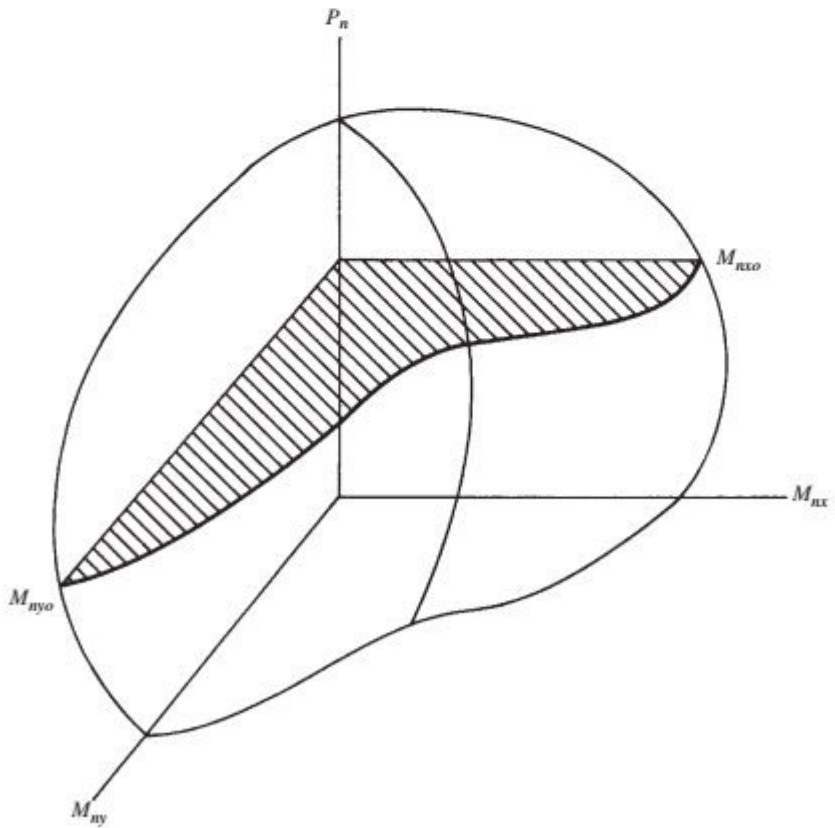


FIGURE 10.22 Interaction surface for biaxially loaded column.

the interaction curve if bending occurs about the x-axis only, whereas that labeled M_{nyo} represents the curve if bending occurs about the y-axis only.

In this figure, for a constant P_n, the hatched plane shown represents the contour of M_n for bending about any axis.

Today, the analysis of columns subject to biaxial bending is primarily done with computers. One of the approximate methods that is useful in analysis and that can be handled with pocket calculators includes the use of the so-called reciprocal interaction equation, which was developed by Professor Boris Bresler of the University of California at Berkeley.[5] This equation, which is shown in Section R10.3.6 of the ACI 318-11 Commentary[6], follows:

$$\frac{1}{P_{ni}} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_o}$$

where

P_{ni} = the nominal axial load capacity of the section when the load is placed at a given eccentricity along both axes

P_{nx} = the nominal axial load capacity of the section when the load is placed at an eccentricity e_x

P_{ny} = the nominal axial load capacity of the section when the load is placed at an eccentricity e_y

P_o = the nominal axial load capacity of the section when the load is placed with a zero eccentricity. It is usually taken as $0.85f'_cA_g + f_yA_s$

The Bresler equation works rather well as long as P_{ni} is at least as large as $0.10P_o$. Should P_{ni} be less than $0.10P_o$, it is satisfactory to neglect the axial force completely and design the section as a member subject to biaxial bending only. This procedure is a little on the conservative side. For this lower part of the interaction curve, it will be remembered that a little axial load increases the moment capacity of the section. The Bresler equation does not apply to axial tension loads. Professor Bresler found that the ultimate loads predicted by his equation for the conditions described do not vary from test results by more than 10%.

Example 10.8 illustrates the use of the reciprocal theorem for the analysis of a column subjected to biaxial bending. The procedure for calculating P_{nx} and P_{ny} is the same as the one used for the prior examples of this chapter.

Example 10.8

Determine the design capacity, P_{ni}, of the short tied column shown in Figure 10.23, which is subjected to biaxial bending. $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, $e_x = 16$ in., and $e_y = 8$ in.

SOLUTION

For Bending about *x*-Axis

$$\gamma = \frac{20 \text{ in.}}{25 \text{ in.}} = 0.80$$

$$\rho_g = \frac{8.00 \text{ in.}^2}{(15 \text{ in.})\,(25 \text{ in.})} = 0.0213$$

$$\frac{e}{h} = \frac{16 \text{ in.}}{25 \text{ in.}} = 0.64$$

[5] Bresler, B., 1960, "Design Criteria for Reinforced Concrete Columns under Axial Load and Biaxial Bending," *Journal ACI*, 57, p. 481.
[6] The reciprocal interaction equation does not appear in ACI 318-14.

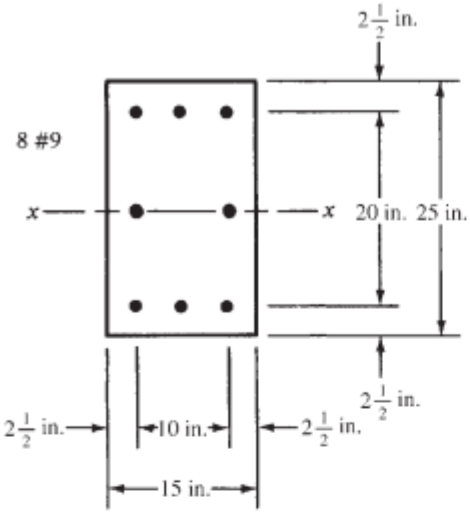


FIGURE 10.23 Column cross section for Example 10.8.

Drawing Line of Constant $e/h = 0.64$ in Graph 8 of Appendix A and Estimating R_n Corresponding to $\rho_g = 0.0213$, Read $R_n \cong 0.185$

$$R_n = \frac{P_{nx}e}{f'_c A_g h} = 0.185$$

$$P_{nx} = \frac{(4\text{ ksi})\,(15\text{ in.} \times 25\text{ in.})\,(25\text{ in.})\,(0.185)}{16\text{ in.}} = 434\text{ k}$$

For Bending about *y*-Axis

$$\gamma = \frac{10\text{ in.}}{15\text{ in.}} = 0.667$$

$$\rho_g = \frac{8.00\text{ in.}^2}{(15\text{ in.})\,(25\text{ in.})} = 0.0213$$

$$\frac{e}{h} = \frac{8\text{ in.}}{15\text{ in.}} = 0.533$$

Drawing Radial Lines of Constant e/h (0.533) in Graphs 6 and 7 of Appendix A and Interpolating between Them for $\gamma = 0.667$

$$R_n = \frac{P_{ny}e}{f'_c A_g h} = 0.163$$

$$P_{ny} = \frac{(4\text{ ksi})\,(15\text{ in.} \times 25\text{ in.})\,(15\text{ in.})\,(0.163)}{8\text{ in.}} = 458\text{ k}$$

Determining Axial Load Capacity of Section

$$P_o = (0.85)\,(4.0\text{ ksi})\,(15\text{ in.} \times 25\text{ in.}) + (8.00\text{ in.}^2)\,(60\text{ ksi}) = 1755\text{ k}$$

Using the Bresler Expression to Determine P_{ni}

$$\frac{1}{P_{ni}} = \frac{1}{P_{nx}} + \frac{1}{P_{ny}} - \frac{1}{P_o}$$

$$\frac{1}{P_{ni}} = \frac{1}{434\text{ k}} + \frac{1}{458\text{ k}} - \frac{1}{1755\text{ k}}$$

Multiplying by 1755 k on both sides, we get

$$\frac{1755\text{ k}}{P_{ni}} = 4.044 + 3.832 - 1$$

$$P_{ni} = \underline{\underline{255.3\text{ k}}}$$

If the moments in the weak direction (y-axis here) are rather small compared to bending in the strong direction (x-axis), it is common to neglect the smaller moment. This practice is probably reasonable as long as e_y is less than about 20% of e_x, because the Bresler expression will show little reduction for P_{ni}. For the example just solved, an e_y equal to 50% of e_x caused the axial load capacity to be reduced by approximately 40%.

Example 10.9 illustrates the design of a column subject to biaxial bending. The Bresler expression, which is of little use in the proportioning of such members, is used to check the capacities of the sections selected by some other procedure. Exact theoretical designs of columns subject to biaxial bending are very complicated and, as a result, are seldom handled with pocket calculators. They are proportioned either by approximate methods or with computer programs.

10.9 Design of Biaxially Loaded Columns

During the past few decades, several approximate methods have been introduced for the design of columns with biaxial moments. For instance, quite a few design charts are available with which satisfactory designs may be made. The problems are reduced to very simple calculations in which coefficients are taken from the charts and used to magnify the moments about a single axis. Designs are then made with the regular uniaxial design charts.[7,8,9]

Another approximate procedure that works fairly well for design office calculations is used for Example 10.9. If this simple method is applied to square columns, the values of both M_{nx} and M_{ny} are assumed to act about both the x-axis and the y-axis (i.e., $M_x = M_y = M_{nx} + M_{ny}$). The steel is selected about one of the axes and is spread around the column, and the Bresler expression is used to check the ultimate load capacity of the eccentrically loaded column.

Should a rectangular section be used where the y-axis is the weaker direction, it would seem logical to calculate $M_y = M_{nx} + M_{ny}$ and to use that moment to select the steel required about the y-axis and spread the computed steel area over the whole column cross section. Although such a procedure will produce safe designs, the resulting columns may be rather uneconomical, because they will often be much too strong about the strong axis. A fairly satisfactory approximation is to calculate $M_y = M_{nx} + M_{ny}$ and multiply it by b/h, and with that moment design the column about the weaker axis.[10]

Example 10.9 illustrates the design of a short square column subject to biaxial bending. The approximate method described in the last two paragraphs is used, and the Bresler expression is used

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[7] Parme, A. L., Nieves, J. M., and Gouwens, A., 1966, "Capacity of Reinforced Rectangular Columns Subject to Biaxial Bending," *Journal ACI*, 63 (11), pp. 911–923.

[8] Weber, D. C., 1966, "Ultimate Strength Design Charts for Columns with Biaxial Bending," *Journal ACI*, 63 (11), pp. 1205–1230.

[9] Row, D. G., and Paulay, T., 1973, "Biaxial Flexure and Axial Load Interaction in Short Reinforced Concrete Columns," *Bulletin of New Zealand Society for Earthquake Engineering*, 6 (2), pp. 110–121.

[10] Fintel, M., ed., 1985, *Handbook of Concrete Engineering*, 2nd ed. (New York: Van Nostrand), pp. 37–39.

Richmond Convention Center, Richmond, Virginia. Notice column size changes.

for checking the results. If this had been a long column, it would have been necessary to magnify the design moments for slenderness effects, regardless of the design method used.

Example 10.9

Select the reinforcing bars needed for the short square tied column shown in Figure 10.24 for the following: $P_D = 100$ k, $P_L = 200$ k, $M_{DX} = 50$ ft-k, $M_{LX} = 110$ ft-k, $M_{DY} = 40$ ft-k, $M_{LY} = 90$ ft-k, $f'_c = 4000$ psi, and $f_y = 60{,}000$ psi.

SOLUTION

Computing Design Values

$$P_u = (1.2)\,(100\text{ k}) + (1.6)\,(200\text{ k}) = 440\text{ k}$$

$$\frac{P_u}{f'_c A_g} = \frac{440\text{ k}}{(4\text{ ksi})\,(484\text{ in.}^2)} = 0.227$$

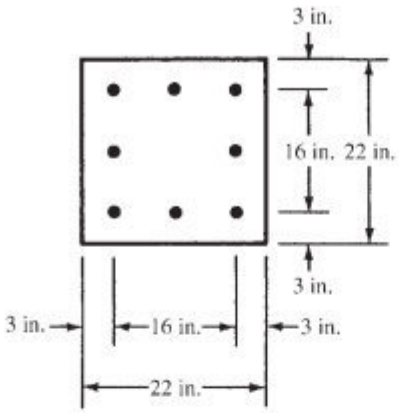


FIGURE 10.24 Column cross section for Example 10.9.

Assume $\phi = 0.65$

$$P_n = \frac{440 \text{ k}}{0.65} = 677 \text{ k}$$

$$M_{ux} = (1.2)\ (50 \text{ ft-k}) + (1.6)\ (110 \text{ ft-k}) = 236 \text{ ft-k}$$

$$M_{nx} = \frac{236 \text{ ft-k}}{0.65} = 363 \text{ ft-k}$$

$$M_{uy} = (1.2)\ (40 \text{ ft-k}) + (1.6)\ (90 \text{ ft-k}) = 192 \text{ ft-k}$$

$$M_{ny} = \frac{192 \text{ ft-k}}{0.65} = 295 \text{ ft-k}$$

As a result of biaxial bending, the design moment about the x- or y-axis is assumed to equal $M_{nx} + M_{ny} = 363 \text{ ft-k} + 295 \text{ ft-k} = 658 \text{ ft-k}$.

Determining Steel Required

$$e_x = e_y = \frac{(12 \text{ in./ft})\ (658 \text{ ft-k})}{677 \text{ k}} = 11.66 \text{ in.}$$

$$\gamma = \frac{16 \text{ in.}}{22 \text{ in.}} = 0.727$$

By interpolation from interaction diagrams with bars on all four faces,

$$\rho_g = 0.0235$$

$$A_s = (0.0235)\ (22 \text{ in.})\ (22 \text{ in.}) = 11.37 \text{ in.}^2$$

Use 8 #11 bars = 12.50 in.^{2}

A review of the column with the Bresler expression gives a $P_{ni} = 804 \text{ k} > 677 \text{ k}$, which is satisfactory. Should the reader go through the Bresler equation here, he or she must remember to calculate the correct e_x and e_y values for use with the interaction diagrams. For instance,

$$e_x = \frac{(12 \text{ in./ft})\ (363 \text{ ft-k})}{677 \text{ k}} = 6.43 \text{ in.}$$

When a beam is subjected to biaxial bending, the following approximate interaction equation may be used for design purposes:

$$\frac{M_x}{M_{ux}} + \frac{M_y}{M_{uy}} \leq 1.0$$

In this expression, M_x and M_y are the design moments, M_{ux} is the design moment capacity of the section if bending occurs about the x-axis only, and M_{uy} is the design moment capacity if bending occurs

about the y-axis only. This same expression may be satisfactorily used for axially loaded members if the design axial load is about 15% or less of the axial load capacity of the section. For a detailed discussion of this subject, the reader is referred to the *Handbook of Concrete Engineering*.[11]

Numerous other methods are available for the design of biaxially loaded columns. One method that is particularly useful to the design profession is the PCA Load Contour Method, which is recommended in the ACI *Design Handbook*.[12]

10.10 Continued Discussion of Capacity Reduction Factors, ϕ

As previously described, the value of ϕ can be larger than 0.65 for tied columns, or 0.75 for spiral columns, if ϵ_t is larger than f_y/E_s. The lower ϕ values are applicable to compression-controlled sections because of their smaller ductilities. Such sections are more sensitive to varying concrete strengths than are tension-controlled sections. The ACI Code, Section 21.2.2 states that ϕ for a particular column may be increased linearly from 0.65 or 0.75 to 0.90 as the net tensile strain, ϵ_t, increases from the compression-controlled strain, f_y/E_s, to the tension-controlled case of 0.005.

For this discussion, Figure 3.5 from Chapter 3 is repeated with slight modification as Figure 10.25. The only difference between the two figures is the addition of a scale called c/d_t on the x-axis. From this figure, you can see the range of ϵ_t values for which ϕ may be increased.

The hand calculation of ϵ_t for a particular column is a long and tedious trial-and-error problem, and space is not taken here to present a numerical example. However, a description of the procedure is presented in the next few paragraphs. The average designer will not want to spend the time necessary to make these calculations and will either just use the smaller ϕ values or make use of a computer program, such as the Excel spreadsheet provided for this chapter. This program uses a routine for computing ϵ_t and ϕ for columns.

The procedure described here can be used to make a long-hand determination of ϵ_t. As a beginning, we assume $c/d_t = 0.60$ where $\epsilon_t = 0.002$ (ACI Code, Section 21.2.2.1 permits ϵ_{ty} to be approximated with 0.002 for Grade 60 reinforcement.). With this value, we can calculate c, a, ϵ_c, ϵ_t, f_s,

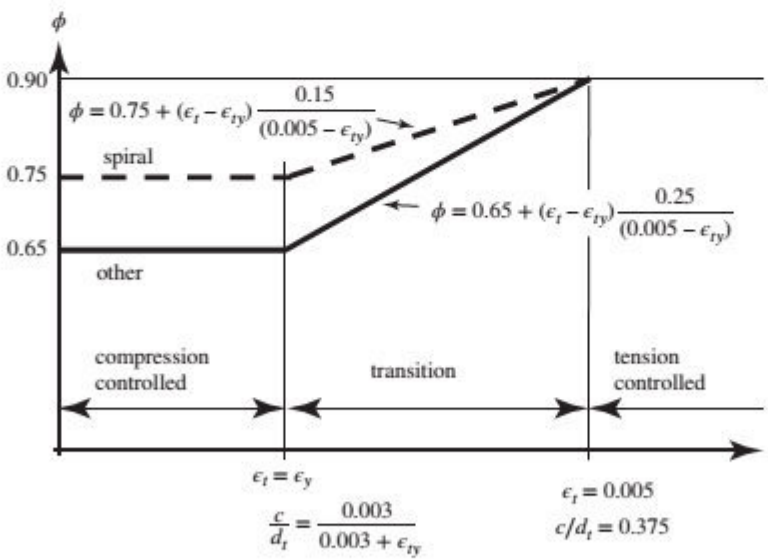


FIGURE 10.25 Variation of ϕ with net tensile ϵ_t and c/d_t.

[11] Fintel, M., *Handbook of Concrete Engineering*, p. 38.

[12] American Concrete Institute, 2007, *Design Handbook in Accordance with the Strength Design Method*, Vol. 2, Columns, (Farmington Hills, MI: ACI), Publication SP-17 (07).

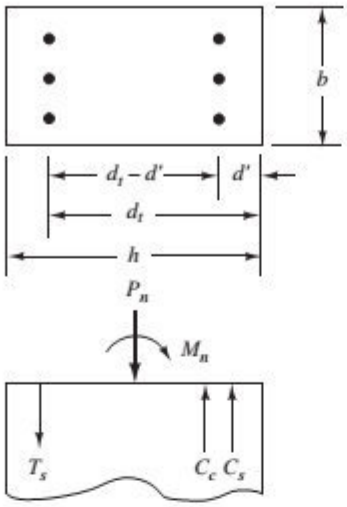


FIGURE 10.26 Notation used for column cross section.

and f'_c for our column. Then, with reference to Figure 10.26, moments can be taken about the centerline of the column and the result solved for M_n and e determined.

$$M_n = T_s\left(\frac{d_t - d'}{2}\right) + C_s\left(\frac{d_t - d'}{2}\right) + C_c\left(\frac{h}{2} - \frac{a}{2}\right)$$

As the next step, c/d_t can be assumed equal to 0.375 (where $\epsilon_t = 0.005$ as shown in Figure 10.25) and another value of ϵ_t is determined. If the ϵ_t of our column falls between the two ϵ_t values we have just calculated, the column falls in the transition zone for ϕ. To determine its value, we can try different c/d_t values between 0.600 and 0.375 until the calculated ϵ_t equals the actual ϵ_t of the column.

If you go through this process one time, you will probably have seen all you want to see of it and will no doubt welcome the fact that the Excel spreadsheet provided for this textbook can be used to determine the value of ϕ for a particular column.

When using the interaction diagrams in Appendix A, it is easy to see the region where the variable ϕ factor applies. In Figure 10.15, note that there are lines labeled f_s/f_y. If the coordinates of K_n and R_n are greater than the value of $f_s/f_y = 1$, the ϕ factor is 0.65 (0.75 for spiral columns). If the coordinates are below the line labeled $\epsilon_t = 0.005$, the ϕ factor is 0.90. Between these lines, the ϕ factor is variable, and you would have to resort to approximate methods or to the spreadsheet provided.

10.11 Computer Example

Example 10.10

Using the Excel Spreadsheet provided for Chapter 10, repeat Example 10.5. Use the theoretical area of reinforcing steel calculated (7.38 in.^{2}) instead of the actual area (8.00 in.^{2}) selected.

SOLUTION

Open the Circular Column worksheet and enter the following:

$P_u = 500$ k, $M_u = 225$ k-ft

Enter $h = 20$ in.

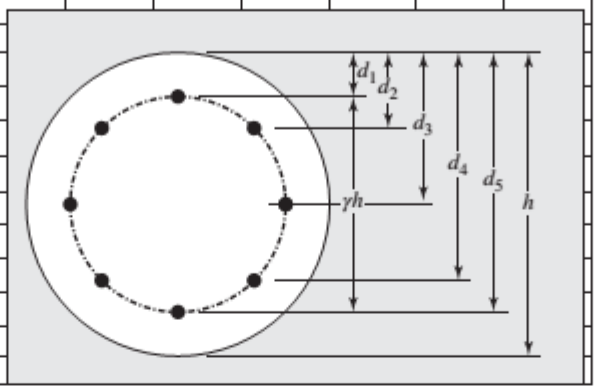
Enter $\gamma = 15$ in./20 in. $= 0.75$

Enter material properties ($f'_c = 4000$ psi, $f_y = 60{,}000$ psi)

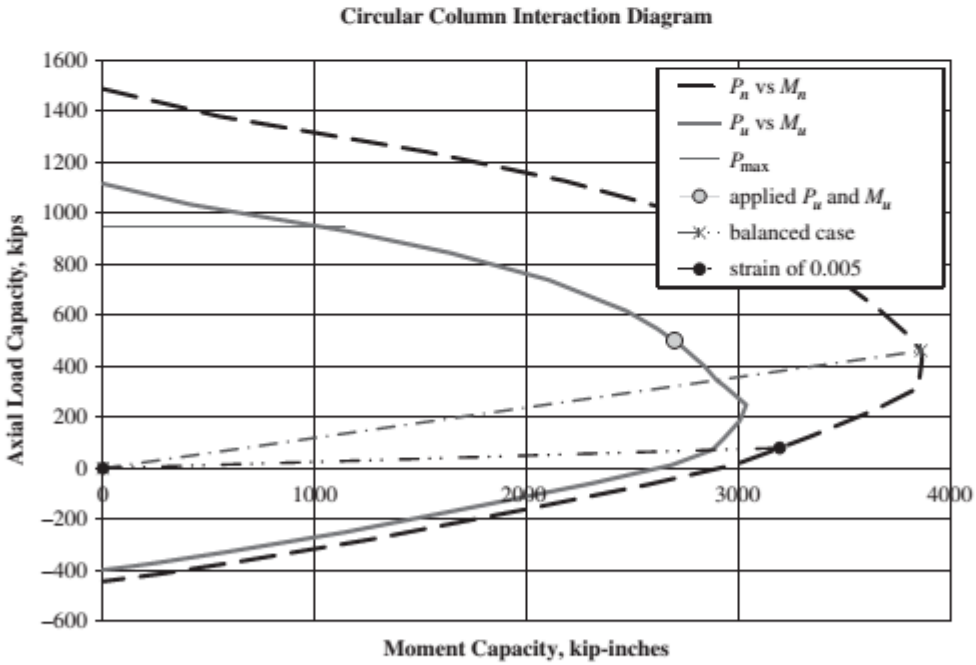
Enter $A_{st} = 7.38$ in.^{2}

Circular Column Capacity

P_u =	500	k							
M_u =	225	k-ft =	2700	k-in.					
h =	20	in.							
γ =	0.75								
f'_c =	4000	psi							
f_y =	60,000	psi							
A_{st} =	7.38	in.^{2}							
A_g =	314.2	in.^{2}							
ρ_t =	0.02349								
β_1 =	0.85								
ϵ_y =	0.00207								
E_s =	29,000	ksi							
c_{bal} =	10.36	in.							
$c_{0.005}$ =	6.5625	in.							



Then select the "Interaction Diagram–Circular" worksheet and observe the location of the "Applied P_u and M_u" relative to the graph for "P_u vs M_u." In this case, the applied load falls on top of the graph, indicating that the theoretical area of reinforcing steel would provide just the correct capacity. If the area of steel were then changed in the input to 8 in.^{2}, the dot would fall inside the contour, indicating excess capacity.

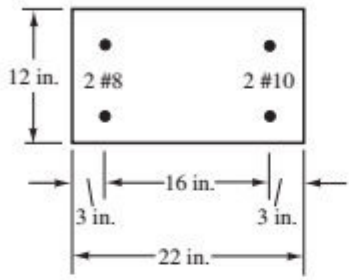


PROBLEMS

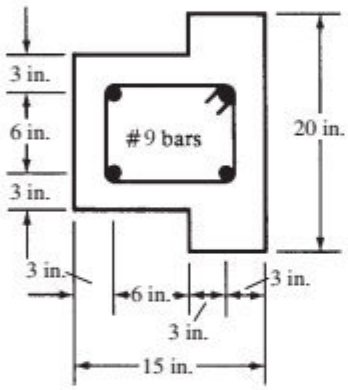
Location of Plastic Centroids

For Problems 10.1 and 10.2, locate the plastic centroids if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

Problem 10.1 (*Ans.* 11.38 in. from left edge)

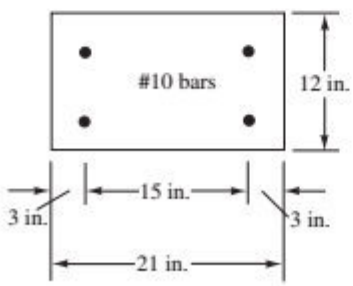


Problem 10.2



Analysis of Column Subjected to Axial Load and Moment

Problem 10.3 Using statics equations, determine the values of P_n and M_n for the column shown, assuming it is strained to -0.003 on its right-hand edge and to $+0.002$ on its left-hand edge. $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. (*Ans.* $P_n = 485.8$ k, $M_n = 335.9$ ft-k)



Problem 10.4 Repeat Problem 10.3 if the strain on the left edge is $+0.001$.

Problem 10.5 Repeat Problem 10.3 if the strain on the left edge is 0.000. (*Ans.* $P_n = 902.9$ k, $M_n = 165.4$ ft-k)

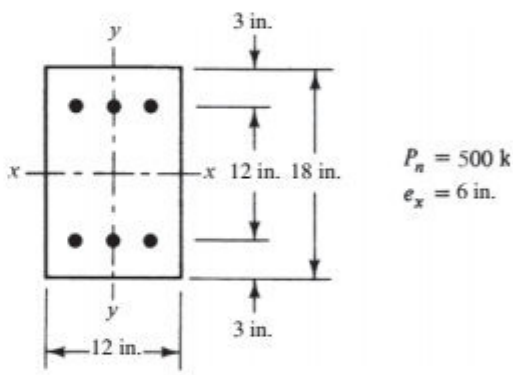
Problem 10.6 Repeat Problem 10.3 if the strain on the left edge is -0.001.

Problem 10.7 Repeat Problem 10.3 if the steel on the left side has a strain in tension of $\epsilon_y = f_y/E_s$ and the right edge is at 0.003. (*Ans.* $P_n = 360.8$ k, $M_n = 368.3$ ft-k)

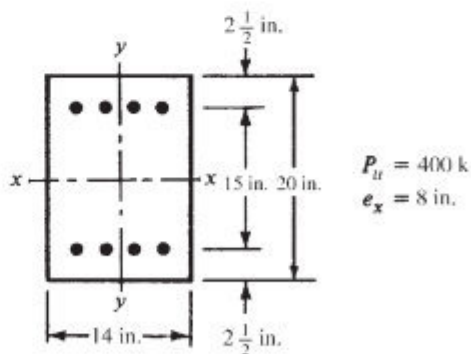
Design of Columns for Axial Load and Moment

For Problems 10.8 to 10.10, use the interaction curves in Appendix A to select reinforcing bars for the short columns shown. $f'_c = 4000$ psi and $f_y = 60{,}000$ psi.

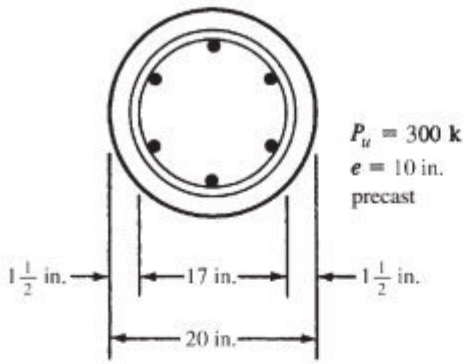
Problem 10.8



Problem 10.9 (*One ans.* 8 #9 bars)

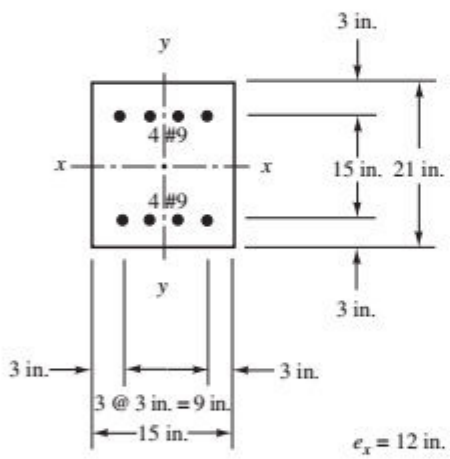


Problem 10.10



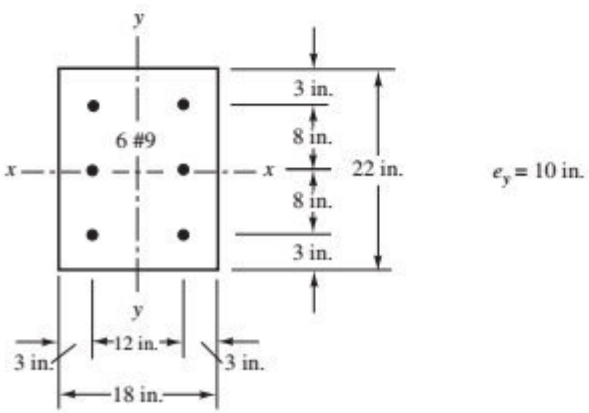
For Problems 10.11 to 10.16, use the interaction diagrams in Appendix A to determine P_n values for the short columns shown, which have bending about one axis and if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.

Problem 10.11 (*Ans*. 328 k)

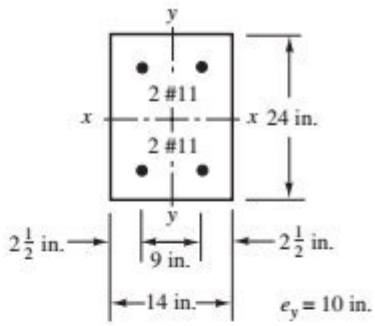


Problem 10.12 Repeat Problem 10.11 if $e_x = 9$ in.

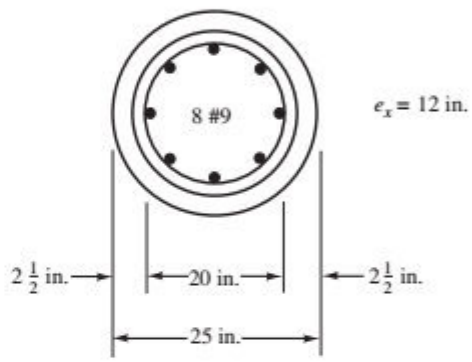
Problem 10.13 (*Ans*. 340 k)



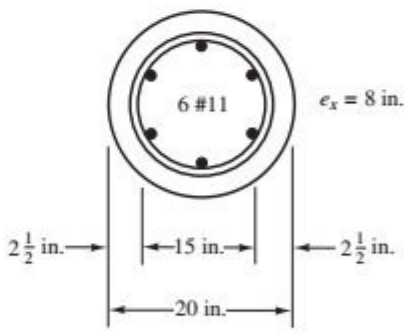
Problem 10.14



Problem 10.15 (*Ans*. 412 k for $\phi = 0.75$; 434 k for $\phi = 0.79$)

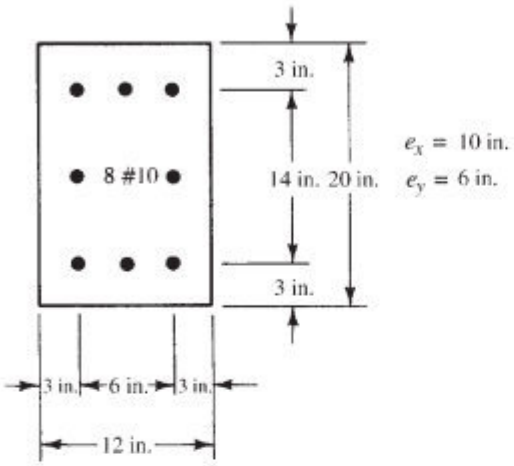


Problem 10.16

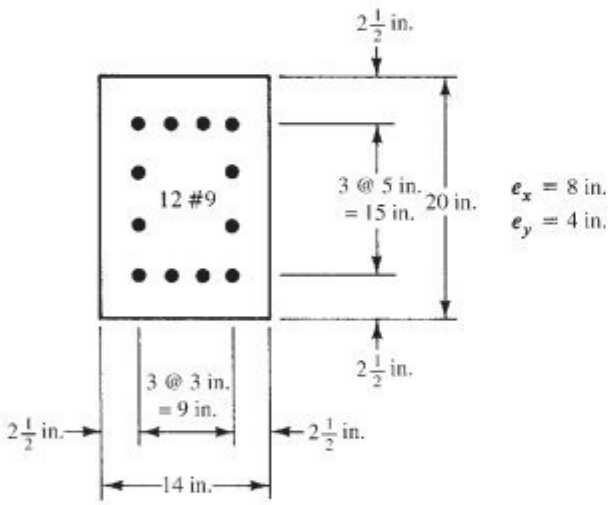


For Problems 10.17 to 10.21, determine P_n values for the short columns shown if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi.

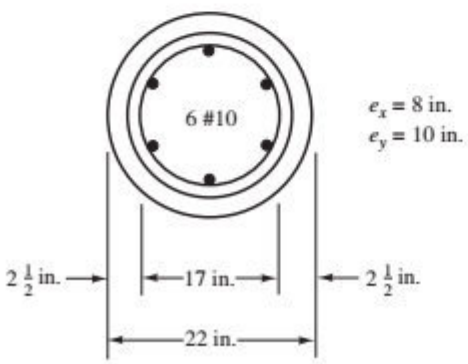
Problem 10.17 (*Ans.* 146 k)



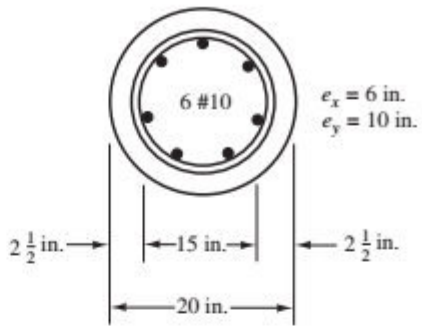
Problem 10.18



Problem 10.19 (*Ans.* 279 k)



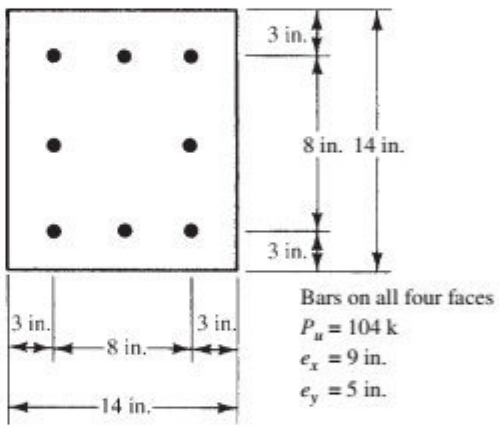
Problem 10.20



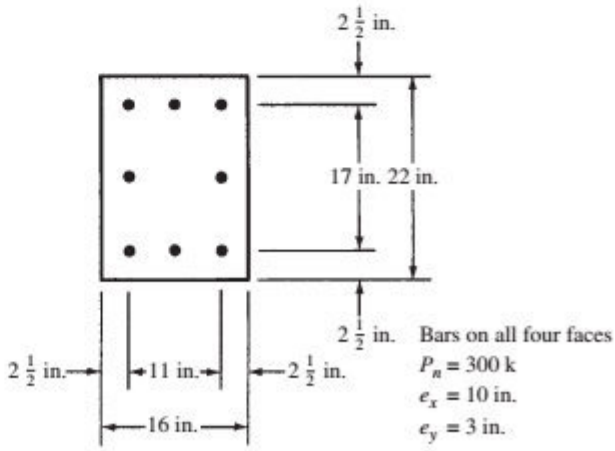
Problem 10.21 Repeat Problem 10.20 if $e_x = 8$ in. and $e_y = 6$ in. (*Ans.* 287 k)

For Problems 10.22 and 10.23, select reinforcing bars for the short columns shown if $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. Check results with the Bresler equation.

Problem 10.22



Problem 10.23 (*One ans.* 8 #9 bars, $P_n = 432$ k)



For Problems 10.24 to 10.27, use the Chapters 9 and 10 Excel spreadsheets.

Problem 10.24 Rework Problem 10.8 using the Spreadsheet provided for Chapter 10.

Problem 10.25 If the column in Problem 10.13 is supporting a load $P_u = 400$ k and $e_x = 0$, how large can M_{uy} be if six #9 bars are used? (*Ans.* 264 ft-k)

Problem 10.26 If the column in Problem 10.15 is to support an axial load $P_u = 400$ k and $e_x = 0$, how many #11 bars must be used to resist a design moment $M_{ux} = 600$ ft-k?

Problem 10.27 If the column in Problem 10.11 has a moment $M_{ux} = 300$ ft-k, what are the limits on P_u? (*Ans.* 420 k $\geq P_u \geq$ 20 k)

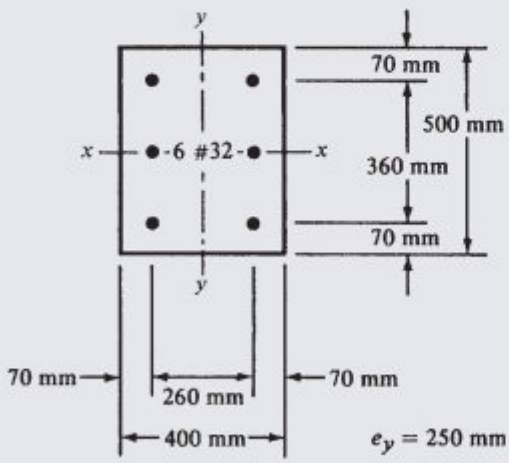
Problem 10.28 Prepare a flowchart for the preparation of an interaction curve for axial compression loads and bending for a short rectangular tied column.

Problems with SI Units

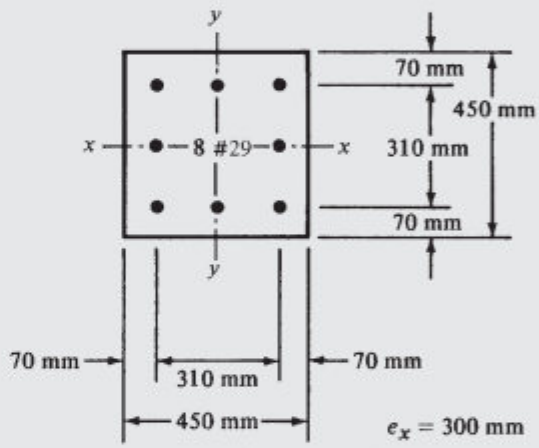
Column interaction curves are not provided in this text for the usual SI concrete strengths (21 MPa, 24 MPa, 28 MPa, etc.) or for the usual steel yield strength (420 MPa). Therefore, the problems that follow are to be solved using the column curves for $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. These diagrams may be applied for the corresponding SI units (27.6 MPa and 413.7 MPa) just as they are for U.S. customary units, but it is necessary to convert the results to SI values.

For Problems 10.29 to 10.31, use the column interaction diagrams in Appendix A to determine P_n values for the short columns shown if $f'_c = 28$ MPa and $f_y = 420$ MPa.

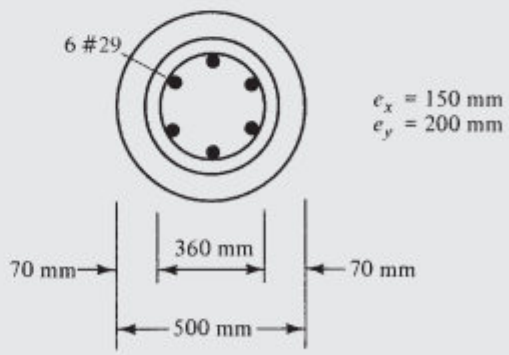
Problem 10.29 (*Ans.* 1855 kN)



Problem 10.30

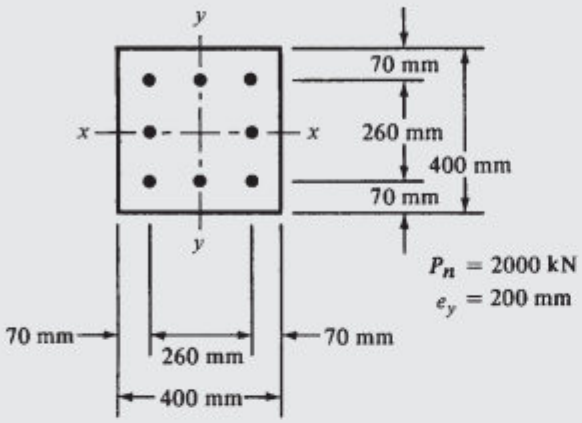


Problem 10.31 (*Ans.* $P_u = 1146$ kN)

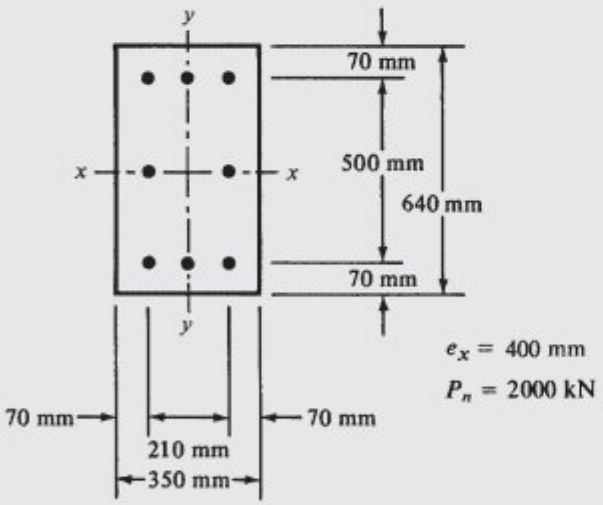


For Problems 10.32 to 10.34, select reinforcing for the short columns shown if $f'_c = 27.6$ MPa and $f_y = 413.7$ MPa. Remember to apply the conversion factor provided before Problem 10.28 when using the interaction curves.

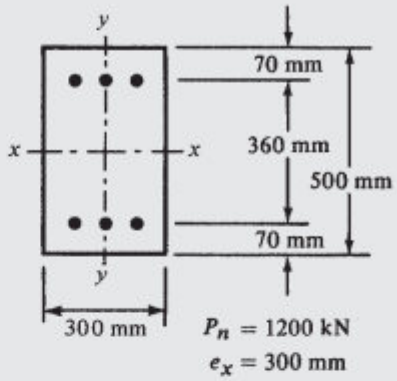
Problem 10.32



Problem 10.33 (*One ans.* 6 #36)



Problem 10.34



Slender Columns

CHAPTER 11

11.1 Introduction

When a column bends or deflects laterally an amount Δ, its axial load will cause an increased column moment equal to $P\Delta$. This moment will be superimposed onto any moments already in the column. Should this $P\Delta$ moment be of such magnitude as to reduce the axial load capacity of the column significantly, the column will be referred to as a *slender column*.

Section 6.7.1.1 of the ACI Code states that the design of a compression member should, desirably, be based on a theoretical analysis of the structure that takes into account the effects of axial loads, duration of loads, varying member sizes, end conditions, and so on. If such a theoretical procedure is not used, the ACI Code, Section 6.6.4.1 provides an approximate method for determining slenderness effects. This method, which is based on the factors just mentioned for an "exact" analysis, results in a moment magnifier, δ, which is to be multiplied by the larger moment at the end of the column denoted as M_2, and that value is used in design. If bending occurs about both axes, δ is to be computed separately for each direction and the values obtained have to be multiplied by the respective moment values.

11.2 Nonsway and Sway Frames

For this discussion, it is necessary to distinguish between frames without sidesway and those with sidesway. In the ACI Code, these are referred to respectively as *nonsway frames* and *sway frames*.

For the building story in question, the columns in nonsway frames must be designed according to Section 6.6.4.5 of the ACI Code, while the columns of sway frames must be designed according to Section 6.6.4.6. As a result, it is first necessary to decide whether we have a nonsway frame or a sway frame. *You must realize that you will rarely find a frame that is completely braced against swaying or one that is completely unbraced against swaying. Therefore, you are going to have to decide which way to handle it.*

The question may possibly be resolved by examining the lateral stiffness of the bracing elements for the story in question. You may observe that a particular column is located in a story where there is such substantial lateral stiffness provided by bracing members, shear walls, shear trusses, and so on that any lateral deflections occurring will be too small to affect the strength of the column appreciably. You should realize that, while examining a particular structure, there may be some nonsway stories and some sway stories.

If we cannot tell by inspection whether we have a nonsway frame or a sway frame, the code provides two ways of making a decision. First, in ACI Code, Section 6.6.4.3a, a story in a frame is said to be a nonsway one if the increase in column end moments from second-order effects is 5% or less of the first-order end moments.

The second method for determining whether a particular frame is braced or unbraced is given in the ACI Code, Section 6.6.4.3a. If the value of the so-called *stability index* (which follows) is ≤ 0.05, the commentary states that the frame may be classified as a nonsway one. (Should V_u be equal to zero, this method will not apply.)

$$Q = \frac{\Sigma P_u \Delta_o}{V_u \ell_c} \qquad \text{(ACI Equation 6.6.4.4.1)}$$

where

ΣP_u = total factored vertical load for all of the columns on the story in question
Δ_o = the elastically determined first-order lateral deflection from V_u at the top of the story in question with respect to the bottom of that story
V_u = the total factored horizontal shear for the story in question
ℓ_c = the height of a compression member in a frame measured from center to center of the frame joints

Despite these suggestions from the ACI, the individual designer is going to have to make decisions as to what is adequate bracing and what is not, depending on the presence of structural walls and other bracing items. For the average-size reinforced concrete building, load eccentricities and slenderness values will be small, and frames will be considered to be braced. *Certainly, however, it is wise in questionable cases to err on the side of the unbraced.*

11.3 Slenderness Effects

The slenderness of columns is based on their geometry and on their lateral bracing. As their slenderness increases, their bending stresses increase, and thus buckling may occur. Reinforced concrete columns generally have small slenderness ratios. As a result, they can usually be designed as short columns without strength reductions because of slenderness. If slenderness effects are considered small, then columns can be considered "short" and can be designed according to Chapter 10. However, if they are "slender," the moment for which the column must be designed is increased or magnified. Once the moment is magnified, the column is then designed according to Chapter 10 using the increased moment.

Several items involved in the calculation of slenderness ratios are discussed in the next several paragraphs. These include unsupported column lengths, effective length factors, radii of gyration, and the ACI Code requirements. The ACI Code, Section 6.2.6 limits second-order effects to not more than 40% of first-order effects.

Unsupported Lengths

The length used for calculating the slenderness ratio of a column, ℓ_u, is its unsupported length. This length is considered to be equal to the clear distance between slabs, beams, or other members that provide lateral support to the column. If haunches or capitals (see Figure 16.1 in Chapter 16) are present, the clear distance is measured from the bottoms of the capitals or haunches.

Effective Length Factors

To calculate the slenderness ratio of a particular column, it is necessary to estimate its effective length. This is the distance between points of zero moment in the column. For this initial discussion, it is assumed that no sidesway or joint translation is possible. Sidesway or joint translation means that one or both ends of a column can move laterally with respect to each other.

If there were such a thing as a perfectly pinned end column, its effective length would be its unsupported length, as shown in Figure 11.1(a). The *effective length factor, k*, is the number that must be multiplied by the column's unsupported length to obtain its effective length. For a perfectly pinned end column, $k = 1.0$.

Columns with different end conditions have entirely different effective lengths. For instance, if there were such a thing as a perfectly fixed end column, its points of inflection (or points of zero moment) would occur at its one-fourth points, and its effective length would be $\ell_u/2$, as shown in Figure 11.1(b). As a result, its k value would equal 0.5.

Obviously, the smaller the effective length of a particular column, the smaller its danger of buckling and the greater its load-carrying capacity. Figure 11.1(c) shows a column with one end fixed and one end pinned. The k factor for this column is approximately 0.70.

Digital Vision/Getty Images, Inc.

Round columns.

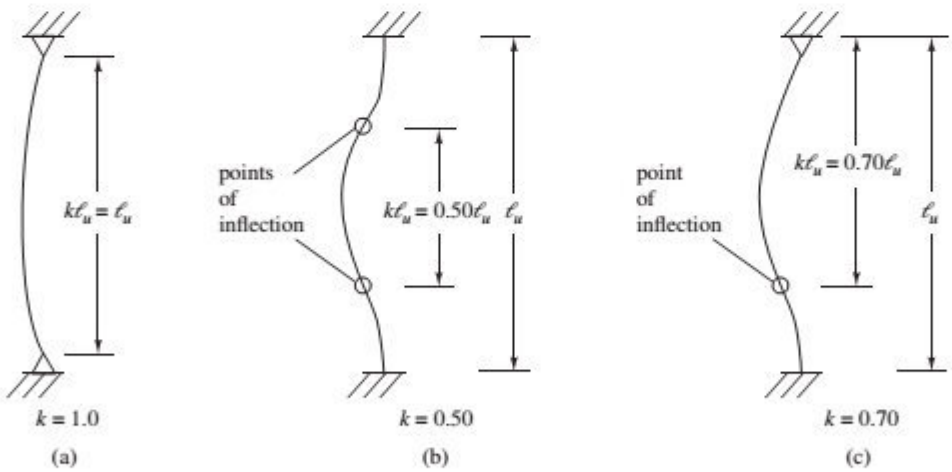
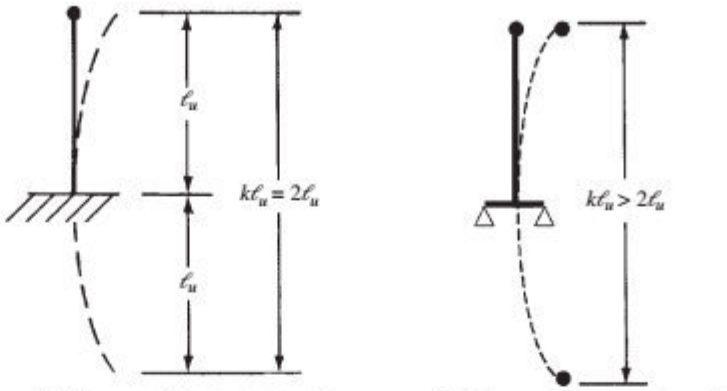


FIGURE 11.1 Effective lengths for columns in braced frames (sidesway prevented).

The concept of effective lengths is simply a mathematical method of taking a column—whatever its end and bracing conditions—and replacing it with an equivalent pinned end-braced column. A complex buckling analysis could be made for a frame to determine the critical stress in a particular column. The k factor is determined by finding the pinned end column with an equivalent length that provides the same critical stress. The k factor procedure is a method of making simple solutions for complicated frame-buckling problems.



(a) Upper end free to rotate and translate, lower end fixed

(b) Upper end free to rotate and translate, lower end partially restrained against rotation

FIGURE 11.2 Columns for unbraced frames.

Reinforced concrete columns serve as parts of frames, and these frames are sometimes *braced* and sometimes *unbraced*. A braced frame is one for which sidesway or joint translation is prevented by means of bracing, shear walls, or lateral support from adjoining structures. An unbraced frame does not have any of these types of bracing supplied and must depend on the stiffness of its own members to prevent lateral buckling. For braced frames, k values can never be greater than 1.0, but for unbraced frames, the k values will always be greater than 1.0 because of sidesway.

An example of an unbraced column is shown in Figure 11.2(a). The base of this particular column is assumed to be fixed, whereas its upper end is assumed to be completely free to both rotate and translate. The elastic curve of such a column will take the shape of the elastic curve of a pinned-end column of twice its length. Its effective length will therefore equal $2\ell_u$, as shown in the figure. In Figure 11.2(b), another unbraced column case is illustrated. The bottom of this column is connected to beams that provide resistance to rotation but not enough to be considered a fixed end. In most buildings, partial rotational restraint is common, not pinned or fixed ends. Section 11.4 shows how to evaluate such partial restraint. For the case shown in Figure 11.2(b), if the beam at the bottom is flexible compared with the column, the k factor approaches infinity. If it is very stiff, k approaches 2.

Section 6.6.4.4.3 of the ACI Code states that the effective length factor is to be taken as 1.0 for compression members in frames braced against sidesway unless a theoretical analysis shows that a lesser value can be used. Should the member be in a frame not braced against sidesway, the value of k will be larger than 1.0 and must be determined with proper consideration given to the effects of cracking and reinforcement on the column stiffness. ACI-ASCE Committee 441 suggests that it is not realistic to assume that k will be less than 1.2 for such columns; therefore, it seems logical to make preliminary designs with k equal to or larger than that value.

11.4 Determining *k* Factors with Alignment Charts

The preliminary procedure used for estimating effective lengths involves the use of the alignment charts shown in Figure 11.3.[1,2] Before computerized analysis, use of such alignment charts was the traditional method for determining effective lengths of columns. The chart of part (a) of the figure is applicable to braced frames, whereas the chart of part (b) is applicable to unbraced frames.

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[1] Structural Stability Research Council, *Guide to Stability Design Criteria for Metal Structures*, 4th ed., T. V. Galambos, ed. (New York: John Wiley & Sons, 1988).

[2] Julian, O. G. and Lawrence, L. S., 1959, "Notes on J and L Nomograms for Determination of Effective Lengths," unpublished. These are also called the Jackson and Moreland Alignment Charts, after the firm with which Julian and Lawrence were associated.

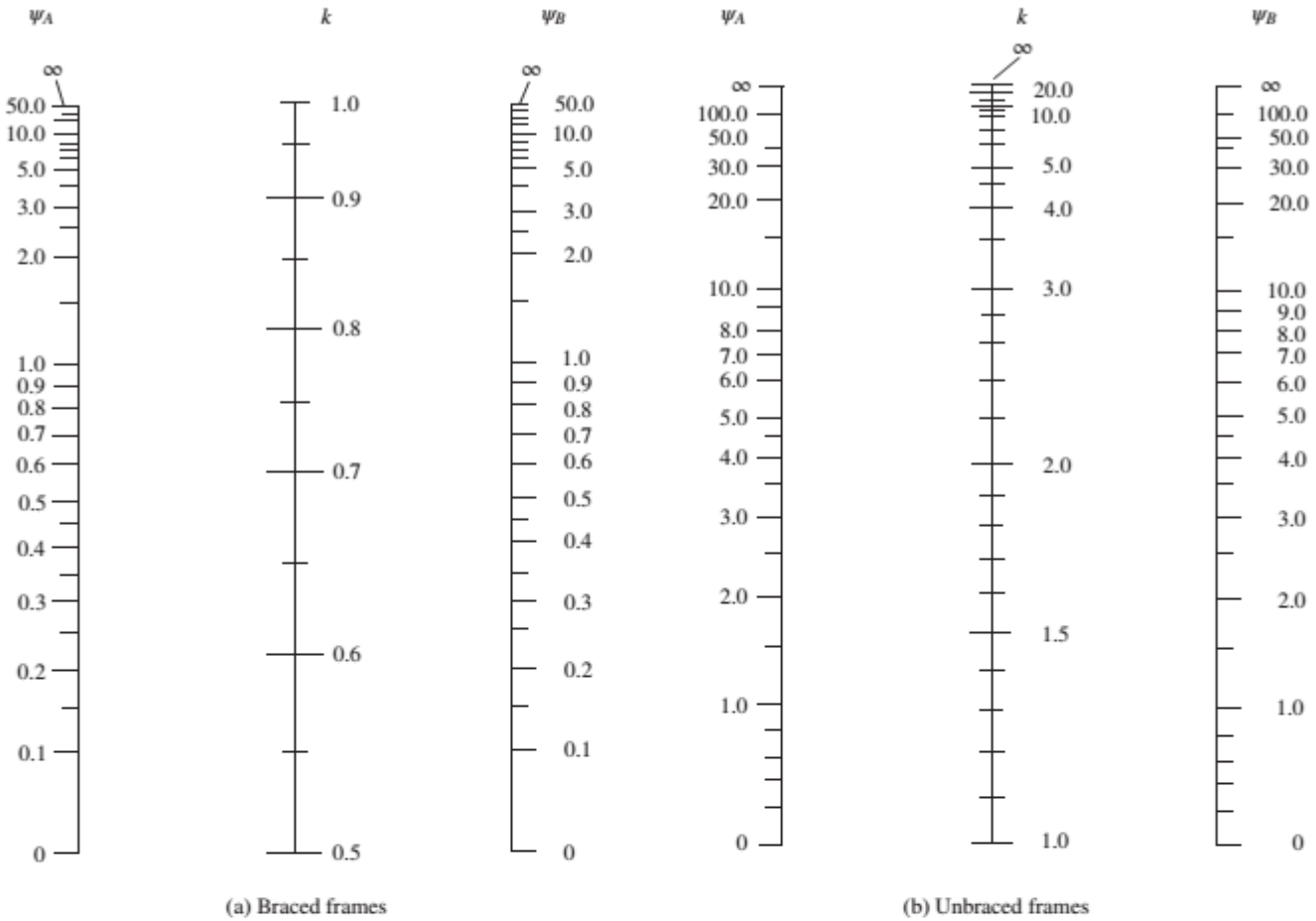

FIGURE 11.3 Effective length factors. ψ = ratio of $\Sigma(EI/\ell)$ of compression members to $\Sigma(EI/\ell)$ of flexural members in a plane at one end of a compression member. k = effective length factor.

To use the alignment charts for a particular column, ψ factors are computed at each end of the column. The ψ factor at one end of the column equals the sum of the stiffness $[\Sigma(EI/\ell)]$ of the columns meeting at that joint, including the column in question, divided by the sum of all the stiffnesses of the beams meeting at the joint. Should one end of the column be pinned, ψ is theoretically equal to ∞, and if fixed, $\psi = 0$. Because a perfectly fixed end is practically impossible to have, ψ is usually taken as 1.0 instead of 0 for assumed fixed ends. When column ends are supported by but not rigidly connected to a footing, ψ is theoretically infinity but usually is taken as about 10 for practical design.

One of the two ψ values is called ψ_A and the other is called ψ_B. After these values are computed, the effective length factor, k, is obtained by placing a straightedge between ψ_A and ψ_B. The point where the straightedge crosses the middle nomograph is k.

It can be seen that the ψ factors used to enter the alignment charts, and thus the resulting effective length factors, are dependent on the relative stiffnesses of the compression and flexural members. If we have a very light flexible column and large stiff girders, the rotation and lateral movement of the column ends will be greatly minimized. The column ends will be close to a fixed condition, and thus the ψ values and the resulting k values will be small. Obviously, if the reverse happens—that is, large stiff columns framing into light flexible girders—the column ends will rotate almost freely, approaching a pinned condition. Consequently, we will have large ψ and k values.

To calculate the ψ values, it is necessary to use realistic moments of inertia. Usually the girders will be appreciably cracked on their tensile sides, whereas the columns will probably have only a few

cracks. If the I values for the girders are underestimated a little, the column k factors will be a little large and thus on the safe side.

Several approximate rules are in use for estimating beam and column rigidities. One common practice of the past for slenderness ratios of up to about 60 or 70 was to use gross moments of inertia for the columns and 50% of the gross moments of inertia for the beams.

In ACI Table 6.6.3.1.1a, it is stated that for determining ψ values for use in evaluating k factors, the rigidity of the beams may be calculated on the basis of $0.35I_g$ to account for cracking and reinforcement, whereas $0.70I_g$ may be used for compression members. This practice is followed for the examples in this chapter. Other values for the estimated rigidity of walls and flat plates are provided in the same section.

11.5 Determining *k* Factors with Equations

Instead of using the alignment charts for determining k values, an alternate method involves the use of relatively simple equations. These equations, which were in the ACI 318-05 Code Commentary (R10.12.1) and taken from the British Standard Code of Practice,[3] are particularly useful with computer programs.

For braced compression members, an upper bound to the effective length factor may be taken as the smaller value determined from the two equations to follow in which ψ_A and ψ_B are the values just described for the alignment charts (commonly called the Jackson and Moreland alignment charts as described in footnote 2 of this chapter). ψ_{min} is the smaller of ψ_A and ψ_B.

$$k = 0.7 + 0.05(\psi_A + \psi_B) \leq 1.0$$
$$k = 0.85 + 0.05\psi_{\text{min}} \leq 1.0$$

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Reinforced concrete columns.

[3] British Standards Institution, 1972, *Code of Practice for the Structural Use of Concrete* (CP110: Part 1), London, 154 pages.

The value of k for unbraced compression members restrained at both ends may be determined from the appropriate one of the following two equations, in which ψ_m is the average of ψ_A and ψ_B:

$$\text{If } \psi_m < 2$$
$$k = \frac{20 - \psi_m}{20}\sqrt{1 + \psi_m}$$
$$\text{If } \psi_m \geq 2$$
$$k = 0.9\sqrt{1 + \psi_m}$$

The value of the effective length factor of unbraced compression members that are hinged at one end may be determined from the following expression, in which ψ is the value at the restrained end:

$$k = 2.0 + 0.3\psi$$

As mentioned in Section 11.3 of this chapter, Section 6.6.4.4.3 of the ACI Code states that k may conservatively be taken to be 1.0 for compression members in frames braced against sidesway unless a theoretical analysis shows that a lesser value can be used. *In the last paragraph of Section R6.6.4.4.3 of the commentary, use of the alignment charts or the equations just presented is said to be satisfactory for justifying k values less than 1.0 for braced frames.*

11.6 First-Order Analyses Using Special Member Properties

After this section, the remainder of this chapter is devoted to an approximate design procedure wherein the effect of slenderness is accounted for by computing moment magnifiers that are multiplied by the column moments. A magnifier for a particular column is a function of its factored axial load, P_u, and its critical buckling load, P_c.

Before moment magnifiers can be computed for a particular structure, it is necessary to make a first-order analysis of the structure. The member section properties used for such an analysis should take into account the influence of axial loads, the presence of cracked regions in the members, and the effect of the duration of the loads. Instead of making such an analysis, ACI Code, Section 6.6.3.1.1 permits use of the following properties for the members of the structure. These properties may be used for both nonsway and sway frames.

(a) *Modulus of elasticity* determined from the following expression given in Section 19.2.2.1 of the ACI Code: $E_c = w_c^{1.5}33\sqrt{f'_c}$ for values of w_c from 90 lb/ft^3 to 155 lb/ft^3 or $57{,}000\sqrt{f'_c}$ for normal-weight concrete.

(b) *Moments of inertia* where I_g = moment of inertia of gross concrete section about centroidal axis neglecting reinforcement (ACI Code, Section 6.6.3.1.1a):

Beams	$0.35I_g$
Columns	$0.70I_g$
Walls—Uncracked	$0.70I_g$
—Cracked	$0.35I_g$
Flat plates and flat slabs	$0.25I_g$

(c) *Area* $1.0A_g$

As an alternative to the above approximate equations for columns and walls, the code permits the following more complex value for moment of inertia:

$$I = \left(0.80 + 25\frac{A_{st}}{A_g}\right)\left(1 - \frac{M_u}{P_u h} - 0.5\frac{P_u}{P_0}\right)I_g \leq 0.875I_g \qquad \text{(ACI Table 6.6.3.1.1b)}$$

P_u and M_u are to be from the load combination under consideration, or they can conservatively be taken as the values of P_u and M_u that result in the lowest value of I. In no case is the value of I for compression members required to be taken less than $0.35I_g$. P_0 is the theoretical concentric axial load strength (see Chapter 9 of this textbook).

For flexural members (beams and flat plates and flat slabs), the following approximate equation is permitted:

$$I = (0.10 + 25\rho)\left(1.2 - 0.2\frac{b_w}{d}\right)I_g \leq 0.5I_g \qquad \text{(ACI Table 6.6.3.1.1b)}$$

For continuous flexural members, it is permitted to use the average value of I from the positive and negative moment sections. In no case is the value of I for flexural members required to be taken less than $0.25I_g$.

Often during the design process, the designer does not know the final values of member section dimensions or steel areas when making calculations such as those in ACI Table 6.6.3.1.1b. This leads to an iterative process where the last cycle of iteration assumes the same member properties as the final design. The ACI Code, Section 6.6.4.2 allows these values to be only within 10% of the final values in the final iteration.

11.7 Slender Columns in Nonsway and Sway Frames

There is a major difference in the behavior of columns in nonsway or braced frames and those in sway or unbraced frames. In effect, each column in a braced frame acts by itself. In other words, its individual strength can be determined and compared to its computed factored loads and moments. In an unbraced or sway frame, a column will probably not buckle individually but will probably buckle simultaneously with all of the other columns on the same level. As a result, it is necessary in a sway frame to consider the buckling strength of all the columns on the level in question as a unit.

For a compression member in a nonsway frame, the effective slenderness ratio, $k\ell_u/r$, is used to determine whether the member is short or slender. For this calculation, ℓ_u is the unbraced length of the member. The effective length factor, k, can be taken as 1.0 unless an analysis provides a lesser value. The radius of gyration, r, is equal to 0.25 times the diameter of a round column and 0.289 times the dimension of a rectangular column in the direction that stability is being considered. The ACI Code, Section 6.2.5.1 permits the approximate value of 0.30 to be used in place of 0.289, and this is done herein. For other sections, the value of r will have to be computed from the properties of the gross sections.

For nonsway frames, slenderness effects may be ignored if the following expression is satisfied:

$$\frac{k\ell_u}{r} \leq 34 - 12\left(\frac{M_1}{M_2}\right) \qquad \text{(ACI Equation 6.2.5b)}$$

In this expression, M_1 is the smaller factored end moment in a compression member. It has a plus sign if the member is bent in single curvature (C shaped) and a negative sign if the member is bent in double curvature (S shaped). M_2 is the larger factored end moment in a compression member, and it always has a plus sign. In this equation, the term $[34 - 12(M_1/M_2)]$ shall not be taken larger than 40, according to ACI Equation 6.2.5c.

For sway frames, slenderness effects may be ignored if

$$\frac{k\ell_u}{r} < 22 \qquad \text{(ACI Equation 6.2.5a)}$$

Should $k\ell_u/r$ for a particular column be larger than the applicable ratio, we will have a slender column. For such a column, the effect of slenderness must be considered. This may be done by using approximate methods or by using a theoretical second-order analysis that takes into account the effect of deflections. Second-order effects cannot exceed 40% of first-order effects (ACI Section 6.2.6).

A second-order analysis is one that takes into account the effect of deflections and also makes use of a reduced tangent modulus. The equations necessary for designing a column in this range are extremely complicated, and, practically, it is necessary to use column design charts or computer programs.

Avoiding Slender Columns

The design of slender columns is appreciably more complicated than the design of short columns. As a result, it may be wise to give some consideration to the use of certain minimum dimensions so that none of the columns will be slender. In this way, they can be almost completely avoided in the average-size building.

If k is assumed to be equal to 1.0, slenderness can usually be neglected in braced frame columns, if ℓ_u/h is kept to 10 or less on the first floor and 14 or less for the floors above the first one. To determine these values, it was assumed that little moment resistance was provided at the footing–column connection and the first-floor columns were assumed to be bent in single curvature. Should the footing–column connection be designed to have appreciable moment resistance, the maximum ℓ_u/h value given above as 10 should be raised to about 14 or equal to the value used for the upper floors.[4]

Should we have an unbraced frame and assume $k = 1.2$, it is probably necessary to keep ℓ_u/h to 6 or less. So for a 10-ft clear floor height, it is necessary to use a minimum h of about 10 ft/6 = 1.67 ft = 20 in. in the direction of bending to avoid slender columns.

Example 11.1 illustrates the selection of the k factor and the determination of the slenderness ratio for a column in an unbraced frame. For calculating I/L values, the authors used 0.70 times the gross moments of inertia for the columns, 0.35 times the gross moments of inertia for the girders, and the full lengths of members center to center of supports.

Example 11.1

(a) Using the alignment charts of Figure 11.3, calculate the effective length factor for column AB of the braced frame of Figure 11.4. Consider only bending in the plane of the frame.

(b) Compute the slenderness ratio of column AB. Is it a short or a slender column? *The maximum permissible slenderness ratio for a short unbraced column is 22, as will be described in Section 11.9 of this chapter.* End moments on the column are $M_1 = 45$ ft-k and $M_2 = 75$ ft-k, resulting in single curvature.

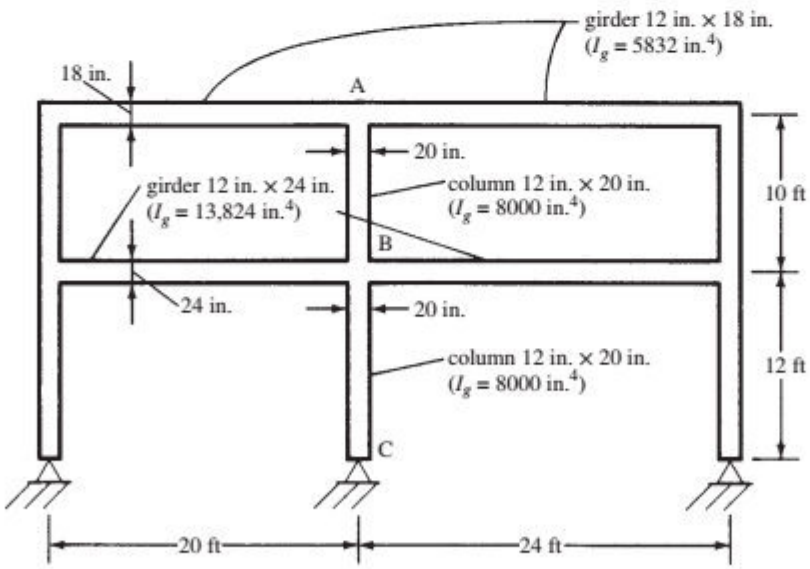
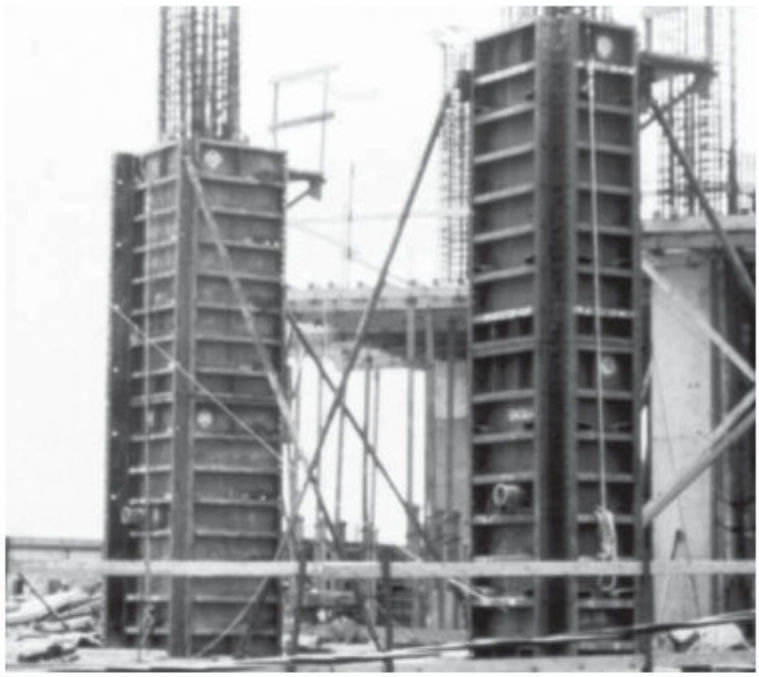


FIGURE 11.4 Frame for Example 11.1.

[4] Neville, G. B., ed., 1984, *Simplified Design Reinforced Concrete Buildings of Moderate Size and Height* (Skokie, IL: Portland Cement Association), pp. 5-10 to 5-12.



Courtesy of Dayton Superior Corporation.

Adjustable steel column forming system.

SOLUTION

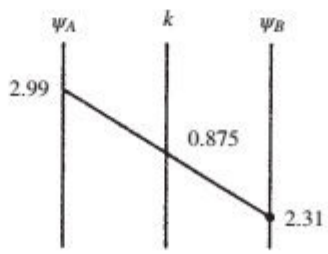
(a) Effective Length Factor for Column AB

Using the reduced moments of inertia from 11.6(b) and applying the method described in Section 11.4

$$\psi_A = \frac{\dfrac{0.7 \times 8000 \text{ in.}^4}{12 \times 10 \text{ ft}}}{\left(\dfrac{0.35 \times 5832 \text{ in.}^4}{12 \times 20 \text{ ft}} + \dfrac{0.35 \times 5832 \text{ in.}^4}{12 \times 24 \text{ ft}}\right)} = 2.99$$

$$\psi_B = \frac{\dfrac{0.7 \times 8000 \text{ in.}^4}{12 \times 10 \text{ ft}} + \dfrac{0.7 \times 8000 \text{ in.}^4}{12 \times 12 \text{ ft}}}{\left(\dfrac{0.35 \times 13{,}824 \text{ in.}^4}{12 \times 20 \text{ ft}} + \dfrac{0.35 \times 13{,}824 \text{ in.}^4}{12 \times 24 \text{ ft}}\right)} = 2.31$$

From Figure 11.3(a)



(b) Is It a Slender Column?

$$\ell_u = 10 \text{ ft} - \frac{9 \text{ in.} + 12 \text{ in.}}{12 \text{ in./ft}} = 8.25 \text{ ft}$$

$$\frac{k\ell_u}{r} = \frac{(0.875) \times (12 \text{ in./ft} \times 8.25 \text{ ft})}{0.3 \times 20 \text{ in.}} = 14.44 < \text{Maximum } \frac{k\ell_u}{r} \text{ for a short column in a braced}$$

$$\text{frame by ACI Equation 6.2.5b} = 34 - 12\left(\frac{+45 \text{ ft-k}}{+75 \text{ ft-k}}\right) = 26.8$$

$\therefore$ It's not a slender column

An experienced designer would first simply assume $k = 1$ and quickly see that $k\ell_u/r = \ell_u/r = 16.5 < 26.5$. There would then be no need to determine k.

If this column were in the same frame but the frame were unbraced, then k would be 1.78 and $k\ell_u/r = 29.37 > 22$. It would be a slender column. The only difference in determining k is the use of Figure 11.3(b) for sway columns instead of Figure 11.3(a) for nonsway columns.

11.8 ACI Code Treatments of Slenderness Effects

The ACI Code permits the determination of second-order effects by one of three methods. The first is by a *nonlinear second-order analysis* (ACI Code, Section 6.8.1). Such an analysis must consider nonlinearity of materials, member curvature and lateral drift, load duration, volume changes in concrete because of creep and shrinkage, and foundation or support interaction. The analysis technique should predict the ultimate loads to within 15% or test results on statically indeterminate reinforced concrete structures (ACI R6.8.1.2). This technique would require sophisticated computer software that has been demonstrated to satisfy the 15% accuracy requirement mentioned previously.

The second method is by an *elastic second-order analysis* (ACI Code, Section 6.7.1.1). This technique is simpler than the nonlinear method because it uses member stiffnesses immediately prior to failure. Values of E_c and moments of inertia and cross-sectional area for columns, beams, walls, flat plates, and flat slabs that are permitted to be used in the elastic second-order analysis are listed in Section 11.6 of the text. This method would also most likely require a computer analysis.

The third method is the *moment magnifier procedure* (ACI Code, Section 6.6.4). Different procedures for this method are given for sway and nonsway structures. The next two sections describe the moment magnifier method for these two cases.

11.9 Magnification of Column Moments in Nonsway Frames

When a column is subjected to moment along its unbraced length, it will be displaced laterally in the plane of bending. The result will be an increased or secondary moment equal to the axial load times the lateral displacement or eccentricity. In Figure 11.5, the load P causes the column moment to be increased by an amount $P\Delta$. This moment will cause δ to increase a little more, with the result that the $P\Delta$ moment will increase, which in turn will cause a further increase in Δ and so on until equilibrium is reached.

We could take the column moments, compute the lateral deflection, increase the moment by $P\Delta$, recalculate the lateral deflection and the increased moment, and so on. Although about two cycles would be sufficient, this would still be a tedious and impractical procedure.

It can be shown[5] that the increased moment can be estimated very well by multiplying the primary moment by $1/(1 - P/P_c)$, where P is the axial load and P_c is the Euler buckling load $\pi^2 EI/(k\ell_u)^2$.

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[5] Timoshenko, S. P., and Gere, J. M., 1961, *Theory of Elastic Stability*, 2nd ed. (New York: McGraw-Hill), pp. 319–356.

FIGURE 11.5 Moment magnification in a nonsway column.

In Example 11.2, this expression is used to estimate the magnified moment in a laterally loaded column. It will be noted that in this problem, the primary moment of 75 ft-k is estimated to increase by 7.4 ft-k. If we computed the deflection from the lateral load, we would get 0.445 in. For this value, $P\Delta = (150)\,(0.445) = 66.75$ in.-k $= 5.6$ ft-k. This moment causes more deflection, which causes more moment, and so on.

Example 11.2

(a) Compute the primary moment in the column shown in Figure 11.6 from the lateral 20-k load.
(b) Determine the estimated total moment, including the theoretical secondary moment from lateral deflection, using the appropriate magnification factor just presented. $E = 3.16 \times 10^3$ ksi. Assume $k = 1.0$ and $\ell_u = 15$ ft.

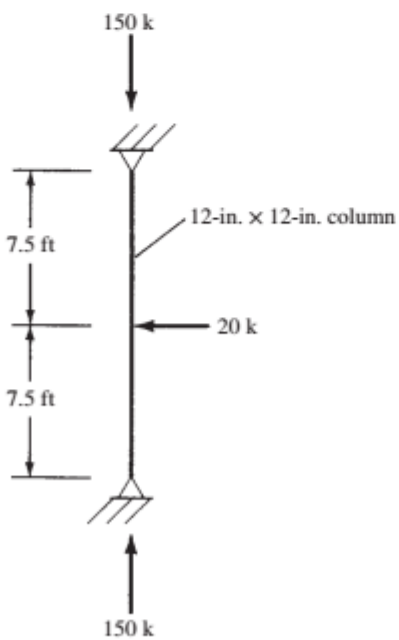

FIGURE 11.6 Column for Example 11.2.

SOLUTION

(a) Primary Moment Resulting from Lateral Load

$$M_u = \frac{(20\text{ k})\,(15\text{ ft})}{4} = \underline{\underline{75\text{ ft-k}}}$$

(b) Total Moment, Including Secondary Moment

$$P_c = \text{Euler buckling load} = \frac{\pi^2 EI}{(k\ell_u)^2}$$

$$= \frac{(\pi^2)\,(3160\text{ ksi})\left(\frac{1}{12} \times 12\text{ in.} \times 12\text{ in.}^3\right)}{(1.0 \times 12\text{ in./ft} \times 15\text{ ft})^2} = 1663.4\text{ k}$$

$$\text{Magnified moment} = \frac{1}{1 - \frac{P}{P_c}} M_2$$

$$= \frac{1}{1 - \frac{150\text{ k}}{1663.4\text{ k}}} 75\text{ ft-k} = (1.1)(75\text{ ft-k}) = \underline{\underline{82.4\text{ ft-k}}}$$

As you have seen, it is possible to calculate approximately the increased moment resulting from lateral deflection by using the $(1 - P/P_c)$ expression. In ACI Code, Section 6.6.4.5, the factored design moment for slender columns with no sway is increased by using the following expression, in which M_c is the magnified or increased moment and M_2 is the larger factored end moment on a compression member:

$$M_c = \delta M_2 \qquad \text{(ACI Equation 6.6.4.5.1)}$$

Should our calculations provide very small moments at both column ends, the code provides an absolutely minimum value of M_2 to be used in design. In effect, it requires the computation of a moment based on a minimum eccentricity of $0.6 + 0.03h$, where h is the overall thickness of the member perpendicular to the axis of bending.

$$M_{2\,\min} = P_u(0.6 + 0.03h) \qquad \text{(ACI Equation 6.6.4.5.4)}$$

Or in SI units

$M_{2\,\min} = P_u(15 + 0.03h)$, where h is in mm, as is the number 15

A moment magnifier, δ, is used to estimate the effect of member curvature between the ends of compression members. It involves a term C_m, which is defined later in this section.

$$\delta = \frac{C_m}{1 - \frac{P_u}{0.75P_c}} \geq 1.0 \qquad \text{(ACI Equation 6.6.4.5.2)}$$

The determination of the moment magnifier, δ, involves the following calculations:

1. $E_c = 57{,}000\sqrt{f'_c}$ for normal-weight concrete (see Section 1.11 for other densities).
2. I_g = gross inertia of the column cross section about the centroidal axis being considered.

3. $E_s = 29 \times 10^6$ psi.

4. I_{se} = moment of inertia of the reinforcement about the centroidal axis of the section. (This value equals the sum of each bar area times the square of its distance from the centroidal axis of the compression member.)

5. The term β_{dns} accounts for the reduction in stiffness caused by sustained axial loads and applies only to nonsway frames. It is defined as the ratio of the maximum factored sustained axial load divided by the total factored axial load associated with the same load combination. It is always assumed to have a plus sign and is never permitted to exceed 1.0.

6. Next, it is necessary to compute an effective value of (EI) designated as $(EI)_{\text{eff}}$. The two expressions given for $(EI)_{\text{eff}}$ in the code were developed so as to account for creep, cracks, and so on. If the column and bar sizes have already been selected or estimated, $(EI)_{\text{eff}}$ can be computed with the following expression, which is particularly satisfactory for columns with high steel percentages.

$$(EI)_{\text{eff}} = \frac{(0.2E_cI_g + E_sI_{se})}{1 + \beta_{dns}} \qquad \text{(ACI Equation 6.6.4.4.4b)}$$

The alternate expression for $(EI)_{\text{eff}}$ that follows is probably the better expression to use when steel percentages are low. Notice also that this expression will be the one used if the reinforcement has not been previously selected.

$$(EI)_{\text{eff}} = \frac{0.4E_cI_g}{1 + \beta_{dns}} \qquad \text{(ACI Equation 6.6.4.4.4a)}$$

A third alternative equation for $(EI)_{\text{eff}}$ is given by code equation 6.6.4.4.4c. In this equation, the moment of intertia of the column, I, is calculated from ACI Table 6.6.3.1.1b.

$$(EI)_{\text{eff}} = \frac{E_cI}{1 + \beta_{dns}} \qquad \text{(ACI Equation 6.6.4.4.4c)}$$

7. The Euler buckling load is computed:

$$P_c = \frac{\pi^2(EI)_{\text{eff}}}{(k\ell_u)^2} \qquad \text{(ACI Equation 6.6.4.4.2)}$$

8. For some moment situations in columns, the amplification or moment magnifier expression provides moments that are too large. One such situation occurs when the moment at one end of the member is zero. For this situation, the lateral deflection is actually about half of the deflection in effect provided by the amplification factor. Should we have approximately equal end moments that are causing reverse curvature bending, the deflection at middepth and the moment there are close to zero. As a result of these and other situations, the code provides a modification factor (C_m) to be used in the moment expression that will result in more realistic moment magnification.

For braced frames without transverse loads between supports, C_m can vary from 0.4 to 1.0 and is determined with the expression at the end of this paragraph. For all other cases, it is to be taken as 1.0. (Remember the sign convention: M_1 is positive for single curvature and is negative for reverse curvature, and M_2 is always positive.)

$$C_m = 0.6 + 0.4\frac{M_1}{M_2} \qquad \text{(ACI Equation 6.6.4.5.3a)}$$

Should $M_{2\,\text{min}}$ as computed with ACI Equation 6.6.4.5.4 be larger than M_2, the value of C_m in this equation shall either be taken as equal to 1.0 or be based on the ratio of the computed end moments M_1/M_2 (ACI Code, Section 6.6.4.5.3). If the braced column is subjected to transverse loads between supports, C_m is taken as 1.0 (ACI Equation 6.6.4.5.3b).

Example 11.3 illustrates the design of a column in a nonsway frame.

Example 11.3

The tied column of Figure 11.7 has been approximately sized to the dimensions 12 in. × 15 in. It is to be used in a frame braced against sidesway. The column is bent in single curvature about its y-axis and has an ℓ_u of 16 ft. If $k = 0.83$, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi normal-weight concrete, determine the reinforcing bars required. Consider only bending in the plane of the frame. Note also that the unfactored dead axial load P_D is 30 k.

SOLUTION

1. Is It a Slender Column?

$$\text{Max } \frac{k\ell_u}{r} \text{ for short columns} = 34 - 12\frac{M_1}{M_2} = 34 - 12\left(\frac{+82 \text{ ft-k}}{+86 \text{ ft-k}}\right)$$
$$= 22.56$$

$$\text{Actual } \frac{k\ell_u}{r} = \frac{(0.83)\ (12 \text{ in./ft} \times 16 \text{ ft})}{0.3 \times 15 \text{ in.}} = 35.41 > 22.56$$
$$\therefore \text{ It's a slender column}$$

2. $E_c = 57{,}000\sqrt{f'_c} = 57{,}000\sqrt{4000}$ psi $= 3{,}605{,}000$ psi $= 3.605 \times 10^3$ ksi

3. $I_g = \left(\frac{1}{12}\right)(12 \text{ in.})(15 \text{ in.})^3 = 3375 \text{ in.}^4$

4. $\beta_{dns} = \dfrac{\text{factored axial dead load}}{\text{factored axial total load}} = \dfrac{(1.2)\ (30 \text{ k})}{110 \text{ k}} = 0.327$

5. Because reinforcement has not been selected, we must use ACI Equation 6.6.4.4.4a for $(EI)_{eff}$.

$$(EI)_{eff} = \frac{0.4E_cI_g}{1 + \beta_d} = \frac{(0.4)\ (3605 \text{ ksi})\ (3375 \text{ in.}^4)}{1 + 0.327} = 3.67 \times 10^6 \text{ k-in.}^2 \qquad \text{(ACI Equation 6.6.4.4.4a)}$$

6. $P_c = \dfrac{\pi^2(EI)_{eff}}{(k\ell_u)^2} = \dfrac{(\pi^2)\ (3.67 \times 10^6 \text{ k-in.}^2)}{(0.83 \times 12 \text{ in./ft} \times 16 \text{ ft})^2} = 1426 \text{ k}$ (ACI Equation 6.6.4.4.2)

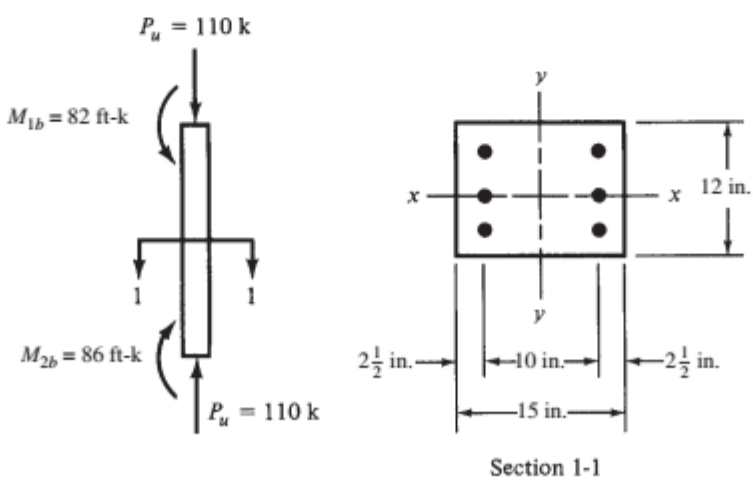

FIGURE 11.7 Column profile and cross section for Example 11.3.

7. $C_m = 0.6 + 0.4\dfrac{M_1}{M_2} = 0.6 + 0.4\left(\dfrac{+82 \text{ ft-k}}{+86 \text{ ft-k}}\right) = 0.981$ (ACI Equation 6.6.4.5.3a)

8. $\delta = \dfrac{C_m}{1 - \dfrac{P_u}{0.75P_c}} = \dfrac{0.981}{1 - \dfrac{110 \text{ k}}{(0.75)\,(1426 \text{ k})}} = 1.09 > 1.0$ (ACI Equation 6.6.4.5.2)

9. $M_{2\,\text{min}} = P_u(0.6 + 0.03h) = 110 \text{ k}(0.6 + 0.03 \times 15 \text{ in.})$
$= 115.5 \text{ in.-k} = 9.6 \text{ ft-k}$ (ACI Equation 6.6.4.5.4)

10. $M_c = \delta M_2 = (1.09)\,(86 \text{ ft-k}) = 93.7 \text{ ft-k}$ (ACI Equation 6.6.4.5.1)

11. Magnified $e = \delta e = \dfrac{(12)\,(93.7 \text{ ft-k})}{110 \text{ k}} = 10.22 \text{ in.}$

12. $\gamma = 10 \text{ in.}/15 \text{ in.} = 0.667$
$\therefore \rho_g$ is determined by interpolation between values presented in Appendix A, Graphs 2 and 3.

$$P_n = \frac{P_u}{\phi} = \frac{110 \text{ k}}{0.65} = 169.23 \text{ k}$$

$$K_n = \frac{P_n}{f'_c A_g} = \frac{169.23 \text{ k}}{(4 \text{ ksi})\,(12 \text{ in.} \times 15 \text{ in.})} = 0.235$$

$$R_n = \frac{P_n}{f'_c A_g}\frac{\delta e}{h} = (0.235)\left(\frac{10.22 \text{ in.}}{15 \text{ in.}}\right) = 0.160$$

$$\rho_g = 0.0160$$

$$A_g = (0.0160)\,(12 \text{ in.})\,(15 \text{ in.}) = 2.88 \text{ in.}^2$$

Use 4 #8 bars (3.14 in.^{2})

Because K_n and R_n are between the radial lines labeled $f_s/f_y = 1.0$ and $\epsilon_t = 0.005$ on the interaction diagrams, the ϕ factor is permitted to be increased from the 0.65 value used. If the spreadsheet for rectangular columns given in Chapters 9 and 10 is used, an area of reinforcing of only 1.80 in.^{2} is found to be sufficient. This significant reduction occurs because of the increased ϕ factor in this region.

Most columns are designed for multiple load combinations, and the designer must be certain that the column is able to resist all of them. Often there are some columns with high axial load and low moment, such as $1.2D + 1.6L$, and others with low axial load and high moment, such as $0.9D + 1.0E$. The first of these is likely to have a ϕ factor of 0.65. The second, however, is more likely to be eligible for the increase in the ϕ factor.

In this example, the authors assumed that the frame was braced, and yet we have said that frames are often in that gray area between being fully braced and fully unbraced. *Assuming a frame is fully braced clearly may be quite unconservative.*

11.10 Magnification of Column Moments in Sway Frames

Tests have shown that even though the lateral deflections in unbraced frames are rather small, their buckling loads are far less than they would be if the frames had been braced. As a result, the buckling strengths of the columns of an unbraced frame can be decidedly increased (perhaps by as much as two or three times) by providing bracing.

If a frame is unbraced against sidesway, it is first necessary to compute its slenderness ratio. If $k\ell_u/r$ is less than 22, slenderness may be neglected (ACI Section 6.2.5). For this discussion, it is assumed that values greater than 22 are obtained.

When sway frames are involved, it is necessary to decide for each load combination which of the loads cause appreciable sidesway (probably the lateral loads) and which do not. The factored end

moments that cause sidesway are referred to as M_{1s} and M_{2s}, and they must be magnified because of the P-Δ effect. The other end moments, resulting from loads that do not cause appreciable sidesway, are M_{1ns} and M_{2ns}. They are determined by first-order analysis and will not have to be magnified.

Section 6.6.4.6.2 of the ACI Code states that the moment magnifier, δ_s, can be determined by one of the following two methods.

1. The moment magnifier may be calculated with the equation given at the end of this paragraph (Equation 6.6.4.6.2a) in which Q is the stability index previously presented in Section 11.2 of this chapter. Should the computed value of δ_s be greater than 1.5, it will be necessary to compute δ_s by ACI Equation 6.6.4.6.2b or by a second-order analysis.

$$\delta_s = \frac{1}{1-Q} \geq 1 \qquad \text{(ACI Equation 6.6.4.6.2a)}$$

2. With the second method and the one used in this chapter, the magnified sway moments may be computed with the following expression:

$$\delta_s = \frac{1}{1-\dfrac{\Sigma P_u}{0.75\Sigma P_c}} \geq 1 \qquad \text{(ACI Equation 6.6.4.6.2b)}$$

In this last equation, ΣP_u is the summation of all the vertical loads in the story in question, and ΣP_c is the sum of all the Euler buckling loads, $P_c = \pi^2(EI)_{\text{eff}}/(k\ell_u)^2$, for all of the sway-resisting columns in the story with k values determined as described in ACI Code, Section 6.6.4.4.3. This formula reflects the fact that the lateral deflections of all the columns in a particular story are equal, and thus the columns are interactive. In calculating P_c, $(EI)_{\text{eff}}$ is determined by either ACI Equation 6.6.4.4.4 or ACI Equation 6.6.4.4.5 using β_{ds} instead of β_{dns}.

Whichever of the preceding methods is used to determine the δ_s values, the design moments to be used must be calculated with the expressions that follow.

$$M_1 = M_{1ns} + \delta_s M_{1s} \qquad \text{(ACI Equation 6.6.4.6.1a)}$$

$$M_2 = M_{2ns} + \delta_s M_{2s} \qquad \text{(ACI Equation 6.6.4.6.1b)}$$

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Reinforced concrete columns and shearwalls supporting structural steel roof.

Sometimes the point of maximum moment in a slender column will fall between its ends. The ACI Commentary (R6.6.4.6.4) says the moment magnification for this case may be evaluated using the procedure described for nonsway frames (ACI Code, Section 6.6.4.5).

Example 11.4 illustrates the design of a slender column subject to sway.

Example 11.4

Select reinforcing bars using the moment magnification method for the 18 in. × 18 in. unbraced column shown in Figure 11.8 if $\ell_u = 17.5$ ft, $k = 1.3$, $f_y = 60$ ksi, and $f'_c = 4$ ksi. A first-order analysis has resulted in the following axial loads and moments:

$$P_D = 200 \text{ k} \qquad M_D = 48 \text{ ft-k}$$
$$P_L = 150 \text{ k} \qquad M_L = 25 \text{ ft-k}$$
$$P_W = 65 \text{ k} \qquad M_W = 120 \text{ ft-k}$$

The loading combinations from Section 4.1 of the text are shown in Table 11.1. The most critical case is Case 3, $U = 1.2D + L + W$. Calculations of P_u and M_u for the other combinations are also shown.

The column in Table 11.1 headed sum P_u is the sum of all of the factored loads in the entire floor for that load combination. Similarly, sum P_c is the sum of the critical buckling of all columns on the floor level. These are used in ACI Equation 6.6.4.6.2b to determine δ_s. The magnified design moments are listed in the last column. The most critical moment and axial load are $P_u = 455$ k, $M_2 = 213.3$ ft-k (Case 3 in Table 11.1). Calculations below are limited to only that load combination.

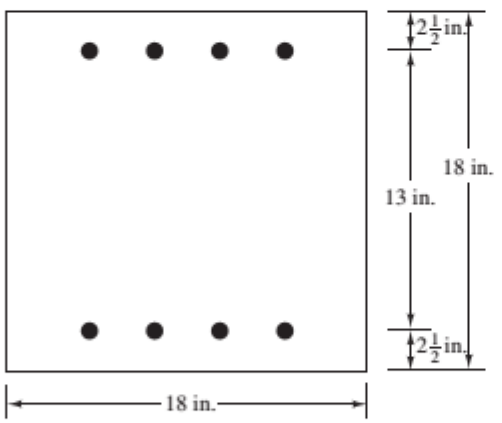


FIGURE 11.8 Column cross section for Example 11.4.

TABLE 11.1 Summary of Load Combination Calculations for Example 11.4

Case	ASCE 7 Equation*	Combination	P_u(k)	M_u (k-ft)	M_{2ns} (ft-k)	M_{2s} (ft-k)	sum P_u (k)	sum P_c (k)	δ_s	$M_2 = M_{2ns} + \delta_s M_{2s}$ (ft-k)
1	5.3.1a	$1.4D$	280	67.2	67.2		3360	50,000	1.053	67.20
2	5.3.1b	$1.2D + 1.6L$	480	97.6	97.6		5760	50,000	1.095	97.60
3	5.3.1d	$1.2D + 1.0W + L$	455	202.6	82.6	120	5460	50,000	1.089	213.30
4		$1.2D - 1.0W + L$	325	−37.4	82.6	−120	3900	50,000	1.062	−44.86
5	5.3.1f	$0.9D + 1.0W$	245	163.2	43.2	120	2940	50,000	1.046	168.74
6		$0.9D - 1.0W$	115	0.9	43.2	−120	1380	50,000	1.021	−79.34

*Section 4.1 of this text.

SOLUTION

Is It a Slender Column? (ACI Code, Section 6.2.5, Equation 6.2.5a)

$$\frac{k\ell_u}{r} = \frac{(1.3)\,(12\text{ in./ft})\,(17.5\text{ ft})}{(0.3)\,(18\text{ in.})} = 50.55 > 22 \quad \underline{\text{Yes}}$$

Calculating the Magnified Moment δ_s

$$\delta_s = \frac{1}{1 - \dfrac{\Sigma P_u}{0.75\Sigma P_c}} \qquad \text{(ACI Equation 6.6.4.6.2b)}$$

$$= \frac{1}{1 - \dfrac{5460\text{ k}}{(0.75)\,(50{,}000\text{ k})}} = 1.089$$

Computing Magnified Moment M_2

$$M_2 = M_{2ns} + \delta_s M_{2s} \qquad \text{(ACI Equation 6.6.4.6.1b)}$$

$$= 82.6\text{ ft-k} + (1.089)\,(120\text{ ft-k}) = 213.3\text{ ft-k}$$

Selecting Reinforcing Bars

$$\gamma = \frac{13\text{ in.}}{18\text{ in.}} = 0.722 \text{ with reference to Figure 11.8}$$

$$P_n = \frac{P_u}{\phi} = \frac{455\text{ k}}{0.65} = 700.0\text{ k (assuming } \phi = 0.65 \text{ to be verified later)}^{[6]}$$

$$e = \frac{(12\text{ in./ft})\,(213.3\text{ ft-k})}{455\text{ k}} = 5.63\text{ in.}$$

$$K_n = \frac{P_n}{f'_c A_g} = \frac{700.0\text{ k}}{(4\text{ ksi})\,(18\text{ in.} \times 18\text{ in.})} = 0.540$$

$$R_n = \frac{P_n}{f'_c A_g}\frac{e}{h} = (0.540)\left(\frac{5.63\text{ in.}}{18\text{ in.}}\right) = 0.169$$

By interpolation between Appendix A, Graphs 3 and 4, we find ρ_z is 0.017.

$$A_s = \rho_z A_g = (0.017)\,(18\text{ in.} \times 18\text{ in.}) = 5.51\text{ in.}^2$$

$$\underline{\text{Use 6 \#9 bars (6.00 in.}^2\text{)}}$$

The interaction diagram shown in Figure 11.9 shows the final column design with all 6 load combinations superimposed. The "dot" with coordinates 455 k, 213.3 ft-k (2560 in.-k) is clearly the most critical because it is located closest to the P_u vs M_u diagram. This load combination is the only one whose moment has been magnified, however.

11.11 Analysis of Sway Frames

The frame of Figure 11.10 is assumed to be unbraced in the plane of the frame. It supports a uniform gravity load, w_u, and a short-term concentrated lateral load, P_w. As a result, it is necessary to consider both the moments resulting from the loads that do not cause appreciable sidesway and the loads that do. It will, therefore, be necessary to compute both δ and δ_s values, if the column proves to be slender.

The M_s values are obviously caused by the lateral load in this case. The reader should realize, however, that if the gravity loads and/or the frame are unsymmetrical, additional M_s or sidesway moments will occur.

[6] Upon entering Graphs 3 and 4 of Appendix A, the value of f_s/f_y corresponding to the coordinates of K_n and R_n is less than one so the value of ϕ is 0.65 as assumed.

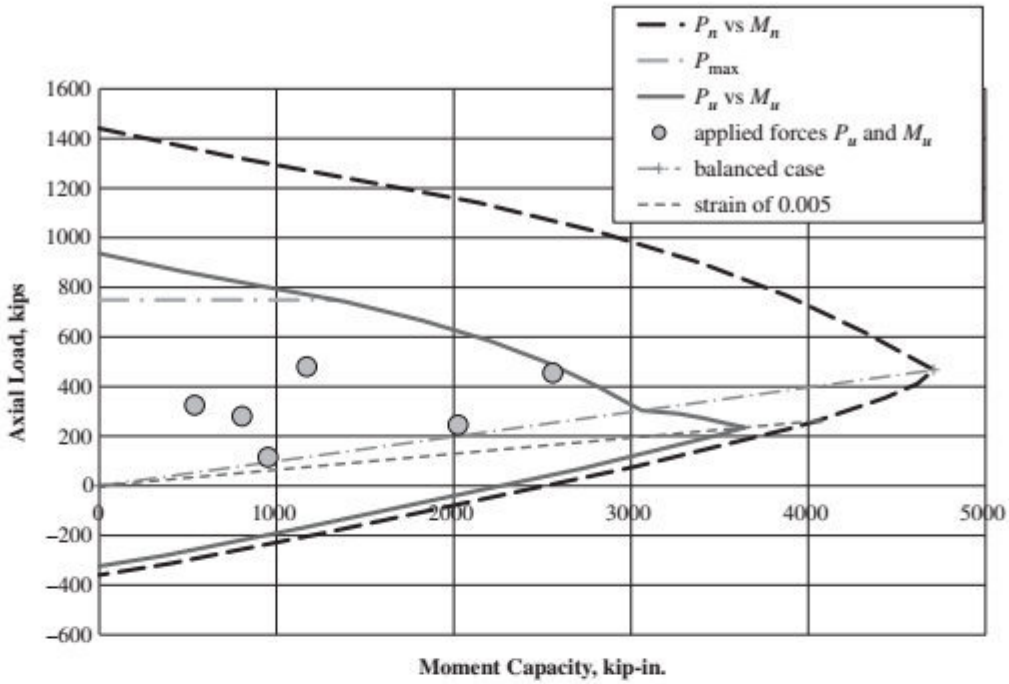


FIGURE 11.9 Interaction diagram for final column selected for Example 11.4.

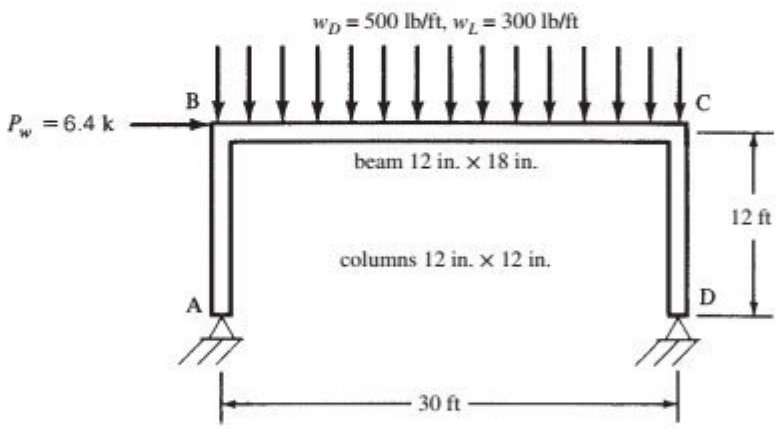


FIGURE 11.10 Sway frame for Example 11.5.

If we have an unbraced frame subjected to short-term lateral wind or earthquake loads, the columns will not have appreciable creep (which would increase lateral deflections and, thus, the $P\Delta$ moments). The effect of creep is accounted for in design by reducing the stiffness EI used to calculate P_c and thus δ by dividing EI by $1 + \beta_{dns}$, as specified in ACI Code, Section 6.6.4.4.4. Both the concrete and steel terms (if present) in ACI Code, Section 6.6.4.4.4 are divided by this value. In the case of sustained lateral load, such as soil backfill or water pressure, ACI Code, Section 6.6.3.1.1 requires that the moments of inertia for compression members in Section 11.6 be divided by $(1 + \beta_{ds})$. The term β_{ds} is the ratio of the maximum factored sustained shear within a story to the maximum factored shear in that story for the same load combination.

To illustrate the computation of the magnified moments needed for the design of a slender column in a sway frame, the authors have chosen the simple frame of Figure 11.10. We hope thereby that the student will not become lost in a forest of numbers, as he or she might if a large frame were considered.

The beam and columns of the frame have been tentatively sized, as shown in the figure. In Example 11.5, the frame is analyzed for each of the conditions specified in ACI Code, Section 5.3.1 using $1.0W$.

In the example, the magnification factors δ and δ_s are computed for each of the loading conditions and used to compute the magnified moments. Notice in the solution that different k values are used for determining δ and δ_s. The k for the δ calculation is determined from the alignment chart of Figure 11.3(a) for braced frames, whereas the k for the δ_s calculation is determined from the alignment chart of Figure 11.3(b) for unbraced frames.

Example 11.5

Determine the moments and axial forces that must be used for the design of column CD of the unbraced frame of Figure 11.10. The dead load includes the self-weight of the beam. Consider only bending in the plane of the frame. The assumed member sizes shown in the figure are used for the analyses given in the problem. $f_y = 60{,}000$ psi and $f'_c = 4000$ psi. For this example, the authors considered the load factor cases of ACI Equations 5.3.1b, 5.3.1d, and 5.3.1f. The wind load was also considered to have the same magnitude in both the positive and negative cases. The dead load of 500 lb/ft includes the beam self-weight.

SOLUTION

1. *Determine the effective length factor for the sway case using $0.35I_g$ for the girder and $0.70I_g$ for the columns.*

$$I_{column} = (0.70)\left(\frac{1}{12}\right)(12 \text{ in.})\,(12 \text{ in.})^3 = 1210 \text{ in.}^4$$

Note that if the lateral load were sustained, I_{column} would be divided by $(1 + \beta_{ds})$.

$$I_{girder} = (0.35)\left(\frac{1}{12}\right)(12 \text{ in.})\,(18 \text{ in.})^3 = 2041 \text{ in.}^4$$

$$\psi_B = \frac{\dfrac{1210 \text{ in.}^4}{12 \text{ ft}}}{\dfrac{2041 \text{ in.}^4}{30 \text{ ft}}} = 1.48$$

$\psi_A = \infty$ for pinned ends (For practical purposes, use 10)

$k = 1.95$ from Figure 11.3(b)

2. *Is it a slender column?*

$$\ell_u = 12 \text{ ft} - \frac{9 \text{ in.}}{12 \text{ in./ft}} = 11.25 \text{ ft}$$

Max $\dfrac{k\ell_u}{r}$ to be a short column = 22 for sway frames

$$\frac{k\ell_u}{r} = \frac{(1.95)\,(12 \text{ in./ft} \times 11.25 \text{ ft})}{0.3 \times 12 \text{ in.}} = 73.12 > 22$$

$\therefore$ It is a slender column.

3. *Consider the loading case $U = 1.2D + 1.6L$ (see Figure 11.11).*

(a) Are Column Moments ≥ ACI Minimum?

$$e_{min} = 0.6 + 0.03 \times 12 \text{ in.} = 0.96 \text{ in.}$$

$M_{2\,min} = (16.2 \text{ k})\,(0.96 \text{ in.}) = 15.55 \text{ in.-k} = 1.30 \text{ ft-k} < 55.89 \text{ ft-k}$ (see Figure 11.11) OK

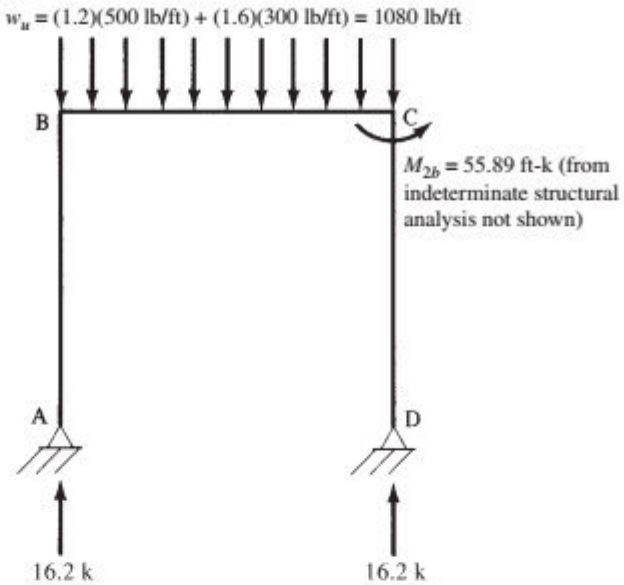


FIGURE 11.11 Loading $1.2D + 1.6L$.

(b) Compute the Magnification Factor δ

$$E_c = 57{,}000\sqrt{4000}\text{ psi} = 3{,}605{,}000\text{ psi} = 3605\text{ ksi}$$

$$\beta_{dns} = \frac{(1.2)\ (500\text{ plf})}{(1.2)\ (500\text{ plf}) + (1.6)\ (300\text{ plf})} = 0.556$$

$$(EI)_{eff} = \frac{(0.4)\ (3605\text{ ksi})\ (1728\text{ in.}^4)}{1 + 0.556} = 1.60 \times 10^6\text{ k-in.}^2$$

Assuming conservatively that $k = 1.0$ for computing P_c

$$P_c = \frac{(\pi^2)\ (1.60 \times 10^6\text{ k-in.}^2)}{(1.0 \times 12\text{ in./ft} \times 11.25\text{ ft})^2} = 866\text{ k}$$

$$C_m = 0.6 + (0.4)\left(\frac{-0\text{ ft-k}}{+55.89\text{ ft-k}}\right) = 0.6$$

$$\delta_{ns} = \frac{0.6}{1 - \dfrac{16.2\text{ k}}{(0.75)\ (866\text{ k})}} = 0.615 < 1.0 \qquad \underline{\text{Use 1.0}}$$

(c) Compute the Magnification Factor δ_s

Because there are no lateral loads and the loads and frame are symmetric, this load case has no sidesway.

(d) Compute the Magnified Moment

$$M_c = (1.0)\ (55.89\text{ ft-k})$$

4. *Consider the loading case $U = (1.2D + 1.0L + 1.0W)$ as specified in ACI Code, Section 5.3.1d. Analysis results are shown in Figure 11.12.*

(a) Are Column Moments ≥ ACI Minimum?

$$e_{min} = 0.6\text{ in.} + 0.03 \times 12\text{ in.} = 0.96\text{ in.}$$

$$M_{2\,min} = (13.5\text{ k})\ (0.96\text{ in.}) = 12.96\text{ in.-k} = 1.08\text{ ft-k} < 46.58\text{ ft-k} \qquad \underline{\text{OK}}$$

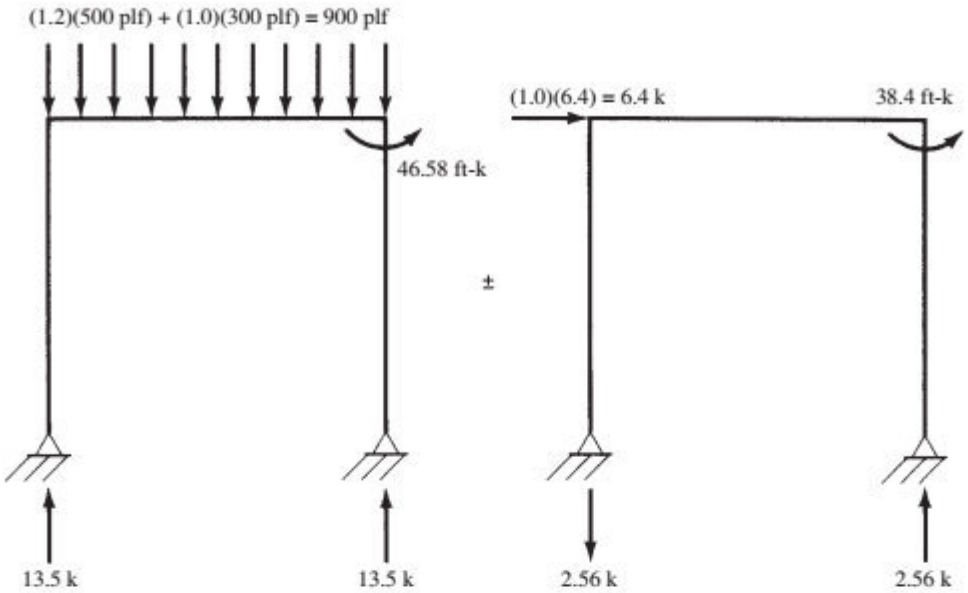


FIGURE 11.12 Nonsway and sway load cases for $U = 1.2D + 1.0L + 1.0W$.

(b) Computing δ_s

$$\beta_{dns} = \frac{1.2D}{1.2D + 1.0L + 1.0W} = \frac{(1.2)(500 \text{ plf})(15 \text{ ft})}{(1.2)(500 \text{ plf})(15 \text{ ft}) + (1.0)(300 \text{ plf})(15 \text{ ft}) + 2560 \text{ lb}} = 0.560$$

$$(EI)_{eff} = \frac{(0.4)(3605 \text{ ksi})(1728 \text{ in.}^4)}{1 + 0.560} = 1.60 \times 10^6 \text{ k-in.}^2$$

$$P_c = \frac{(\pi^2)(1.60 \times 10^6 \text{ k-in.}^2)}{(1.95 \times 12 \text{ in./ft} \times 11.25 \text{ ft})^2} = 227.9 \text{ k}$$

$$\delta_s = \frac{1}{1 + \dfrac{\Sigma P_u}{0.75 \Sigma P_c}} = \frac{1}{1 - \dfrac{(2)(13.5 \text{ k}) + 2.56 \text{ k} - 2.56 \text{ k}}{0.75 \times 2 \times 227.9 \text{ k}}} = 1.086$$

(c) Compute the Magnified Moment

$$M_c = (1.0)(46.58 \text{ ft-k}) + (1.086)(38.4 \text{ ft-k}) = \underline{\underline{88.27 \text{ ft-k}}}$$

5. *Consider the loading case* $0.9D + 1.0W$. *Analysis results are shown in Figure 11.13.*

(a) Are Column Moments ≥ ACI Minimum?

$$e_{min} = 0.6 \text{ in.} + 0.03 \times 12 \text{ in.} = 0.96 \text{ in.}$$

$$M_{2\,min} = (6.75 \text{ k})\,(0.96 \text{ in.}) = 6.48 \text{ in.-k} = 0.54 \text{ ft-k} < 23.29 \text{ ft-k} \quad \underline{\underline{\text{OK}}}$$

(b) Computing δ

$$\beta_{dns} = \frac{0.9D}{0.9D + 1.0W} = \frac{(0.9)(300 \text{ plf})(15 \text{ ft})}{(0.9)(300 \text{ plf})(15 \text{ ft}) + 2560 \text{ lb}} = 0.613$$

$$(EI)_{eff} = \frac{(0.4)\,(3605 \text{ ksi})\,(1728 \text{ in.}^4)}{1 + 0.613} = 1.54 \times 10^6 \text{ k-in.}^2$$

$$P_c = \frac{(\pi^2)\,(1.54 \times 10^6 \text{ k-in.}^2)}{(1.00 \times 12 \text{ in./ft} \times 11.25 \text{ ft})^2} = 834 \text{ k}$$

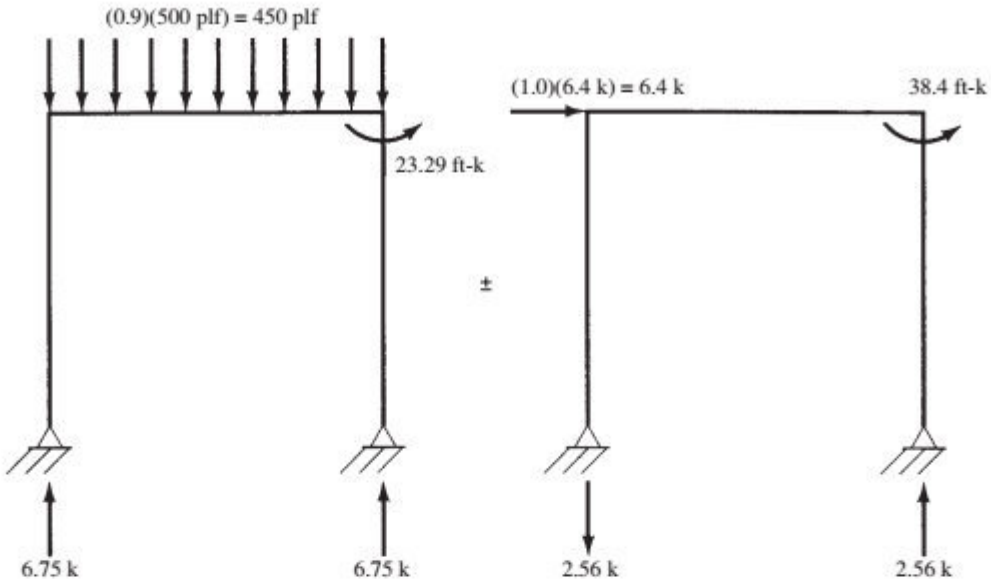


FIGURE 11.13 Nonsway and sway load cases for $U = 0.9D + 1.0W$.

$$C_m = 0.6 + 0.4\left(\frac{-0 \text{ ft-k}}{23.29 \text{ ft-k}}\right) = 0.6$$

$$\delta = \frac{0.6}{1 - \dfrac{6.75 \text{ k}}{0.75 \times 834 \text{ k}}} = 0.61 < 1.0 \qquad \underline{\text{Use 1.0}}$$

(c) Computing δ

$$\beta_{ds} = \frac{(0.9)(500 \text{ plf})(15 \text{ ft})}{(0.9)(500 \text{ plf})(15 \text{ ft}) + 2560 \text{ lb}} = 0.725$$

$$EI_{\text{eff}} = \frac{0.4(3,605 \text{ ksi})(1728 \text{ in.}^4)}{1 + 0.725} = 1.44 \times 10^6 \text{ k-in.}^2$$

$$P_c = \frac{\pi^2(1.44 \times 10^6 \text{ k-in.}^2)}{[1.95(12 \text{ in./ft} \times 11.25 \text{ ft})]^2} = 205 \text{ k}$$

$$C_m = 1.0$$

$$\delta = \frac{1}{1 - \dfrac{6.75 \text{ k} + 2.56 \text{ k} - 2.56 \text{ k}}{0.75(2)(205 \text{ k})}} = 1.022$$

(d) Calculate Moment

$$M_c = (1.0)(23.29 \text{ k-ft}) + (1.022)(38.4 \text{ k-ft}) = 62.53 \text{ k-ft}$$

6. *Summary of axial loads and moments to be used in design*

Loading I: $P_u = 16.2$ k, $M_c = 55.89$ ft-k

Loading II: $P_u = 13.5 \text{ k} + 2.56 \text{ k} = 16.06$ k, $M_c = 88.27$ ft-k

Loading III: $P_u = 6.75 \text{ k} + 2.56 \text{ k} = 9.31$ k, $M_c = 62.53$ ft-k

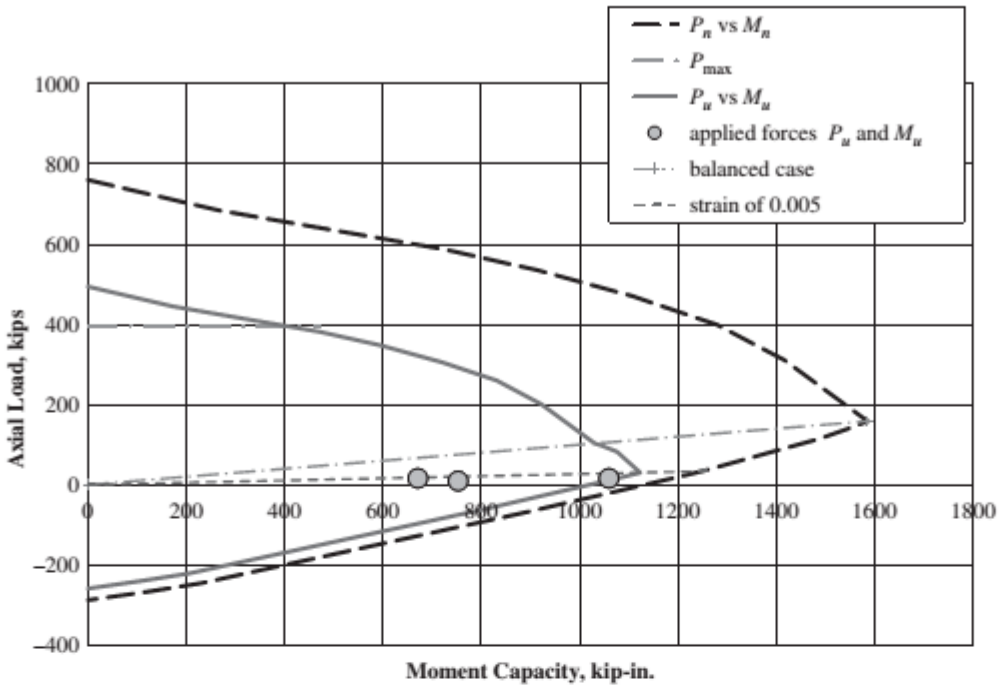


FIGURE 11.14 Interaction diagram for Example 11.5: 12 in. × 12 in. with $A_{sl} = 4.80$ in.^{2}.

Note: Should the reader now wish to determine the reinforcing steel needed for the above loads and moments, he or she will find that the steel percentage of 0.0333 is required using equal steel in two faces. This can be achieved using 4 #10 bars (2 in each opposite face) or 6 #9 bars. Six #8 bars will work if the value of $(EI)_{eff}$ is revised using ACI Equation 6.6.4.4.4b. Figure 11.14 shows the interaction diagram for this column with the theoretically required steel area of 4.80 in.^{2}. The three load combinations are superimposed on the diagram. It is clear that Loading II controls the amount of steel required.

11.12 Computer Examples

The Excel spreadsheet provided for Chapter 11 computes the effective length factor, k, for both braced and unbraced frames. It uses the same method as in Example 11.1, except that it uses the equations from Section 11.5 instead of the Jackson-Moreland Alignment Chart to determine k.

Example 11.6

Repeat Example 11.1 using the Excel spreadsheet for Chapter 11.

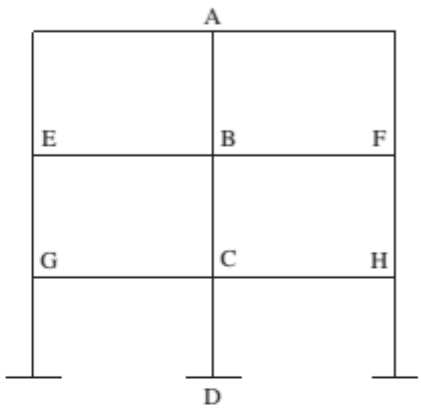
SOLUTION

Open the Chapter 11 spreadsheet and select the *k* factor tab. Enter the values of the cells highlighted in yellow (only in the Excel spreadsheets, not in the printed example). Note that a value of $b = 0$ is entered for member AB, since it does not exist. The software determines a value of $\Psi_A = 2.99$ and $\Psi_B = 2.31$.

These values are the same as those calculated by hand in Example 11.1. The value of k from the spreadsheet is 1.72 if the frame is unbraced compared with 1.74 from the Jackson-Moreland Alignment Chart. If the frame were braced, the software would give a value of $k = 0.96$. The Jackson-Moreland Alignment Chart gives a value of 0.87. The equations in Section 11.5 do not agree well with the chart in the case of braced frames.

This spreadsheet calculates the k factor for column BC.
Enter the values of shaded cells (highlighted in yellow in the Excel Spreadsheet).
If you do not have all the members shown, enter $b = 0$ for the missing member.

Columns				Girders			
	f'_c =	4000	psi		f'_c =	4000	
Col AB	b	0	in.	Beam EB	b	12	in.
	h	12	in.		h	18	in.
	col ht.	15	ft		span	20	
	$0.70I_g$	0	in.^{4}		$0.35I_g$	2041	in.^{4}
Col BC	b	12	in.^{2}	Beam BF	b	12	
	h	20			h	18	
	col ht.	10	ft		span	24	in.
	$0.70I_g$	5600			$0.35I_g$	2041	in.^{4}
Col CD	b	12	in.	Beam CG	b	12	in.
	h	20	in.^{4}		h	24	
	col ht.	12	ft		span	20	ft
	$0.70I_g$	5600			$0.35I_g$	4838	in.^{4}
				Beam CH	b	12	k
					h	24	k
					span	24	
					$0.35I_g$	4838	in.^{4}



$$\Psi_B = \frac{E_c \times 0.70I_g/l_{AB} + E_c \times 0.70I_g/l_{BC}}{E_c \times 0.35I_g/l_{BE} + E_c \times 0.35I_g/l_{BF}} \quad 2.993$$

$$\Psi_C = \frac{E_c \times 0.70I_g/l_{BC} + E_c \times 0.70I_g/l_{CD}}{E_c \times 0.35I_g/l_{CG} + E_c \times 0.35I_g/l_{CH}} \quad 2.315$$

$$\Psi_{min} = 2.315$$

$$\Psi_m = 2.654$$

Braced

$k = 0.7 + 0.05(\Psi_A + \Psi_B) \le 1.0 = 0.965$

$k = 0.85 + 0.05\Psi_{min} \le 1.0 = 0.966$

Braced

$k = 0.965$

Unbraced

$k = 1.72$

Example 11.7

Repeat Example 11.3 using the Excel spreadsheet for Chapter 11.

SOLUTION

Open the Chapter 11 spreadsheet and select the "slender column braced rect." tab. Enter the values in the cells highlighted in yellow. The value of A_s entered is zero since the column is not yet designed. The software automatically uses ACI Equation 6.6.4.4.4a in this case. If a value for A_s is entered, the

software compares the value of $(EI)_{eff}$ from ACI Equations 6.6.4.4.4a and 6.6.4.4.4b and uses the larger. The final value of δ is 1.093, which is in agreement with the solution obtained in Example 11.3. The final magnified moment, δM_2, equals 93.97 ft-k. To complete the design, the Column Design spreadsheets for Chapters 9 and 10 can be used.

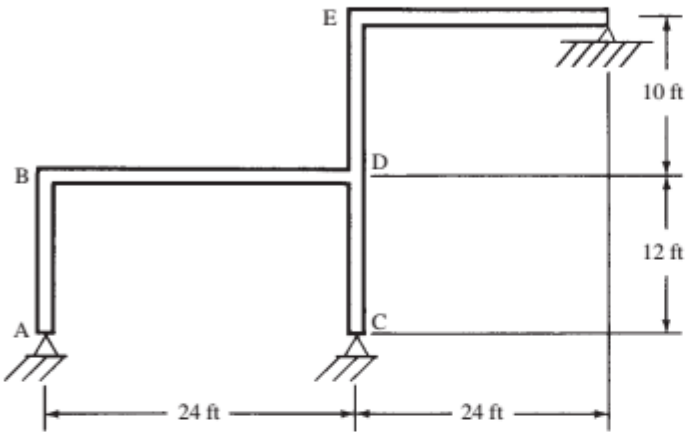
Slender Column Braced - Rectangular			
	f'_c	4000	psi
	γ	145	pcf
Column	b	12	in.
	h	15	in.
	ℓ_u	16	ft
	A_{st}	0	in.^{2}
	ρ_g	0	
Col loads	P_D	30	k
	P_L	46.25	k
	M_{2D}	10	ft-k
	M_{2L}	46.25	ft-k
	P_u	110	k
	M_2	86	ft-k
	M_{1D}	10	ft-k
	M_{1L}	43.75	ft-k
	M_1	82	ft-k
	C_m	0.9814	
	k	0.83	
	$k\ell_u/r$	35.41	
$34 - 12M_1/M_2$		22.56	
Is $k\ell_u/r < 34 - 12M_1/M_2$		consider slenderness	
	E_c	3644	psi
	n	7.96	
	I_g	3375	in.^{4}
	γ	0.67	
	β_{dns}	0.327	
	n	0.500	
	$0.4E_cI_gn$	4920	k-in.^{2}
$EI = 0.4EIn/1 + \beta_d$		3707	k-in.^{2}
	P_c	1440	k
	$M_{2\,\min}$	9.625	ft-k
		—	
	δ	1.093	
	δM_2	93.97	ft-k

Use the Chapter 10 spreadsheet to design this column for

P_u	110	k
δM_2	93.97	ft-k

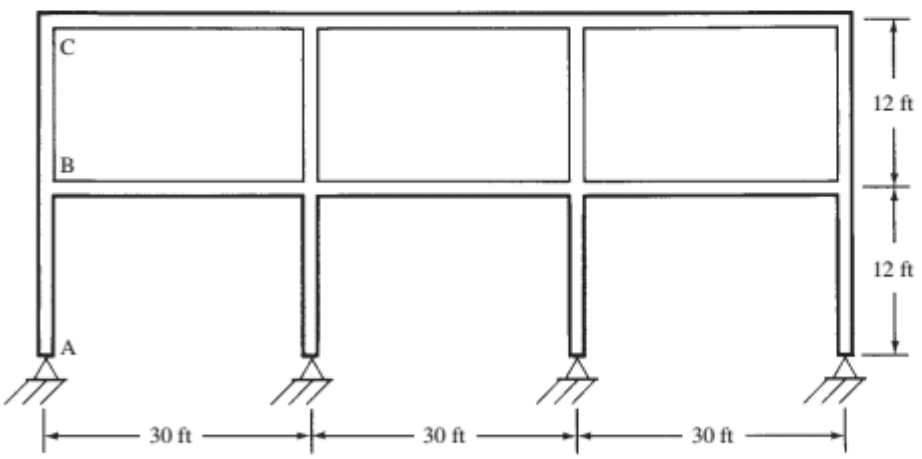
PROBLEMS

Problem 11.1 Using the alignment charts of Figure 11.3, determine the effective length factors for columns CD and DE for the braced frame shown. Assume that the beams are 12 in. × 20 in. and the columns are 12 in. × 16 in. Use 0.70 gross moments of inertia of columns and 0.35 gross moments of inertia of beams. Assume ψ_A and $\psi_C = 10$. (*Ans.* 0.94, 0.89)



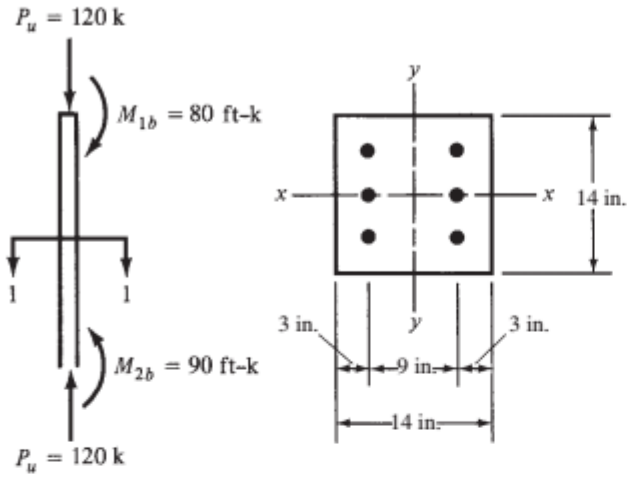
Problem 11.2 Repeat Problem 11.1 if the column bases are fixed.

Problem 11.3 Using the alignment charts of Figure 11.3, determine the effective length factors for columns AB and BC of the braced frame shown. Assume that all beams are 12 in. × 20 in. and all columns are 12 in. × 16 in. Use 0.70 gross moments of inertia of columns and 0.35 gross moments of inertia of beams. Assume that the far ends of beams are pinned, and use $\psi_A = 10$. (*Ans.* 0.94, 0.90)



Problem 11.4 Repeat Problem 11.3 if the frame is unbraced.

Problem 11.5 The tied column shown is to be used in a braced frame. It is bent about its y-axis with the factored moments shown and ℓ_u is 16 ft. If $k = 1.0$, $f_y = 60{,}000$ psi, and $f'_c = 4000$ psi, select the reinforcing bars required. Assume $P_D = 60$ k. (*One ans.* $\delta = 1.21$, 6 #7 bars)



Problem 11.6 Repeat Problem 11.5 with $(EI)_{\text{eff}}$ based on using 6 #8 bars.

Problem 11.7 Repeat Problem 11.5 if the column is bent in reverse curvature, and its length is 18 ft. Use ACI Equation 6.6.4.4.4a for $(EI)_{\text{eff}}$. (*One ans.* $\delta = 1.0$, 6 #6 bars)

For Problems 11.8 to 11.12 and the braced tied columns given, select reinforcing bars (placed in two faces) if the distance from the column edge to the c.g. of the bars is 2.5 in., $f_y = 60$ ksi, and $f'_c = 4$ ksi. Rectangular columns are bent about their strong axis. Use ACI Equation 6.6.4.4.4a for $(EI)_{\text{eff}}$.

Problem no.	Column size $b \times h$ (in.)	ℓ_u(ft)	k	P_u (k)	Not factored P_D (k)	Factored M_{1b} (ft-k)	Factored M_{2b} (ft-k)	Curvature	
11.8	14 × 14	12	1.0	400	100	75	85	Single	
11.9	15 × 15	16	1.0	500	120	100	140	Double	(*One ans.* $\delta = 1.0$, 8 #10 bars)
11.10	16 × 18	15	0.85	250	40	100	120	Single	
11.11	12 × 18	16	0.80	500	130	90	120	Single	(*One ans.* $\delta = 1.20$, 8 #9 bars)
11.12	14 × 18	18	0.90	600	150	100	130	Double	

Problem 11.13 Repeat Problem 11.9 using single curvature. (*One ans.* $\delta = 1.87$, 8 #14 bars)

For Problems 11.14 to 11.17 and the unbraced tied columns given, select reinforcing bars (placed in two faces) if the distance from the column edge to the c.g. of the bars is 2.5 in., $f_y = 60$ ksi, and $f'_c = 4$ ksi. Rectangular columns are bent about their strong axis. None of the wind load is considered sustained. Use ACI Equation 6.6.4.4.4a for $(EI)_{\text{eff}}$.

Problem no.	Column size $b \times h$ (in.)	ℓ_u (ft)	k	P_u (k) for loads not considered sway	P_u (k) due to wind	M_{1ns} (ft-k)	M_{2ns} (ft-k)	M_u (ft-k) due to wind	$\sum P_u$ (k) for all columns on floor	$\sum P_c$ (k) for all columns on floor	
11.14	16 × 20	15	1.3	500	110	80	90	100	12,000	34,000	
11.15	14 × 18	12	1.4	300	80	70	75	80	10,000	30,000	(*One ans.* 6 #11 bars)
11.16	16 × 20	16	1.65	500	140	110	140	120	16,500	80,000	
11.17	12 × 18	12	1.5	440	140	90	120	110	14,000	36,200	(*One ans.* 10 #11 bars)

Problems in SI Units

For Problems 11.18 to 11.20 and the braced tied columns given, select reinforcing bars (placed in two faces) if the distances from column edges to c.g. of bars are 75 mm each. To be able to use Appendix A graphs, use $f_y = 413.7$ MPa and $f'_c = 27.6$ MPa. Use ACI Equation 6.6.4.4.4a for $(EI)_{eff}$.

Problem no.	Column size $b \times d$ (mm)	ℓ_u (m)	k	P_u (kN)	P_D not factored (kN)	M_{1b} factored (kN-m)	M_{2b} factored (kN-m)	Curvature	
11.18	450 × 450	4	1.0	1800	400	80	100	Single	
11.19	300 × 400	5	0.92	2200	500	110	125	Double	(*One ans.* 6 #32 bars)
11.20	300 × 500	6	0.88	2400	550	120	140	Single	

For Problems 11.21 to 11.23 and the unbraced tied columns given, select reinforcing bars (placed on two faces) if the distance from the column edge to the c.g. of the bars is 75 mm, $f_y = 420$ MPa, and $f'_c = 28$ MPa.

Problem no.	Column size $b \times d$ (in.)	ℓ_u(m)	k	P_u (kN) not considered sway	P_u (kN) due to wind	M_{1ns} (kN-m)	M_{2ns} (kN-m)	M_u (kN-m) due to wind	$\sum P_u$ (kN) for all columns on floor	$\sum P_c$ (kN) for all columns on floor	
11.21	300 × 400	5	1.2	400	1200	40	50	60	40 000	110 000	(*One ans.* 6 #32 bars)
11.22	300 × 500	4	1.3	500	1800	50	60	70	44 000	125 000	
11.23	350 × 600	6	1.35	600	2500	65	90	110	50 000	156 000	(*One ans.* 6 #32 bars)

Footings

CHAPTER 12

12.1 Introduction

Footings are structural members used to support columns and walls and transmit their loads to the underlying soils. Reinforced concrete is a material admirably suited for footings and is used as such for both reinforced concrete and structural steel buildings, bridges, towers, and other structures.

The permissible pressure on the soil beneath a footing is normally a few tons per square foot. The compressive stresses in the walls and columns of an ordinary structure may run as high as a few hundred tons per square foot. It is, therefore, necessary to spread these loads over sufficient soil areas to permit the soil to support the loads safely.

Not only is it desired to transfer the superstructure loads to the soil beneath in a manner that will prevent excessive or uneven settlements and rotations, but it is also necessary to provide sufficient resistance to sliding and overturning.

To accomplish these objectives, it is necessary to transmit the supported loads to a soil of sufficient strength and then to spread them out over an area such that the unit pressure is within a reasonable range. If it is not possible to dig a short distance and find a satisfactory soil, it will be necessary to use piles or caissons to do the job. These latter subjects are not discussed as they do not fall within the scope of this text.

The closer a foundation is to the ground surface, the more economical it will be to construct. There are two reasons, however, that may keep the designer from using very shallow foundations. First, it is necessary to locate the bottom of a footing below the ground freezing level to avoid vertical movement or heaving of the footing as the soil freezes and expands in volume. This depth varies from about 3 ft to 6 ft in the northern states and less in the southern states. Second, it is necessary to excavate a sufficient distance so that a satisfactory bearing material is reached, and this distance may on occasion be quite a few feet.

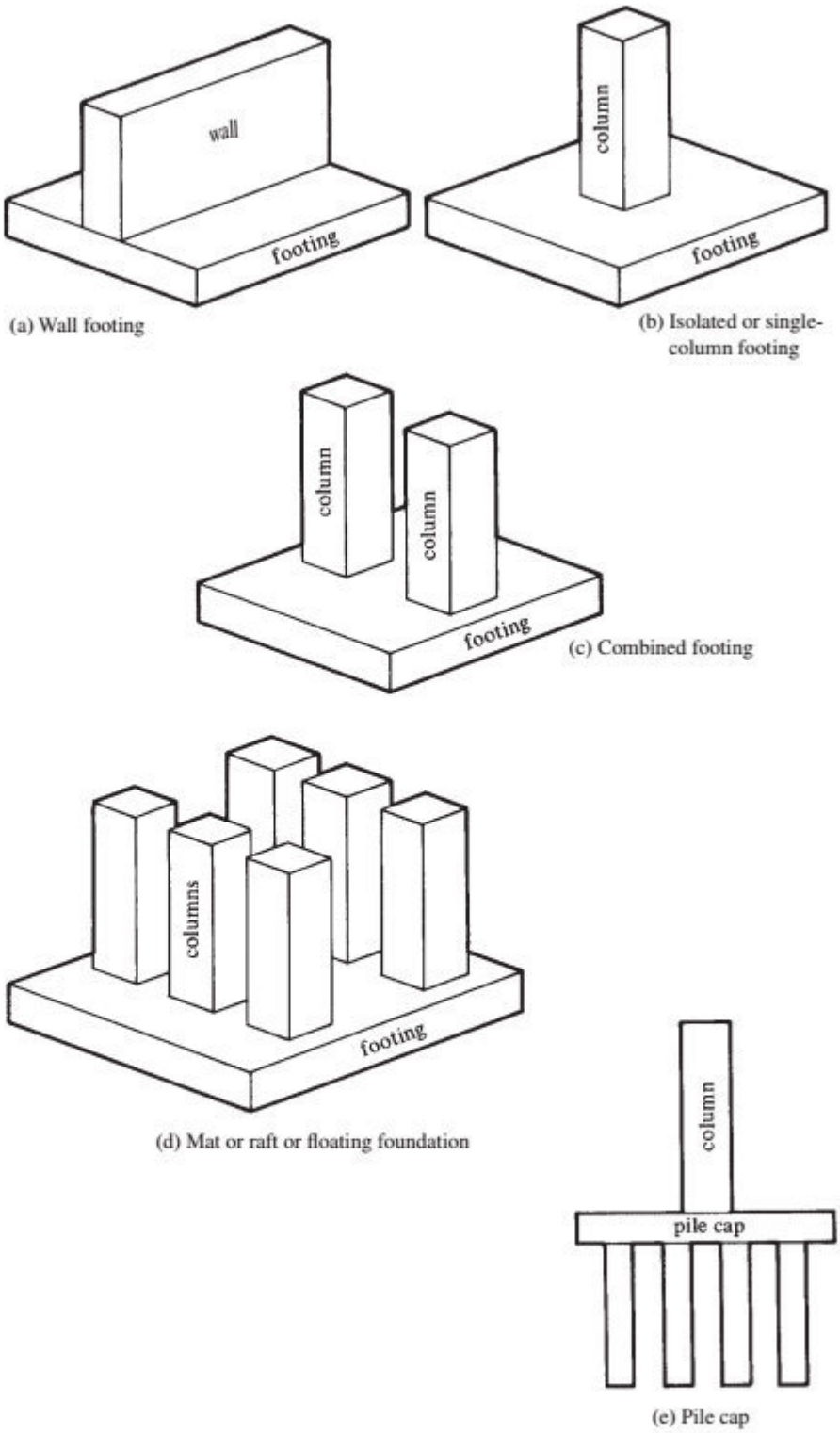
12.2 Types of Footings

Among the several types of reinforced concrete footings in common use are the wall, isolated, combined, raft, and pile-cap types. These are briefly introduced in this section; the remainder of the chapter is used to provide more detailed information about the simpler types of this group.

1. A *wall footing*, as shown in Figure 12.1(a), is simply an enlargement of the bottom of a wall that will sufficiently distribute the load to the foundation soil. Wall footings are normally used around the perimeter of a building and perhaps for some of the interior walls.

2. An *isolated or single-column footing*, as shown in Figure 12.1(b), is used to support the load of a single column. These are the most commonly used footings, particularly where the loads are relatively light and the columns are not closely spaced.

3. *Combined footing* is used to support two or more column loads, as shown in Figure 12.1(c). A combined footing might be economical where two or more heavily loaded columns are so spaced that normally designed single-column footings would run into each other. Single-column footings are usually square or rectangular and, when used for columns located right at property lines, extend across those lines. A footing for such a column combined with one for an interior column can be designed to fit within the property lines.



(a) Wall footing

(b) Isolated or single-column footing

(c) Combined footing

(d) Mat or raft or floating foundation

(e) Pile cap

FIGURE 12.1 Types of footings.

Steve Dunwell/Getty Images, Inc.

Pouring concrete, Big Dig, Boston, Massachusetts.

4. A *mat or raft or floating foundation*, as shown in Figure 12.1(d), is a continuous reinforced concrete slab over a large area used to support many columns and walls. This kind of foundation is used where soil strength is low or column loads are large but where piles or caissons are not used. For such cases, isolated footings would be so large that it is more economical to use a continuous raft or mat under the entire area. The cost of the formwork for a mat footing is far less than is the cost of the forms for a large number of isolated footings. If individual footings are designed for each column and if their combined area is greater than half of the area contained within the perimeter of the building, it is usually more economical to use one large footing or mat. The raft or mat foundation is particularly useful in reducing differential settlements between columns—the reduction being 50% or more. For these types of footings, the excavations are often rather deep. The goal is to remove an amount of earth approximately equal to the building weight. If this is done, the net soil pressure after the building is constructed will theoretically equal what it was before the excavation was made. Thus, the building will *float* on the raft foundation.
5. *Pile caps*, as shown in Figure 12.1(e), are slabs of reinforced concrete used to distribute column loads to groups of piles.

12.3 Actual Soil Pressures

The soil pressure at the surface of contact between a footing and the soil is assumed to be uniformly distributed as long as the load above is applied at the center of gravity of the footing, Figure 12.2(a). This assumption is made even though many tests have shown that soil pressures are unevenly distributed due to variations in soil properties, footing rigidity, and other factors.

As an example of the variation of soil pressures, footings on sandy and clayey soils are considered. When footings are supported by sandy soils, the pressure is larger under the center of the footing and smaller near the edge, Figure 12.2(b). The sand at the edges of the footing does not have a great deal of lateral support and tends to move from underneath the footing edges, with the result that more of the load is carried near the center of the footing. Should the bottom of a footing be located at some distance from the ground surface, a sandy soil will provide fairly uniform support because it is restrained from lateral movement.

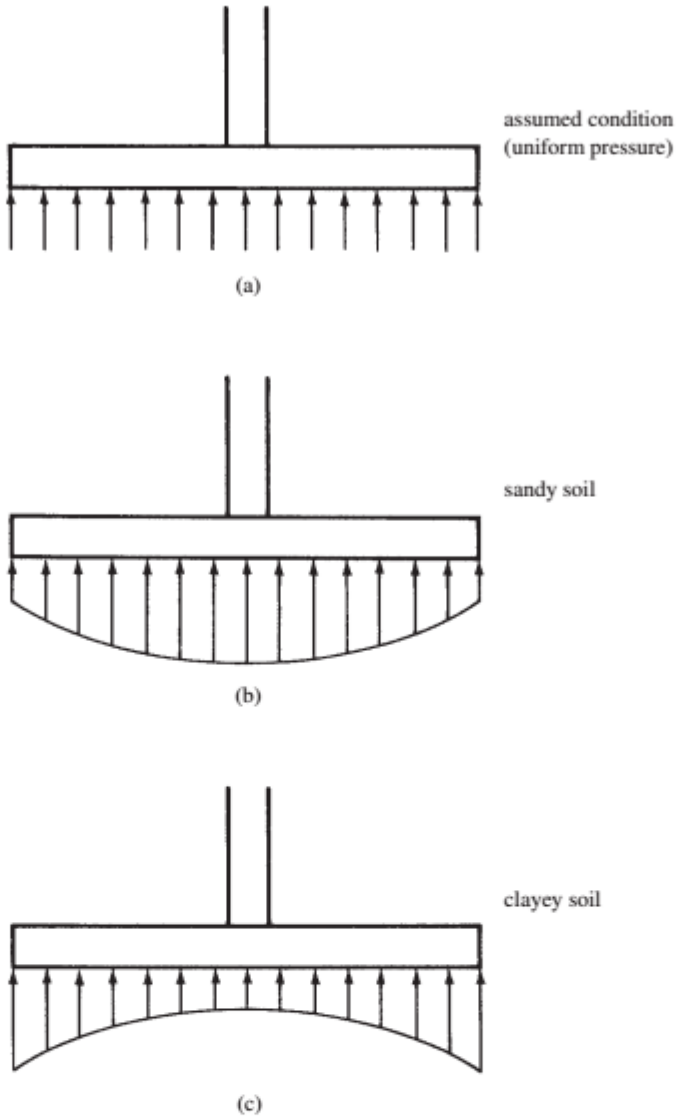


FIGURE 12.2 Soil conditions.

Just the opposite situation is true for footings supported by clayey soils. The clay under the edges of the footing sticks to or has cohesion with the surrounding clayey soil. As a result, more of the load is carried at the edge of the footing than near the middle [see Figure 12.2(c)].

The designer should clearly understand that the assumption of uniform soil pressure underneath footings is made for reasons of simplifying calculations and may very well have to be revised for some soil conditions.

Should the load be eccentrically applied to a footing with respect to the center of gravity of the footing, the soil pressure is assumed to vary uniformly in proportion to the moment, as discussed in Section 12.12 and illustrated in Figure 12.23.

12.4 Allowable Soil Pressures

The allowable soil pressures to be used in designing the footings for a particular structure are preferably obtained by using the services of a geotechnical engineer. He or she will determine safe values from the principles of soil mechanics on the basis of test borings, load tests, and other experimental investigations. Other issues may enter into the determination of the allowable soil pressures, such as

the sensitivity of the building frame to deflection of the footings. Also, cracking of the superstructure resulting from settlement of the footings would be much more important in a performing arts center than a warehouse.

Because such investigations often may not be feasible, most building codes provide certain approximate allowable bearing pressures that can be used for the types of soils and soil conditions occurring in that locality. Table 12.1 shows a set of allowable values that are typical of such building codes. It is thought that these values usually provide factors of safety of approximately three against severe settlements.

Section 13.3.1.1 of the ACI Code states that the required area of a footing is to be determined by dividing the anticipated total load, including the footing weight, by a permissible soil pressure or permissible pile capacity determined using the principles of soil mechanics. It will be noted that this total load is the *unfactored* load, and yet the design of footings described in this chapter is based on strength design, where the loads are multiplied by the appropriate load factors. It is obvious that an ultimate load cannot be divided by an allowable soil pressure to determine the bearing area required.

The designer can handle this problem in two ways. He or she can determine the bearing area required by summing up the actual or unfactored dead and live loads and dividing them by the allowable soil pressure. Once this area is determined and the dimensions are selected, an ultimate soil pressure can be computed by dividing the factored or ultimate load by the area provided. The remainder of the footing can then be designed by the strength method using this ultimate soil pressure. This simple procedure is used for the footing examples here.

The 1971 ACI Commentary (15.2) provided an alternative method for determining the footing area required that will give exactly the same answers as the procedure just described. By this latter method, the allowable soil pressure is increased to an ultimate value by multiplying it by a ratio equal to that used for increasing the magnitude of the service loads. For instance, the ratio for D and L loads would be

$$\text{Ratio} = \frac{1.2D + 1.6L}{D + L}$$

Or for $D + L + W$, and so on

$$\text{Ratio} = \frac{1.2D + 1.6W + 1.0L + 0.5(L_r \text{ or } S \text{ or } R)}{D + L + W + (L_r \text{ or } S \text{ or } R)}$$

The resulting ultimate soil pressure can be divided into the ultimate column load to determine the area required.

TABLE 12.1 Maximum Allowable Soil Pressure

	Maximum Allowable Soil Pressure	
Class of Material	**U.S. Customary Units ($kips/ft^2$)**	**SI Units (kN/m^2)**
Rock	20% of ultimate crushing strength	20% of ultimate crushing strength
Compact coarse sand, compact fine sand, hard clay, or sand clay	8	385
Medium stiff clay or sandy clay	6	290
Compact inorganic sand and silt mixtures	4	190
Loose sand	3	145
Soft sand clay or clay	2	95
Loose inorganic sand–silt mixtures	1	50
Loose organic sand–silt mixtures, muck, or bay mud	0	0

12.5 Design of Wall Footings

The theory used for designing beams is applicable to the design of footings with only a few modifications. The upward soil pressure under the wall footing of Figure 12.3 tends to bend the footing into the deformed shape shown. The footings will be designed as shallow beams for the moments and shears involved. In beams where loads are usually only a few hundred pounds per foot and spans are fairly large, sizes are almost always proportioned for moment. In footings, loads from the supporting soils may run several thousand pounds per foot and spans are relatively short. As a result, shears will almost always control depths.

It appears that the maximum moment in this footing occurs under the middle of the wall, but tests have shown that this is not correct because of the rigidity of such walls. If the walls are of reinforced concrete with their considerable rigidity, it is considered satisfactory to compute the moments at the faces of the walls (ACI Code, Section 13.2.7.1). Should a footing be supporting a masonry wall with its greater flexibility, the code states that the moment should be taken at a section halfway from the face of the wall to its center. (For a column with a steel base plate, the critical section for moment is to be located halfway from the face of the column to the edge of the plate.)

To compute the bending moments and shears in a footing, it is necessary to compute only the net upward pressure, q_u, caused by the factored wall loads above. In other words, the weight of the footing and soil on top of the footing can be neglected. These items cause an upward pressure equal to their downward weights, and they cancel each other for purposes of computing shears and moments. In a similar manner, it is obvious that there are no moments or shears existing in a book lying flat on a table.

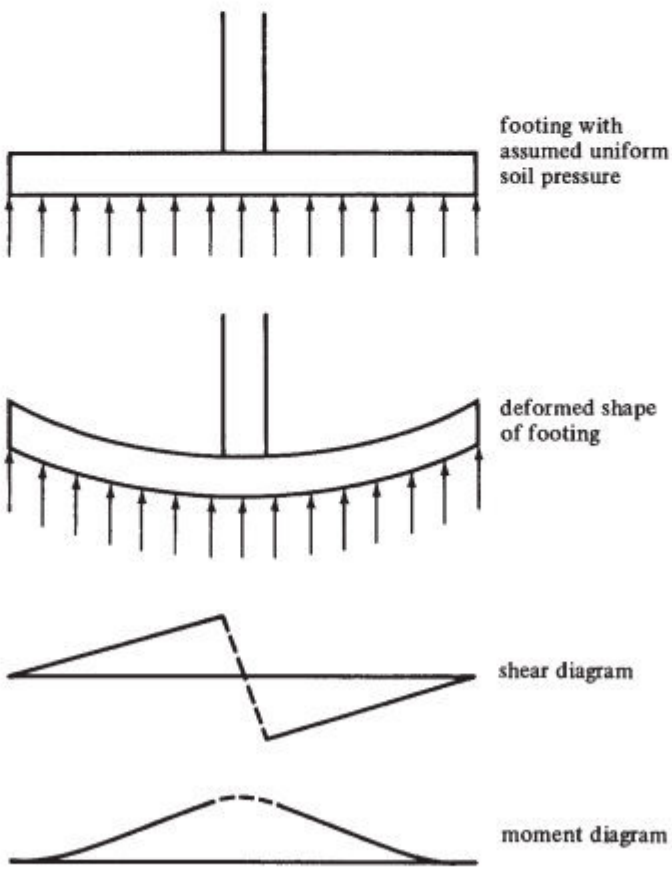


FIGURE 12.3 Shear and moment diagram for wall footing with uniform soil pressure.

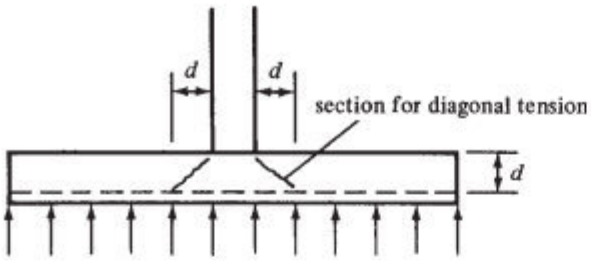


FIGURE 12.4 Critical section for shear in a wall footing.

Should a wall footing be loaded until it fails in shear, the failure will not occur on a vertical plane at the wall face but rather at an angle of approximately 45° with the wall, as shown in Figure 12.4. Apparently the diagonal tension, which one would expect to cause cracks in between the two diagonal lines, is opposed by the squeezing or compression caused by the downward wall load and the upward soil pressure. Outside this zone, the compression effect is negligible in its effect on diagonal tension. Therefore, for non-prestressed sections, shear may be calculated at a distance d from the face of the wall (ACI Code, Section 13.2.7.2) because of the loads located outside the section.

The use of stirrups in footings is usually considered impractical and uneconomical. For this reason, the effective depth of wall footings is selected so that V_u is limited to the design shear strength, ϕV_c, that the concrete can carry without web reinforcing, that is, $\phi 2\lambda\sqrt{f'_c}b_w d$ (from ACI Code, Section 22.5.5.1 and ACI Equation 22.5.5.1). Although the equation for v_c contains the term λ, it would be unusual to use lightweight concrete to construct a footing. The primary advantage for using lightweight concrete and its associated additional cost is to reduce the weight of the concrete superstructure. It would not be economical to use it in a footing. For this reason, the term λ will not be included in the example problems for this chapter. The following expression is used to select the depths of wall footings:

$$d = \frac{V_u}{(\phi)\left(2\lambda\sqrt{f'_c}\right)(b_w)}$$

or in SI units

$$d = \frac{3V_u}{\phi\lambda\sqrt{f'_c}b_w}$$

The design of wall footings is conveniently handled by using 12-in. widths of the wall, as shown in Figure 12.5. Such a practice is followed for the design of a wall footing in Example 12.1. It should be noted that the ACI Code states that the depth of a footing above the bottom reinforcing bars may not be less than 6 in. for footings on soils (Section 13.3.1.2) and 12 in. for those on piles (Section 13.4.2.1). Thus, total minimum practical depths are at least 10 in. for regular spread footings and 16 in. for pile caps.

In SI units, it is convenient to design wall footings for 1-m-wide sections of the walls. The depth of such footings above the bottom reinforcing bars may not be less than 150 mm for footings on soils or 300 mm for those on piles. As a result, minimum footing depths are at least 250 mm for regular spread footings and 400 mm for pile caps.

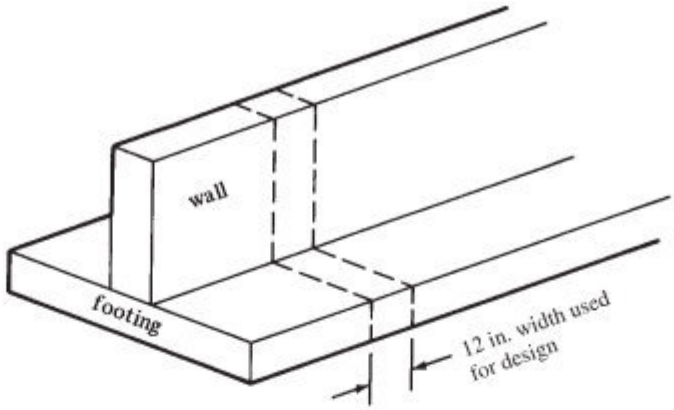


FIGURE 12.5 One-foot design strip width for wall footing.

The design of a wall footing is illustrated in Example 12.1. Although the example problems and homework problems of this chapter use various f'_c values, 3000-psi and 4000-psi concretes are commonly used for footings and are generally quite economical. Occasionally, when it is very important to minimize footing depths and weights, stronger concretes may be used. For most cases, however, the extra cost of higher-strength concrete will appreciably exceed the money saved with the smaller concrete volume. The exposure category of the footing may control the concrete strength. ACI Code, Section 19.3.1.1 requires that concrete exposed to sulfate have minimum f'_c values, depending on the sulfur concentration in the soil.

The determination of a footing depth is a trial-and-error problem. The designer assumes an effective depth, d, computes the d value required for shear, tries another d, computes the d value required for shear, and so on, until the assumed value and the calculated value are within about 1 in. of each other.

You probably get upset when a footing size is assumed here. You say, "Where in the world did you get that value?" We think of what seems like a reasonable footing size and start there. We compute the d value required for shear and probably find we've missed the assumed value quite a bit. We then try another value roughly two-thirds to three-fourths of the way from the trial value to the computed value (for wall footings) and compute the d value. (For column footings, we probably go about halfway from the trial value to the computed value.) Two trials are usually sufficient. We have the advantage over you in that often in preparing the sample problems for this textbook, we made some scratch-paper trials before arriving at the assumed values used.

Example 12.1

Design a wall footing to support a 12-in.-wide reinforced concrete wall with a dead load $D = 20$ k/ft and a live load $L = 15$ k/ft. The bottom of the footing is to be 4 ft below the final grade, the soil weighs 100 lb/ft^3, the allowable soil pressure, q_a, is 4 ksf, and there is no appreciable sulfur content in the soil. $f_y = 60$ ksi and $f'_c = 3$ ksi, normal-weight concrete.

SOLUTION

Assume a 12-in.-thick footing ($d = 8.5$ in.). The cover is determined by referring to the ACI Code, Section 20.6.1.3, which says that for concrete cast against and permanently exposed to the earth, a minimum of 3 in. clear distance outside any reinforcing is required. In severe exposure conditions, such as high sulfate concentration in the soil, the cover must be suitably increased (ACI Code, Section 20.6.1.4.1).

The footing weight is (12 in./12 in./ft)(150 pcf) = 150 psf, and the soil fill on top of the footing is (36 in./12 in./ft)(100 pcf) = 300 psf. So 450 psf of the allowable soil pressure (q_a) is used to support the

footing itself and the soil fill on top. The remaining soil pressure is available to support the wall loads. It is called q_e, the effective soil pressure.

$$q_e = 4000 \text{ psf} - \left(\frac{12 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{36 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 3550 \text{ psf}$$

$$\text{Width of footing required} = \frac{20 \text{ k} + 15 \text{ k}}{3.55 \text{ ksf}} = 9.86 \text{ ft} \qquad \underline{\text{Say 10 ft 0 in.}}$$

Bearing Pressure for Strength Design for a 12 in. width

$$q_u = \frac{(1.2)(20 \text{ k}) + (1.6)(15 \text{ k})}{10.00 \text{ ft}^2} = 4.80 \text{ ksf}$$

Depth Required for Shear at a Distance *d* from Face of Wall (Figure 12.6)

$$V_u = \left(\frac{10.00 \text{ ft}}{2} - \frac{6 \text{ in.}}{12 \text{ in./ft}} - \frac{8.5 \text{ in.}}{12 \text{ in./ft}}\right)(4.80 \text{ ksf}) = 18.20 \text{ k}$$

$$d = \frac{18{,}200 \text{ lb}}{(0.75)(2\sqrt{3000} \text{ psi})(12 \text{ in.})} = 18.46 \text{ in.}$$

$$h = 18.46 \text{ in.} + 3.5 \text{ in.} = 21.96 \text{ in.} > 12 \text{ in.} \qquad \underline{\text{Try again}}$$

Assume 20-in. Footing (d = 16.5 in.)

$$q_e = 4000 \text{ psf} - \left(\frac{20 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{28 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 3517 \text{ psf}$$

$$\text{Width required} = \frac{20 \text{ k} + 15 \text{ k}}{3.517 \text{ ksf}} = 9.95 \text{ ft} \qquad \underline{\text{Say 10 ft 0 in.}}$$

Bearing Pressure for Strength Design

$$q_u = \frac{(1.2)\,(20 \text{ k}) + (1.6)\,(15 \text{ k})}{10.00 \text{ ft}^2} = 4.80 \text{ ksf}$$

Depth Required for Shear

$$V_u = \left(\frac{10.00 \text{ ft}}{2} - \frac{6 \text{ in.}}{12 \text{ in./ft}} - \frac{16.50 \text{ in.}}{12 \text{ in./ft}}\right)(4.80) = 15.0 \text{ k}$$

$$d = \frac{15{,}000 \text{ lb}}{(0.75)(2\sqrt{3000} \text{ psi})(12 \text{ in.})} = 15.21 \text{ in.}$$

$$h = 15.21 \text{ in.} + 3.5 \text{ in.} = 18.71 \text{ in.} \qquad \underline{\text{Use 20 in. total depth}}$$

(*A subsequent check of a 19-in. footing shows it will not quite work.*)

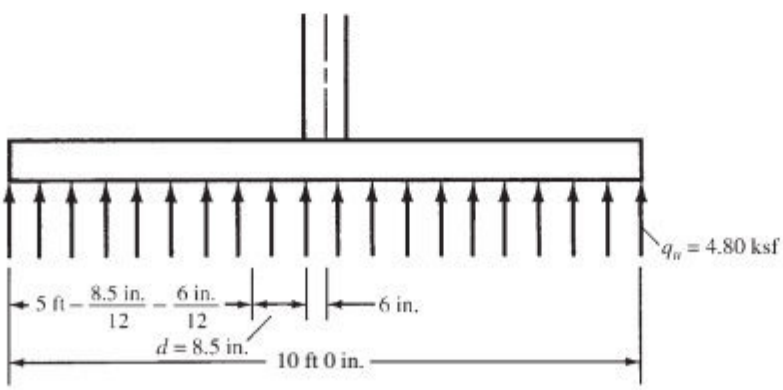


FIGURE 12.6 Critical flexure and shear locations for Example 12.1.

Determine Steel Area (Using $d = 16.5$ in.)

Taking moments at face of wall (Figure 12.6),

$$\text{Cantilever length} = \frac{10.00 \text{ ft}}{2} - \frac{6 \text{ in.}}{12 \text{ in./ft}} = 4.50 \text{ ft}$$

$$M_u = (4.50 \text{ ft})\,(4.80 \text{ klf})\,(2.25 \text{ ft}) = 48.6 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(48{,}600 \text{ ft-lb})}{(0.9)\,(12 \text{ in.})\,(16.5 \text{ in.})^2} = 198.3 \text{ psi}$$

From Appendix A, Table A.12, $\rho = 0.00345$ (by interpolation). Because this value of ρ is less than 0.0136 (from Table A.7), the section is tension controlled and $\phi = 0.9$ as assumed.

$$A_s = (0.00345)(12 \text{ in.})(16.5 \text{ in.}) = 0.68 \text{ in.}^2 \qquad \text{Use \#7 @ 10 in. } (0.72 \text{ in.}^2 \text{ from Table A.6})$$

Development Length

From Table 7.1 in Chapter 7

$$\psi_t = \psi_e = \psi_s = \lambda = 1.0$$

c_b = side cover = 3.50 in. ← controls

c_b = one-half of center-to-center spacing of bars $= \frac{1}{2} \times 10 = 5$ in.

Letting $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{3.5 \text{ in.} + 0 \text{ in.}}{0.875 \text{ in.}} = 4.0 > 2.5 \qquad \therefore \text{ Use } 2.5$$

$$\frac{\ell_d}{d_b} = \frac{3}{40} \frac{f_y}{\lambda \sqrt{f'_c}} \frac{\psi_t \psi_e \psi_s}{\dfrac{c + K_{tr}}{d_b}}$$

$$= \left(\frac{3}{40}\right)\left(\frac{60{,}000 \text{ psi}}{(1)\sqrt{3000} \text{ psi}}\right)\frac{(1.0)(1.0)(1.0)}{2.5} = 32.86 \text{ diameters}$$

$$\frac{\ell_d}{d_b}\frac{A_{s\,\text{reqd}}}{A_{s\,\text{furn}}} = (32.86)\left(\frac{0.68 \text{ in.}^2}{0.72 \text{ in.}^2}\right) = 31.03 \text{ diameters}$$

$$\ell_d = (31.03)(0.875 \text{ in.}) = 27.15 \text{ in.} \qquad \text{Say 28 in.}$$

$$\begin{pmatrix}\text{Available development length}\\ \text{assuming bars are cut off}\\ \text{3 in. from edge of footing}\end{pmatrix} = \frac{10 \text{ ft}}{2} - 6 \text{ in.} - 3 \text{ in.}$$

$$= \begin{pmatrix}\text{4 ft 3 in. from face of}\\ \text{wall at section of}\\ \text{maximum moment}\end{pmatrix} > 28 \text{ in.} \qquad \text{OK}$$

(The bars should be extended to a point not less than 3 in. or more than 6 in. from the edge of the footing.)

Longitudinal Temperature and Shrinkage Steel (Perpendicular to the #7 Bars)

$$A_s = (0.0018)(12 \text{ in.})(20 \text{ in.}) = 0.432 \text{ in.}^2 \text{ (from Appendix A, Table A.6)} \qquad \text{Use \#5 bars @ 8 in.}$$

12.6 Design of Square Isolated Footings

Single-column footings usually provide the most economical column foundations. Such footings are generally square in plan, but they can just as well be rectangular or even circular or octagonal. Rectangular footings are used where such shapes are dictated by the available space or where the cross sections of the columns are very pronounced rectangles.

Most footings consist of slabs of constant thickness, such as the one shown in Figure 12.7(a), but if calculated thicknesses are greater than 3 ft or 4 ft, it may be economical to use stepped footings, as illustrated in Figure 12.7(b). The shears and moments in a footing are obviously larger near the column, with the result that greater depths are required in that area as compared to the outer parts of the footing. For very large footings, such as those used for bridge piers, stepped footings can give appreciable savings in concrete quantities.

Occasionally, sloped footings, shown in Figure 12.7(c), are used instead of stepped ones, but labor costs can be a problem. Whether stepped or sloped, it is considered necessary to place the concrete for the entire footing in a single pour to ensure the construction of a monolithic structure, thus avoiding horizontal shearing weakness. If this procedure is not followed, it is desirable to use keys or shear friction reinforcement between the parts to ensure monolithic action. In addition, when sloped or stepped footings are used, it is necessary to check stresses at more than one section in the footing. For example, steel area and development length requirements should be checked at steps as well as at the faces of walls or columns.

Before a column footing can be designed, it is necessary to make a few comments regarding shears and moments. This is done in the paragraphs to follow, while a related subject, load transfer from columns to footings, is discussed in Section 12.8.

Rhodes Annex–Clemson University, 2008.

Single-column footing prior to placement of column reinforcing bars.

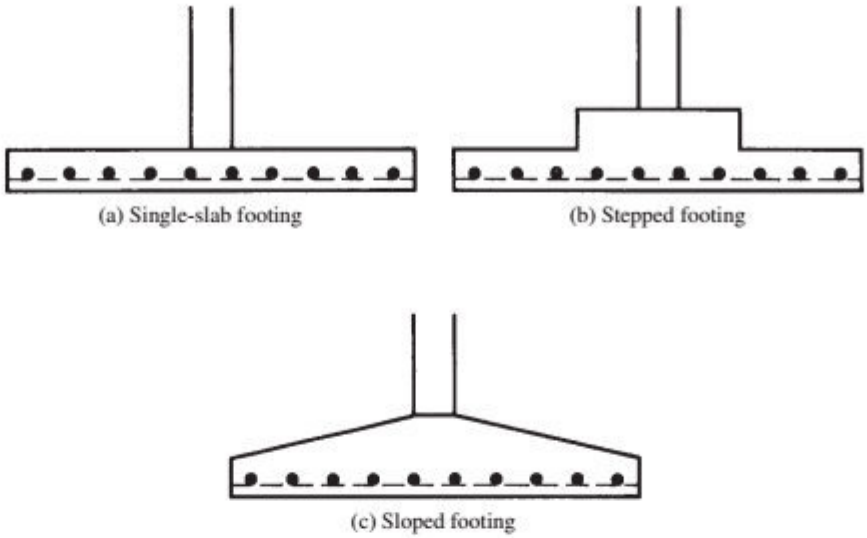


FIGURE 12.7 Shapes of isolated footings.

Shears

Two shear conditions must be considered in column footings, regardless of their shapes. The first of these is *one-way* or *beam shear*, which is the same as that considered in wall footings in the preceding section. For this discussion, reference is made to the footing of Figure 12.8. The total shear (V_{u1}) to be taken along section 1–1 equals the net soil pressure, q_u, times the hatched area outside the section. In the expression to follow, b_w is the whole width of the footing. The maximum value of V_{u1} if stirrups are not used equals ϕv_c, which is $\phi 2\sqrt{f'_c}b_w d$, and the maximum depth required is as follows:

$$d = \frac{V_{u1}}{\phi 2\sqrt{f'_c}b_w}$$

The second shear condition is *two-way* or *punching shear* (see Figure 12.9). The compression load from the column tends to spread out into the footing, opposing diagonal tension in that area, with the result that a square column tends to punch out a piece of the slab, which has the shape of a truncated pyramid. Section 22.6.4.1 of the ACI Code states that the critical section for two-way shear is located at a distance $d/2$ from the face of the column.

The shear force, V_{u2}, consists of all the net upward pressure, q_u, on the hatched area shown, that is, on the area outside the part tending to punch out. In the expressions to follow, b_o is the perimeter around the punching area, equal to $4(a + d)$ for the square column shown in Figure 12.9.

The shear stress for two-way shear is calculated from the equation:

$$v_u = \frac{V_{u2}}{b_0 d} \qquad \text{(Equation 12.1)}$$

Equation 12.1 does not appear in ACI 318-14, but is implied from ACI Code, Section 22.6.1.4.

The nominal two-way shear strength of the concrete, v_c, is specified as the smallest value obtained by substituting into the applicable equations that follow.

The first expression is the usual punching shear strength

$$v_c = 4\lambda\sqrt{f'_c} \qquad \text{(ACI Equation 22.6.5.2a)}$$

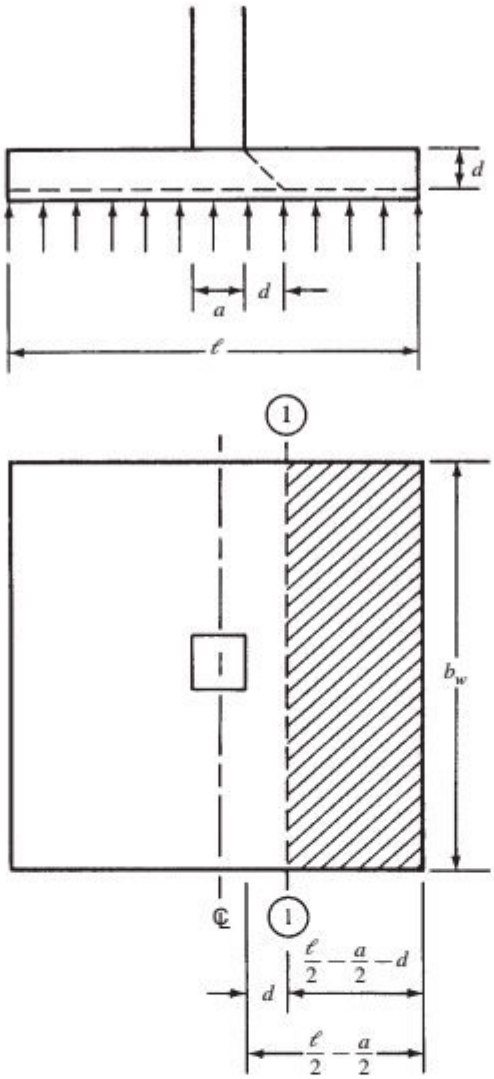


FIGURE 12.8 One-way or beam shear.

Tests have shown that when rectangular footing slabs are subjected to bending in two directions and when the long side of the loaded area is more than two times the length of the short side, the shear strength $v_c = 4\lambda\sqrt{f'_c}$ may be much too high. In the expression to follow, β_c is the ratio of the long side of the column to the short side of the column, concentrated load, or reaction area.

$$v_c = \left(2 + \frac{4}{\beta_c}\right)\lambda\sqrt{f'_c} \qquad \text{(ACI Equation 22.6.5.2b)}$$

The shear stress in a footing increases as the ratio b_o/d decreases. To account for this fact, ACI Equation 22.6.5.2c was developed. The equation includes a term, α_s, that is used to account for variations in the ratio. In applying the equation, α_s is to be used as 40 for interior columns (where the perimeter is 4-sided), 30 for edge columns (where the perimeter is 3-sided), and 20 for corner columns (where the perimeter is 2-sided).

$$v_c = \left(\frac{\alpha_s d}{b_o} + 2\right)\lambda\sqrt{f'_c} \qquad \text{(ACI Equation 22.6.5.2c)}$$

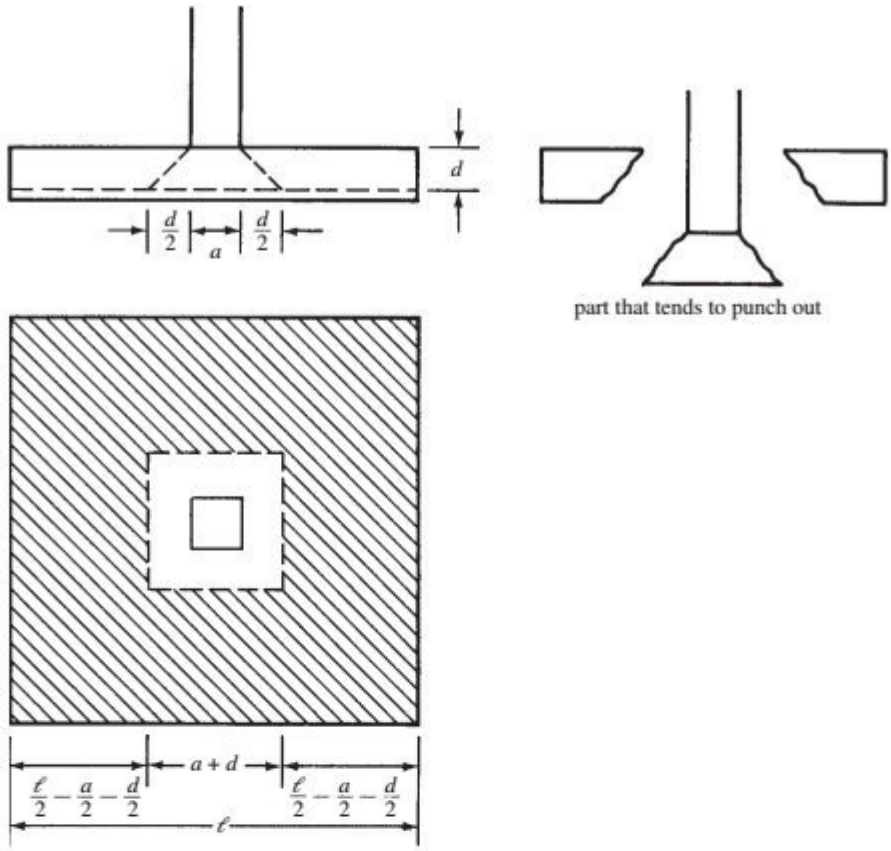


FIGURE 12.9 Two-way or punching shear.

The d required for two-way shear is the largest value obtained from the following expressions:

$$d = \frac{V_{u2}}{\phi 4\lambda\sqrt{f'_c}b_o} \qquad \text{(Equation 12.2)}$$

$$d = \frac{V_{u2}}{\phi\left(2 + \frac{4}{\beta_c}\right)\lambda\sqrt{f'_c}b_o} \quad \text{Equation 12.3 is not applicable unless } \beta_c > 2. \qquad \text{(Equation 12.3)}$$

$$d = \frac{V_{u2}}{\phi\left(\frac{\alpha_s d}{b_o} + 2\right)\lambda\sqrt{f'_c}b_o} \quad \text{Equation 12.4 is not applicable unless } b_o > \alpha_s d \qquad \text{(Equation 12.4)}$$

Or in SI units, with f'_c in MPa and b_o and d in mm,

$$d = \frac{6V_{u2}}{\phi\lambda\sqrt{f'_c}b_o}$$

$$d = \frac{6V_{u2}}{\phi\left(1 + \frac{8}{\beta_c}\right)\lambda\sqrt{f'_c}b_o}$$

$$d = \frac{12V_{u2}}{\phi\left(\frac{\alpha_s d}{b_o} + 2\right)\lambda\sqrt{f'_c}b_o}$$

Moments

The bending moment in a square reinforced concrete footing with a square column is the same about both axes because of symmetry. If the column is not square, the moment will be larger in the direction of the shorter column dimension. It should be noted, however, that the effective depth of the footing cannot be the same in the two directions because the bars in one direction rest on top of the bars in the other direction. The effective depth used for calculations might be the average for the two directions or, more conservatively, the value for the bars on top. This lesser value is used for the examples in this text. Although the result is some excess of steel in one direction, it is felt that the steel in either direction must be sufficient to resist the moment in that direction. It should be clearly understood that having an excess of steel in one direction will not make up for a shortage in the other direction at a 90° angle.

The critical section for bending is taken at the face of a reinforced concrete column or halfway between the middle and edge of a masonry wall or at a distance halfway from the edge of the base plate and the face of the column if structural steel columns are used (ACI Code, Section 13.2.7.1).

The determination of footing depths by the procedure described here will often require several cycles of a trial-and-error procedure. There are, however, many tables and handbooks available with which footing depths can be accurately estimated. One of these is the previously mentioned *CRSI Design Handbook*. In addition, there are many rules of thumb used by designers for making initial thickness estimates, such as 20% of the footing width or the column diameter plus 3 in. Computer programs, such as the spreadsheet provided for this chapter, easily handle this problem.

The reinforcing steel area calculated for footings will often be appreciably less than the minimum values $(200b_w d/f_y)$ and $(3\sqrt{f'_c}\,b_w d/f_y)$ specified for flexural members in ACI Code, Section 9.6.1.2. In Section 7.6.1.1, however, the code states that in slabs of uniform thickness, the minimum area and maximum spacing of reinforcing bars in the direction of bending need only be equal to those required for shrinkage and temperature reinforcement. The maximum spacing (Section 7.7.2.3) of this reinforcement may not exceed the lesser of three times the footing thickness, or 18 in. Many designers feel that the combination of high shears and low ρ values that often occurs in footings is not a good situation. Because of this, they specify steel areas at least as large as the flexural minimums of ACI Code, Section 9.6.1.2. This is the practice we also follow herein.

Example 12.2 illustrates the design of an isolated column footing.

Example 12.2

Design a square column footing for a 16-in. square tied interior column that supports a dead load $P_D = 200$ k and a live load $P_L = 160$ k. The column is reinforced with eight #8 bars, the base of the footing is 5 ft below grade, the soil weight is 100 lb/ft^3, $f_y = 60{,}000$ psi, $f'_c = 3000$ psi, and $q_a = 5000$ psf.

SOLUTION

After Two Previous Trials Assume 24-in. Footing (d = 19.5 in. Estimated to c.g. of *Top Layer* of Flexural Steel)

$$q_e = 5000 \text{ psf} - \left(\frac{24 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{36 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ psf}) = 4400 \text{ psf}$$

$$\text{Area required} = \frac{200 \text{ k} + 160 \text{ k}}{4.400 \text{ ksf}} = 81.82 \text{ ft}^2$$

Use 9-ft-0-in. × 9-ft-0-in. footing = 81.0 ft^2

Bearing Pressure for Strength Design

$$q_u = \frac{(1.2)(200 \text{ k}) + (1.6)(160 \text{ k})}{81.0 \text{ ft}^2} = 6.12 \text{ ksf}$$

Depth Required for Two-Way or Punching Shear (Figure 12.10)

$$b_o = (4)\,(35.5 \text{ in.}) = 142 \text{ in.}$$

$$V_{u2} = (81.0 \text{ ft}^2 - 2.96 \text{ ft}^2)\,(6.12 \text{ ksf}) = 442.09 \text{ k}$$

From Equation 12.2

$$d = \frac{442{,}090 \text{ lb}}{(0.75)\,(4\sqrt{3000} \text{ psi})\,(142 \text{ in.})} = 18.95 \text{ in.} < 19.5 \text{ in.} \qquad \underline{\text{OK}}$$

From Equation 12.3

$$d = \frac{442{,}090 \text{ lb}}{0.75\left(\dfrac{40 \text{ in.} \times 19.5 \text{ in.}}{142 \text{ in.}} + 2\right)\sqrt{3000} \text{ psi}\,(142 \text{ in.})} = 10.12 \text{ in.} < 19.5 \text{ in.} \qquad \underline{\text{OK}}$$

From the limitations stated for Equation 12.4, it does not apply since b_o is less than $\alpha_s d$ [142 in. < (40) (19.5 in.) = 780 in.]

Because both values of d are less than the assumed value of 19.5 in., punching shear is OK.

Depth Required for One-Way Shear (Figure 12.11)

$$V_{u1} = (9.00 \text{ ft})\,(2.208 \text{ ft})\,(6.12 \text{ ksf}) = 121.62 \text{ k}$$

$$d = \frac{121{,}620 \text{ lb}}{(0.75)\,(2\sqrt{3000} \text{ psi})\,(108 \text{ in.})} = 13.71 \text{ in.} < 19.5 \text{ in.} \qquad \underline{\text{OK}}$$

Use 24 in. total depth

$$M_u = (3.83 \text{ ft})\,(9.00 \text{ ft})\,(6.12 \text{ ksf})\left(\frac{3.83 \text{ ft}}{2}\right) = 404 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(404{,}000 \text{ ft-lb})}{(0.9)\,(108 \text{ in.})\,(19.5 \text{ in.})^2} = 131.2 \text{ psi} \qquad \therefore \rho = 0.00225 < \rho_{min} \text{ for flexure}$$

$$\text{Use } \rho = \text{larger of } \frac{200}{60{,}000 \text{ psi}} = 0.0033 \leftarrow \text{controls}$$

$$\text{or } \frac{3\sqrt{3000} \text{ psi}}{60{,}000 \text{ psi}} = 0.00274$$

$$A_s = (0.0033)\,(108 \text{ in.})\,(19.5 \text{ in.}) = 6.95 \text{ in.}^2 \text{ over the entire 108 in. width.}$$

Use 9 #8 bars in both directions (7.07 in.^{2})

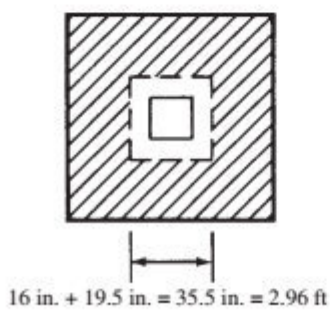


FIGURE 12.10 Two-way shear for Example 12.2.

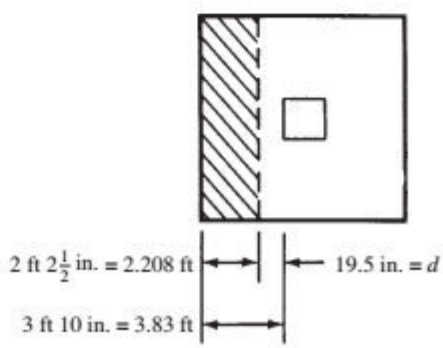


FIGURE 12.11 One-way shear for Example 12.2.

Development Length

From Table 6.1 in Chapter 6

$$\psi_t = \psi_e = \psi_s = \lambda = 1.0$$

Assuming bars spaced 12 in. on center

leaving 6 in. on each side

c_b = bottom cover = 3.5 in.

c_b = one-half of center-to-center spacing of bars = $\left(\frac{1}{2}\right)$ (12 in.) = 6 in.

Letting $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{3.5 \text{ in.} + 0 \text{ in.}}{1.0 \text{ in.}} = 3.5 > 2.5 \qquad \therefore \text{Use } 2.5$$

$$\frac{\ell_d}{d_b} = \frac{3}{40} \frac{f_y}{\lambda \sqrt{f'_c}} \frac{\psi_t \psi_e \psi_s}{\frac{c + K_{tr}}{d_b}}$$

$$= \frac{3}{40} \frac{60{,}000 \text{ psi}}{(1.0)\sqrt{3000} \text{ psi}} \frac{(1.0)\,(1.0)\,(1.0)}{2.5}$$

$$= 32.86 \text{ diameters}$$

$$\frac{\ell_d}{d_b} \frac{A_{s\,\text{reqd}}}{A_{s\,\text{furn}}} = (32.86)\left(\frac{6.95 \text{ in.}^2}{7.07 \text{ in.}^2}\right) = 32.30 \text{ diameters}$$

$$\ell_d = (32.30)\,(1.00 \text{ in.}) = 32.30 \text{ in.} \qquad \text{Say } 33 \text{ in.}$$

$$< \text{available } \ell_d = 4 \text{ ft } 6 \text{ in.} - \frac{16 \text{ in.}}{2} - 3 \text{ in.}$$

$$= 43 \text{ in.} \qquad \text{OK}$$

12.7 Footings Supporting Round or Regular Polygon-Shaped Columns

Sometimes footings are designed to support round columns or regular polygon-shaped columns. If such is the case, Section 13.2.7.3 of the ACI Code states that the column may be replaced with a square member that has the same area as the round or polygonal one. The equivalent square is then used for locating the critical sections for moment, shear, and development length.

12.8 Load Transfer from Columns to Footings

All forces acting at the base of a column must be satisfactorily transferred into the footing. Compressive forces can be transmitted directly by bearing, whereas uplift or tensile forces must be transferred to the supporting footing or pedestal by means of developed reinforcing bars or by mechanical connectors (which are often used in precast concrete).

Courtesy of EFCO Corp.

Pier bases for bridge from Prince Edward Island to mainland New Brunswick and Nova Scotia, Canada.

A column transfers its load directly to the supporting footing over an area equal to the cross-sectional area of the column. The footing surrounding this contact area, however, supplies appreciable lateral support to the directly loaded part, with the result that the loaded concrete in the footing can support more load. Thus, for the same grade of concrete, the footing can carry a larger bearing load than can the base of the column.

In checking the strength of the lower part of the column, only the concrete is counted. The column-reinforcing bars at that point cannot be counted because they are not developed unless dowels are provided or unless the bars themselves are extended into the footing.

At the base of the column, the permitted bearing strength is $\phi\ (0.85f'_cA_1)$ (where ϕ is 0.65), but it may be multiplied by $\sqrt{A_2/A_1} \leq 2$ for bearing on the footing (ACI Code, Section 22.8.3.2). In these expressions, A_1 is the column area, and A_2 is the area of the portion of the supporting footing that is geometrically similar and concentric with the columns. (See Figure 12.12.)

If the computed bearing force is higher than the smaller of the two allowable values in the column or the footing, it will be necessary to carry the excess with dowels or with column bars extended into the footing. Instead of using dowels, it is also possible to increase the size of the column or increase f'_c. Should the computed bearing force be less than the allowable value, no dowels or extended reinforcing

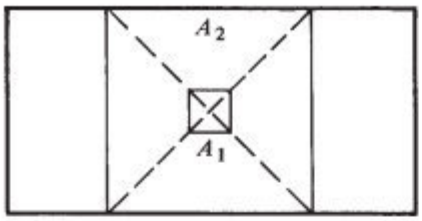


FIGURE 12.12 Bearing strength area modification.

are theoretically needed, but Section 16.3.4.1 of the ACI Code states that there must be a minimum area of dowels furnished equal to not less than 0.005 times the gross cross-sectional area of the column or pedestal.

The development length of the bars must be sufficient to transfer the compression to the supporting member, as per the ACI Code, Section 25.4.9. In no case may the area of the designed reinforcement or dowels be less than the area specified for the case where the allowable bearing force was not exceeded. As a practical matter in placing dowels, it should be noted that regardless of how small a distance they theoretically need to be extended down into the footing, they are usually bent at their ends and set on the main footing reinforcing bars, as shown in Figure 12.13. There the dowels can be tied firmly in place and not be in danger of being pushed through the footing during construction, as might easily happen otherwise. The bent part of the bar does not count as part of the compression development length (ACI Code, Section 25.4.1.2). The reader should again note that the bar details shown in this figure are not satisfactory for seismic areas because the bars should be bent inward and not outward.

An alternative to the procedure described in the preceding paragraph is to place the footing concrete without dowels and then to push straight dowels down into the concrete while it is still in a plastic state. This practice is permitted by the ACI Code in its Section 26.9.2e and is especially useful for plain concrete footings (to be discussed in Section 12.14 of this chapter). It is essential that the dowels be maintained in their correct position as long as the concrete is plastic. Before the licensed design professional approves the use of straight dowels as described here, he or she must be satisfied that the dowels will be properly placed and the concrete is satisfactorily compacted around them.

The ACI Code normally does not permit the use of lapped splices for #14 and #18 compression bars because tests have shown that welded splices or other types of connections are necessary. Nevertheless, based on years of successful use, Section 16.3.5.4 of the ACI Code states that #14 and #18 bars may be lap spliced with dowels (no larger than #11) to provide for force transfer at the base of columns or walls or pedestals. These dowels must extend into the supported member a distance of not less than ℓ_{dc} of #14 or #18 bars or the compression lap splice length of the dowels, whichever is greater, and into the footing a distance not less than ℓ_{dc} of the dowels.

If the computed development length of dowels is greater than the distance available from the top of the footing down to the top of the tensile steel, three possible solutions are available. One or more of the following alternatives may be selected:

1. A larger number of smaller dowels may be used. The smaller diameters will result in smaller development lengths.
2. A deeper footing may be used.
3. A cap or pedestal may be constructed on top of the footing to provide the extra development length needed.

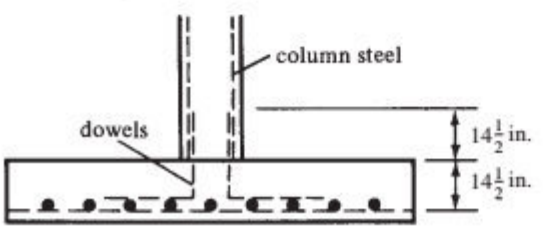


FIGURE 12.13 Dowel configuration.

Should bending moments or uplift forces have to be transferred to a footing such that the dowels would be in tension, the development lengths must satisfy the requirements for tension bars. For tension development length into the footing, a hook at the bottom of the dowel may be considered effective.

If there is moment or uplift, it will be necessary for the designer to conform to the splice requirements of Section 10.7.5.2 of the ACI Code in determining the distance the dowels must be extended up into the wall or column.

Examples 12.3 and 12.4 provide brief examples of column-to-footing load transfer calculations for vertical forces only. Consideration is given to lateral forces and moments in Sections 12.12 and 12.13 of this chapter.

Example 12.3

Design for load transfer from a 16-in. × 16-in. column to a 9-ft-0-in. × 9-ft-0-in. footing if $P_D = 200$ k, $P_L = 160$ k, $f'_c = 3000$ psi for the footing and 4000 psi for the column, and $f_y = 60{,}000$ psi. The footing concrete is normal weight, but the column is constructed with sand–lightweight concrete.

SOLUTION

Bearing force at base of column = (1.2) (200 k) + (1.6) (160 k) = 496 k

Allowable bearing force in concrete at base of column

$$= \phi(0.85f'_cA_1) = (0.65)\ (0.85)\ (4.0\text{ ksi})\ (16\text{ in.} \times 16\text{ in.})$$

$$= 566\text{ k} > 496\text{ k} \qquad \therefore \text{ column bearing is OK}$$

Allowable bearing force in footing concrete $= \phi(0.85f'_cA_1)\sqrt{A_2/A_1}$

$$\sqrt{\frac{A_2}{A_1}} = \sqrt{\frac{9\text{ ft} \times 9\text{ ft}}{1.33\text{ ft} \times 1.33\text{ ft}}} = 6.75 > 2.0$$

$$= (0.65)\ (0.85)\ (3.0)\ (16\text{ in.} \times 16\text{ in.})\ (\text{Use } 2)^{[1]} = 848.6\text{ k} > 496\text{ k} \quad \therefore \text{ Footing bearing is OK}$$

Minimum A_s for dowels = (0.005) (16 in. × 16 in.) = 1.28 in.^{2} Use 4 #6 bars (1.77 in.^{2})

Development Lengths of Dowels (ACI 25.4.9)

For the column, using $\lambda = 0.85$ for sand–lightweight concrete, conservatively taking $\psi_r = 1.0$ (ignoring confining effect of column ties) and $f'_c = 4000$ psi,

$$\ell_d = \frac{\psi_r d_b f_y}{(50)\lambda\sqrt{f'_c}} = \frac{(0.75\text{ in.})\ (60{,}000\text{ psi})}{(50)0.85\sqrt{4000}\text{ psi}} = 16.74\text{ in.}$$

For the footing, $\lambda = 1.0$, $\psi_r = 1.0$ (no confining steel) and $f'_c = 3000$ psi,

$$\ell_d = \frac{\psi_r d_b f_y}{(50)\lambda\sqrt{f'_c}} = \frac{(0.75\text{ in.})\ (60{,}000\text{ psi})}{(50)1.0\sqrt{3000}\text{ psi}} = 16.43\text{ in.}$$

[1] $\sqrt{A_2/A_1} < 2.0$. (ACI Code, Section 22.8.3.2)

In addition, the development must not be less than either

$$\ell_d = 0.0003 d_b f_y = 0.0003(0.75 \text{ in.})\,(60{,}000 \text{ psi}) = 13.50 \text{ in.} \quad \text{or}$$

$$\ell_d = 8.00 \text{ in.}$$

In summary, the dowels must extend upward into the column at least 16.74 in. and down into the footing at least 16.43 in. *Use four #6 dowels extending 17 in. up into the column and 17 in. down into the footing and set on top of the reinforcing mat, as shown in Figure* 12.13.

Example 12.4

Design for load transfer from a 14-in. × 14-in. column to a 13-ft-0-in. × 13-ft-0-in. footing with a P_u of 800 k, $f'_c = 3000$ psi in the footing and 5000 psi in the column, both normal weight, and $f_y = 60{,}000$ psi. The column has eight #8 bars.

SOLUTION

Bearing force at base of column = P_u = 800 k

= ϕ (allowable bearing force in concrete) + ϕ (strength of dowels)

Design bearing strength in concrete at base of column

$$= (0.65)\,(0.85)\,(5.0 \text{ ksi})\,(14 \text{ in.} \times 14 \text{ in.}) = 541.5 \text{ k} < 800 \text{ k} \qquad \underline{\text{No good}}$$

Design bearing strength on footing concrete

$$\sqrt{\frac{A_2}{A_1}} = \sqrt{\frac{(13 \text{ ft})\,(13 \text{ ft})}{(1.17 \text{ ft})\,(1.17 \text{ ft})}} = 11.1 > 2$$

$$= (0.65)\,(0.85)\,(3.0 \text{ ksi})\,(14 \text{ in.} \times 14 \text{ in.})\,(\text{Use } 2)^{[2]} = 649.7 \text{ k} < 800 \text{ k} \qquad \underline{\text{No good}}$$

Therefore, the dowels must be designed for excess load.

Excess load = 800 k – 541.5 k = 258.5 k

$$A_s \text{ of dowels} = \frac{258.5 \text{ k}}{(0.9)\,(60 \text{ ksi})} = \underline{4.79 \text{ in.}^2} \text{ or } (0.005)\,(14 \text{ in.})\,(14 \text{ in.}) = 0.98 \text{ in.}^2$$

$$\underline{\text{Use 8 \#7 bars } (4.80 \text{ in.}^2)}$$

Development Length of #7 Dowels into Column

$$\ell_d = \frac{(0.875 \text{ in.})\,(60{,}000 \text{ psi})}{(50)\,(1.0)\sqrt{5000} \text{ psi}} = 14.85 \text{ in.}$$

$$\ell_d = (0.0003)\,(60{,}000 \text{ psi})\,(0.875 \text{ in.}) = 15.75 \text{ in.} \leftarrow \text{controls}$$

$$\ell_d = 8 \text{ in.}$$

[2] $\sqrt{A_2/A_1} < 2.0$. (See Section 12.8 and ACI Code, Section 22.8.3.2)

Development Length of #7 Dowels into Footing (Different from Column Values because f'_c Values Are Different)

$$\ell_d = \frac{(0.875 \text{ in.})\,(60{,}000 \text{ psi})}{(50)\,(1.0)\sqrt{3000 \text{ psi}}} = 19.42 \text{ in.} \leftarrow \text{controls}$$

$$\ell_d = (0.0003)\,(0.875 \text{ in.})\,(60{,}000 \text{ psi}) = 15.75 \text{ in.}$$

Use eight #7 dowels extending 16 in. *up into the column and* 20 in. *down into the footing.*

12.9 Rectangular Isolated Footings

As previously mentioned, isolated footings may be rectangular in plan if the column has a very pronounced rectangular shape or if the space available for the footing forces the designer into using a rectangular shape. Should a square footing be feasible, it is normally more desirable than a rectangular one because it will require less material and will be simpler to construct.

The design procedure is almost identical with the one used for square footings. After the required area is calculated and the lateral dimensions are selected, the depths required for one-way and two-way shear are determined by the usual methods. One-way shear will very often control the depths for rectangular footings, whereas two-way shear normally controls the depths of square footings.

The next step is to select the reinforcement in the long direction. These longitudinal bars are spaced uniformly across the footing, but such is not the case for the short-span reinforcement. In Figure 12.14, it can be seen that the support provided by the footing to the column will be concentrated near the middle of the footing, and thus the moment in the short direction will be concentrated somewhat in the same area near the column.

As a result of this concentration effect, it seems only logical to concentrate a large proportion of the short-span reinforcement in this area. Section 13.3.3.3 of the ACI Code states that a certain minimum percentage of the total short-span reinforcement should be placed in a band width equal to the length of the shorter direction of the footing. The amount of reinforcement in this band is to be determined with the following expression, in which β is the ratio of the length of the long side to the width of the short side of the footing:

$$\frac{\text{Reinforcement in band width}}{\text{Total reinforcement in short direction}} = \frac{2}{\beta + 1} = \gamma_s \qquad \text{(ACI Equation 13.3.3.3)}$$

The remaining reinforcement in the short direction should be uniformly spaced over the ends of the footing, but the authors feel it should at least meet the shrinkage and temperature requirements of the ACI Code, Section 24.4.3.2.

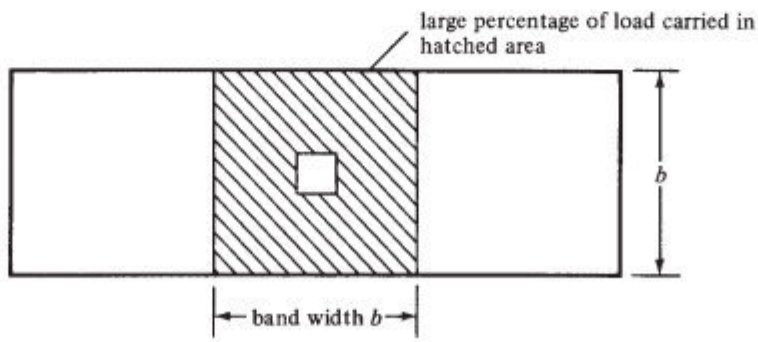


FIGURE 12.14 Band width for steel in the short direction for rectangular isolated footings.

Example 12.5 presents the partial design of a rectangular footing in which the depths for one- and two-way shears are determined and the reinforcement selected.

Example 12.5

Design a rectangular footing for an 18-in. square interior column with a dead load of 185 k and a live load of 150 k. Make the length of the long side equal to twice the width of the short side, $f_y = 60{,}000$ psi, $f'_c = 4000$ psi, normal weight, and $q_a = 4000$ psf. Assume the base of the footing is 5 ft 0 in. below grade.

SOLUTION

Assume 24-in. Footing (d = 19.5 in.)

$$q_e = 4000 \text{ psf} - \left(\frac{24 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{36 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 3400 \text{ psf}$$

$$\text{Area required} = \frac{185 \text{ k} + 150 \text{ k}}{3.4 \text{ ksf}} = 98.5 \text{ ft}^2 \qquad \underline{\text{Use 7 ft 0 in.} \times \text{14 ft 0 in.} = 98.0 \text{ ft}^2}$$

$$q_u = \frac{(1.2)(185 \text{ k}) + (1.6)(150 \text{ k})}{98.0 \text{ ft}^2} = 4.71 \text{ ksf}$$

Checking Depth for One-Way Shear (Figure 12.15)

$b = 7$ ft

$$V_{u1} = (7.0 \text{ ft})(4.625 \text{ ft})(4.71 \text{ ksf}) = 152.49 \text{ k}$$

$$d = \frac{152{,}490 \text{ lb}}{(0.75)(1.0)(2\sqrt{4000} \text{ psi})(84 \text{ in.})} = 19.14 \text{ in.}, h = d + 4.5 \text{ in.} = 23.64 \text{ in.} \qquad \underline{\text{Use 24 in.}}$$

Checking Depth for Two-Way Shear (Figure 12.16)

$$b_o = (4)(37.5 \text{ in.}) = 150 \text{ in.}$$

$$V_{u2} = [98.0 \text{ ft}^2 - (3.125 \text{ ft})^2](4.71 \text{ ksf}) = 415.58 \text{ k}$$

$$d = \frac{415{,}580 \text{ lb}}{(0.75)(1.0)(4\sqrt{4000} \text{ psi})(150 \text{ in.})} = 14.60 \text{ in.} < 19.5 \text{ in.} \qquad \underline{\text{OK}}$$

$$d = \frac{415{,}580 \text{ lb}}{0.75\left(\frac{40 \text{ in.} \times 19.5 \text{ in.}}{150 \text{ in.}} + 2\right)(\sqrt{4000} \text{ psi})(150 \text{ in.})} = 8.11 \text{ in.} < 19.5 \text{ in.} \qquad \underline{\text{OK}}$$

From the limitations stated for Equation 12.4, it does not apply since b_o is less than $\alpha_s d$ [142 in. < (40)(19.5) in. = 780 in.]

If either value of d in the last two equations had exceeded the assumed value of 19.5 in., it would have been necessary to increase the trial value of d and start over.

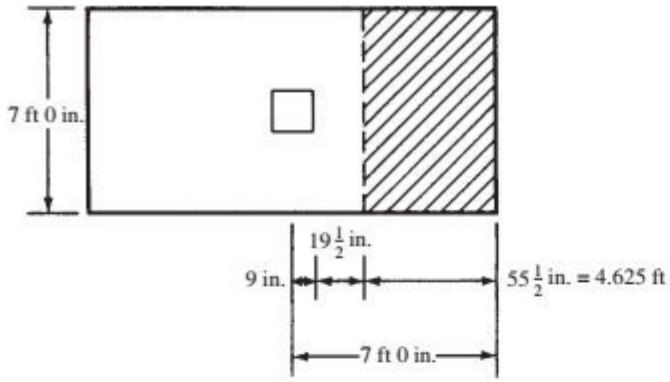


FIGURE 12.15 One-way shear area for Example 12.5.

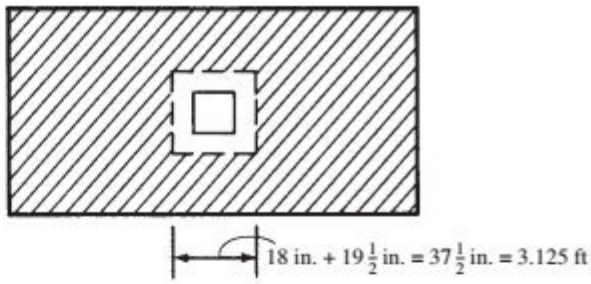


FIGURE 12.16 Two-way shear area for Example 12.5.

Design of Longitudinal Steel

$$\text{Lever arm} = \frac{14 \text{ ft}}{2} - \frac{9 \text{ in.}}{12 \text{ in./ft}} = 6.25 \text{ ft}$$

$$M_u = (6.25 \text{ ft})\,(7.0 \text{ ft})\,(4.71 \text{ ksf})\left(\frac{6.25 \text{ ft}}{2}\right) = 643.9 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(643{,}900 \text{ ft-lb})}{(0.90)\,(84 \text{ in.})\,(19.5 \text{ in.})^2} = 268.8 \text{ psi}$$

$$\rho = 0.00467 \text{ (from Appendix A, Table A.13)}$$

$$A_s = (0.00467)\,(84 \text{ in.})\,(19.5 \text{ in.}) = 7.65 \text{ in.}^2 \qquad \text{Use 10 \#8 bars } (7.85 \text{ in.}^2)$$

Design of Steel in Short Direction (Figure 12.17)

$$\text{Lever arm} = \frac{7 \text{ ft}}{2} - \frac{9 \text{ in.}}{12 \text{ in./ft}} = 2.75 \text{ ft}$$

$$M_u = (2.75 \text{ ft})\,(14.0 \text{ ft})\,(4.71 \text{ ksf})\left(\frac{2.75 \text{ ft}}{2}\right) = 249.3 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(249{,}300 \text{ ft-lb})}{(0.90)\,(168 \text{ in.})\,(19.5 \text{ in.})^2} = 52 \text{ psi}$$

$$\text{Use } \rho = \text{larger of } \frac{200}{60{,}000 \text{ psi}} = 0.0033 \leftarrow$$

$$\text{or } \frac{3\sqrt{4000} \text{ psi}}{60{,}000 \text{ psi}} = 0.00316$$

$$A_s = (0.0033)\,(168 \text{ in.})\,(19.5 \text{ in.}) = 10.81 \text{ in.}^2 \qquad \text{Use 18 \#7 bars } (10.82 \text{ in.}^2)$$

$$\frac{\text{Reinforcement in band width}}{\text{Total reinforcement in short direction}} = \frac{2}{2+1} = \frac{2}{3}$$

$$\text{Use } \frac{2}{3} \times 18 = 12 \text{ bars in band width}$$

Subsequent check of required development lengths is OK.

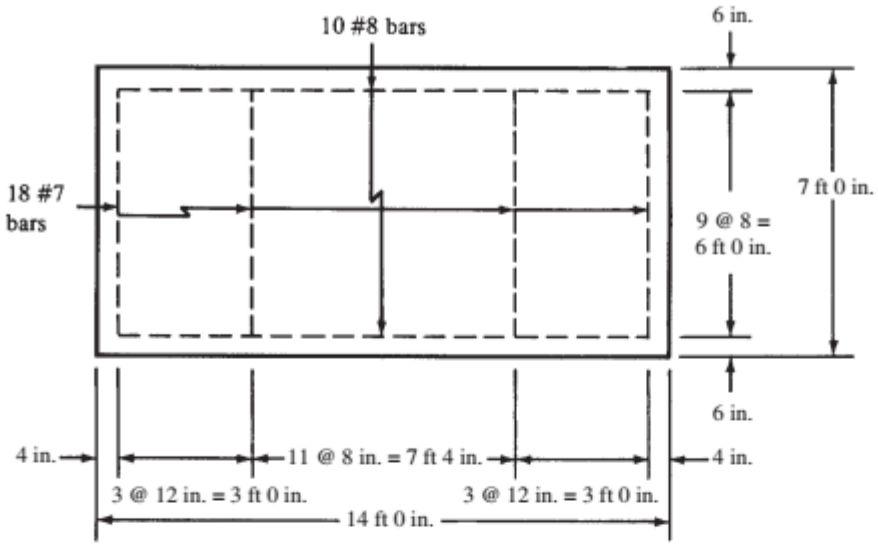


FIGURE 12.17 Two-way footing bar spacing diagram.

12.10 Combined Footings

Combined footings support more than one column. One situation in which they may be used is when the columns are so close together that isolated individual footings would run into each other [see Figure 12.18(a)]. Another frequent use of combined footings occurs where one column is very close to a property line, causing the usual isolated footing to extend across the line. For this situation, the footing for the exterior column may be combined with the one for an interior column, as shown in Figure 12.18(b).

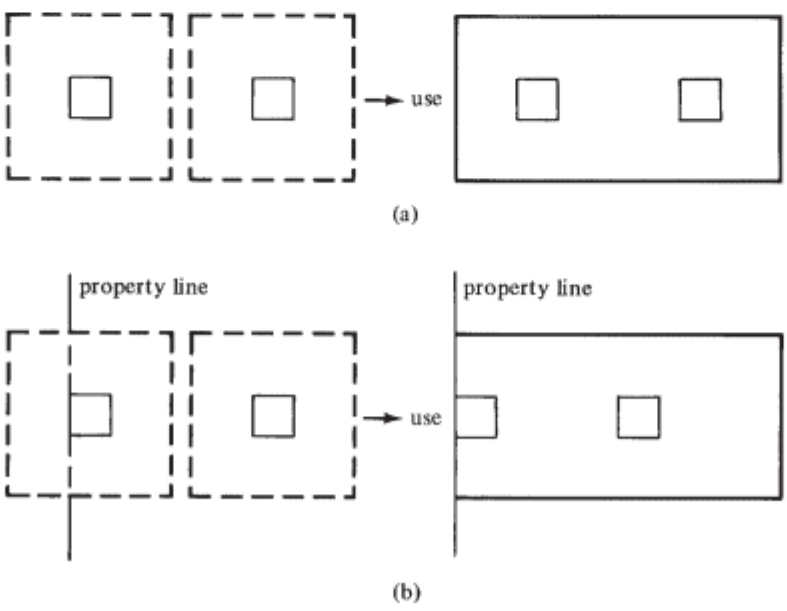


FIGURE 12.18 Use of combined footings.

Rhodes Annex–Clemson University, 2008.

Combined footing (two column) after footing concrete is placed showing column reinforcing bars ready for splicing.

On some occasions, where a column is close to a property line and where it is desired to combine its footing with that of an interior column, the interior column will be so far away as to make the idea impractical economically. For such a case, counterweights, or "deadmen," may be provided for the outside column to take care of the eccentric loading.

Because it is desirable to make bearing pressures uniform throughout the footing, the centroid of the footing should be made to coincide with the centroid of the column loads to attempt to prevent uneven settlements. This can be accomplished with combined footings that are rectangular in plan. Should the interior column load be greater than that of the exterior column, the footing may be so proportioned that its centroid will be in the correct position by extending the inward projection of the footing, as shown in the rectangular footing of Figure 12.18(b).

Other combined footing shapes that will enable the designer to make the centroids coincide are the trapezoid and strap or T footings shown in Figure 12.19. Footings with these shapes are usually economical when there are large differences between the magnitudes of the column loads or where the spaces available do not lend themselves to long rectangular footings. When trapezoidal footings are used, the longitudinal bars are usually arranged in a fan shape with alternate bars cut off some distance from the narrow end.

You probably realize that a problem arises in establishing the centroids of loads and footings when deciding whether to use service or factored loads. The required centroid of the footing will be slightly different for the two cases. The authors determine the footing areas and centroids with the service loads (ACI Code, Section 13.3.1.1), but the factored loads could be used with reasonable results, too. The important point is to be consistent throughout the entire problem.

The design of combined footings has not been standardized as have the procedures used for the previous problems worked in this chapter. For this reason, practicing reinforced concrete designers use slightly varying approaches. One of these methods is described in the paragraphs to follow.

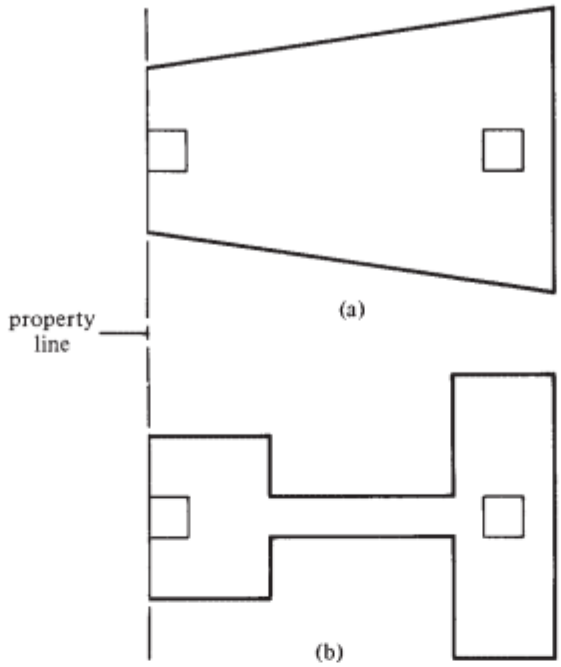


FIGURE 12.19 Trapezoidal and strap or T footings.

First, the required area of the footing is determined for the service loads, and the footing dimensions are selected so that the centroids coincide. The various loads are then multiplied by the appropriate load factors, and the shear and moment diagrams are drawn along the long side of the footing for these loads. After the shear and moment diagrams are prepared, the depth for one- and two-way shear is determined, and the reinforcing bars in the long direction is selected.

In the short direction, it is assumed that each column load is spread over a width in the long direction equal to the column width plus $d/2$ on each side if that much footing is available. Then the steel is designed, and a minimum amount of steel for temperature and shrinkage is provided in the remaining part of the footing.

The ACI Code does not specify an exact width for these transverse strips, and designers may make their own assumptions as to reasonable values. The width selected will probably have very little influence on the transverse bending capacity of the footing, but it can affect appreciably its punching or two-way shear resistance. If the flexural reinforcement is placed within the area considered for two-way shear, this lightly stressed reinforcement will reduce the width of the diagonal shear cracks and will also increase the aggregate interlock along the shear surfaces.

Space is not taken here to design completely a combined footing, but Example 12.6 is presented to show those parts of the design that are different from the previous examples of this chapter. A comment should be made about the moment diagram. If the length of the footing is not selected so that its centroid is located exactly at the centroid of the column loads, the moment diagrams will not close well at all because the numbers are very sensitive. Nevertheless, it is considered good practice to round off the footing lateral dimensions to the nearest 3 in. Another factor that keeps the moment diagram from closing is the fact that the average load factors of the various columns will be different if the column loads are different. We could improve the situation a little by taking the total column factored loads and dividing the result by the total working loads to get an average load factor. This value (which works out to be 1.375 in Example 12.6) could then be multiplied by the total working load at each column and used for drawing the shear and moment diagrams.

Example 12.6

Design a rectangular combined footing for the two columns shown in Figure 12.20. $q_a = 5$ ksf, $f'_c = 3000$ psi, normal weight, and $f_y = 60$ ksi. The bottom of the footing is to be 6 ft below grade.

SOLUTION

Assume 27-in. Footing (d = 22.5 in.)

$$q_e = 5000 \text{ psf} - \left(\frac{27 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{45 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 4287 \text{ psf}$$

$$\text{Area required} = \frac{(120 \text{ k} + 100 \text{ k}) + (200 \text{ k} + 150 \text{ k})}{4.287 \text{ ksf}} = 132.96 \text{ ft}^2$$

Locate Center of Gravity of Column Service Loads

$$x \text{ from c.g. of left column} = \frac{(200 \text{ k} + 150 \text{ k})\,(12 \text{ ft})}{(120 \text{ k} + 100 \text{ k}) + (200 \text{ k} + 150 \text{ k})} = 7.37 \text{ ft}$$

$$\text{Distance from property line to c.g.} = 0.75 \text{ ft} + 7.37 \text{ ft} = 8.12 \text{ ft}$$

$$\text{Length of footing} = (2 \times 8.12 \text{ ft}) = 16.24 \text{ ft, say } 16 \text{ ft } 3 \text{ in.}$$

$$\text{Required footing width} = \frac{\text{Area required}}{\text{length}} = \frac{132.96 \text{ ft}^2}{16.25 \text{ ft}} = 8.182 \text{ ft}$$

Use 16-ft-3-in. × 8-ft-3-in. footing ($A = 134$ ft^2).

$$q_u = \frac{(1.2)\,(320 \text{ k}) + (1.6)\,(250 \text{ k})}{134 \text{ ft}^2} = 5.85 \text{ ksf}$$

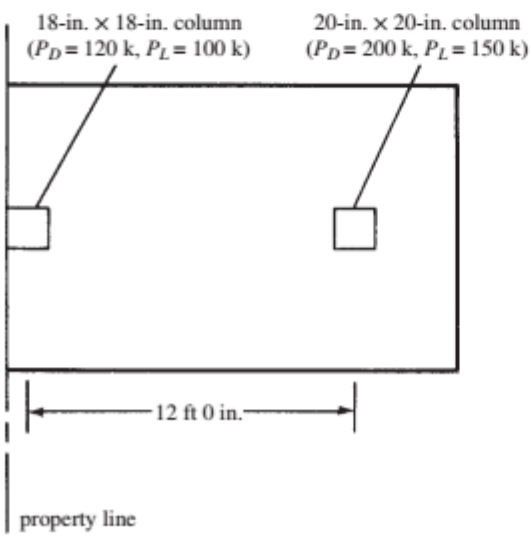


FIGURE 12.20 Plan view of combined footing in Example 12.6.

Depth Required for One-Way Shear

From the shear diagram in Figure 12.21, the largest shear force is 271.1 k at the left face of the right column. At a distance d to the left of this location, the value of shear is

$$V_{u1} = 271.1 \text{ k} - 48.26 \text{ klf}\left(\frac{22.5 \text{ in.}}{12 \text{ in./ft}}\right) = 180.61 \text{ k}$$

$$d = \frac{V_{u1}}{\phi 2\lambda\sqrt{f'_c}b} = \frac{180{,}610 \text{ lb}}{0.75\,(2)\,(1)\sqrt{3000} \text{ psi}\,(8.25 \text{ ft})\,(12 \text{ in./ft})} = 22.2 \text{ in.} < 22.5 \text{ in.} \quad \underline{\underline{\text{OK}}}$$

Depth Required for Two-Way Shear (Equations 12.3 and 12.4 Not Shown because They Do Not Control)

$$V_{u2} \text{ at right column} = 480 \text{ k} - \left(\frac{42.5 \text{ in.}}{12 \text{ in./ft}}\right)^2 (5.85 \text{ ksf}) = 406.6 \text{ k}$$

$$d = \frac{406{,}600 \text{ lb}}{(0.75)\,[(4)\,(1.0)\sqrt{3000} \text{ psi}]\,(4 \times 42.5 \text{ in.})}$$

$$= 14.56 \text{ in.} < 22.5 \text{ in.} \quad \underline{\underline{\text{OK}}}$$

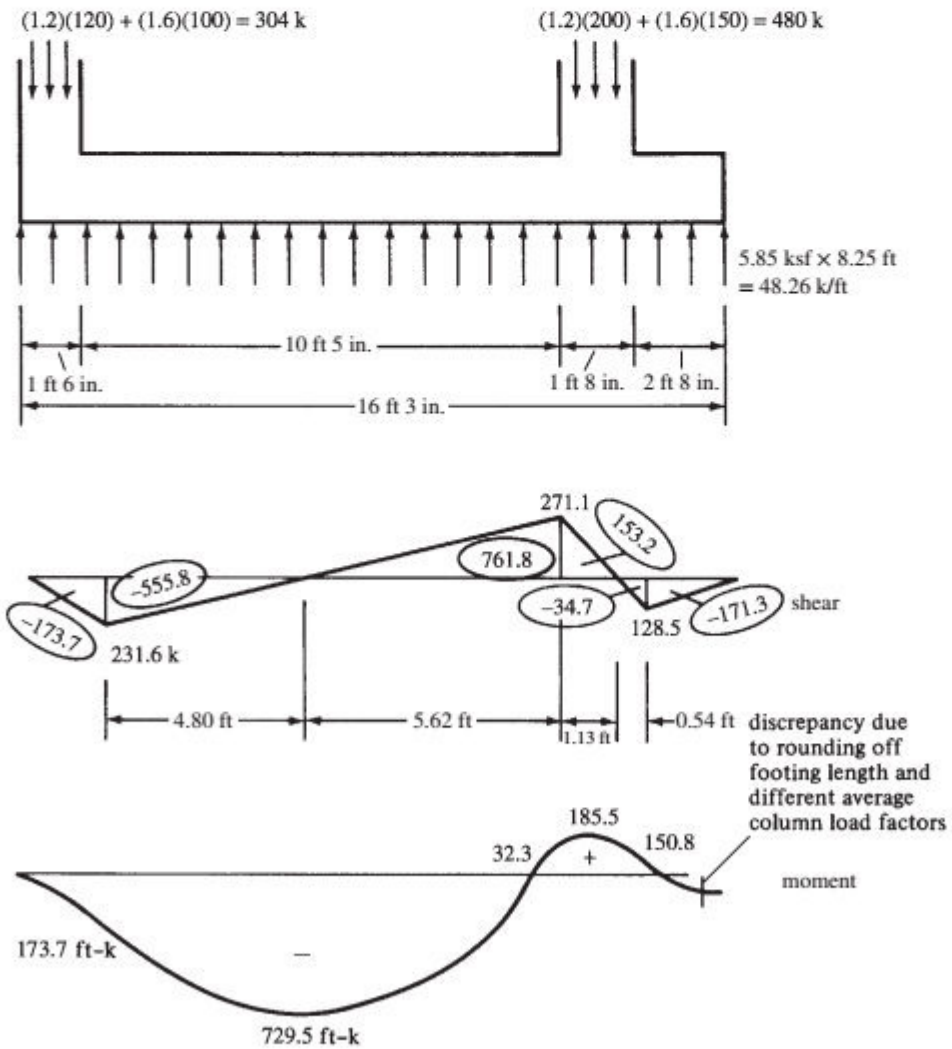


FIGURE 12.21 Shear and moment diagrams for combined footing in Example 12.6.

$$V_{u2} \text{ at left column} = 304 \text{ k} - \left(\frac{29.25 \text{ in.} \times 40.5 \text{ in.}}{144 \text{ in.}^2/\text{ft}^2}\right)(5.85 \text{ ksf}) = 255.9 \text{ k}$$

$$d = \frac{255{,}900 \text{ lb}}{(0.75)\,[(4)\,(1.0)\sqrt{3000} \text{ psi}]\,(2 \times 29.25 \text{ in.} + 40.5 \text{ in.})}$$

$$= 15.73 \text{ in.} < 22.5 \text{ in.} \quad \underline{\text{OK}}$$

For this footing, one-way shear is more critical than two-way shear. This is not unusual for combined footings.

Design of Longitudinal Steel

$$M_u = -729.5 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(729{,}500 \text{ ft-lb})}{(0.90)\,(99 \text{ in.})\,(22.5 \text{ in.})^2} = 194.1 \text{ psi}$$

$$\rho = 0.00337 \text{ (from Appendix A, Table A.12 by interpolation)}$$

$A_s = (0.00337)\,(99 \text{ in.})\,(22.5 \text{ in.}) = 7.51 \text{ in.}^2$ Say 10 #8 bars (7.85 in.^{2})

$+M_u = +171.3$ ft-k (computed from right end of shear diagram, Figure 12.21)

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(171{,}300 \text{ ft-lb})}{(0.90)\,(99 \text{ in.})\,(22.5 \text{ in.})^2} = 45.6 \text{ psi}$$

$$\text{use } \rho = \rho_{\min}$$

Use larger of $\frac{200}{f_y} = 0.00333$ or $\frac{3\sqrt{3000} \text{ psi}}{60{,}000 \text{ psi}} = 0.00274$

$A_s = (0.00333)\,(99 \text{ in.})\,(22.5 \text{ in.}) = 7.42 \text{ in.}^2$ Use 8 #9 bars (8.00 in.^{2})

Design of Short-Span Steel under Interior Column (Figure 12.22)

$$\text{Assuming steel spread over width} = \text{column width} + (2)\left(\frac{d}{2}\right)$$

$$= 20 \text{ in.} + (2)\left(\frac{22.5 \text{ in.}}{2}\right) = 42.5 \text{ in.}$$

Referring to Figure 12.22 and calculating M_u:

$$q_u = \frac{480 \text{ k}}{8.25 \text{ ft}} = 58.18 \text{ k/ft}$$

$$M_u = (3.29 \text{ ft})\,(58.18 \text{ k/ft})\left(\frac{3.29 \text{ ft}}{2}\right) = 314.9 \text{ ft-k}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(314{,}900 \text{ ft-lb})}{(0.90)\,(42.5 \text{ in.})\,(22.5 \text{ in.})^2} = 195.1 \text{ psi}$$

$$p = 0.00339 \text{ (from Appendix A, Table A.12)}$$

$A_s = (0.00339)\,(4.25 \text{ in.})\,(22.5 \text{ in.}) = 3.24 \text{ in.}^2$ Use 6 #7 bars (3.61 in.^{2})

Development lengths (not shown) check out OK.

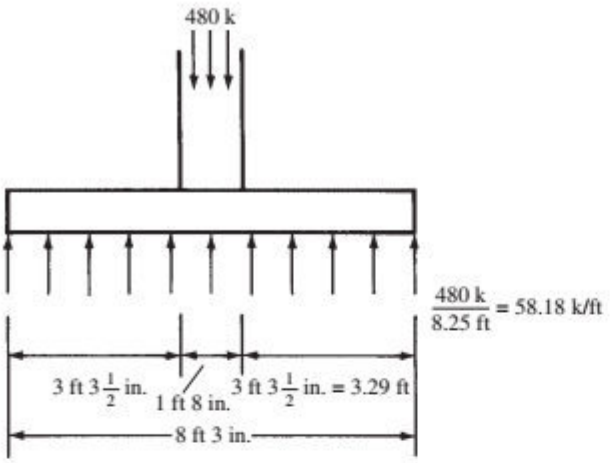


FIGURE 12.22 Soil stress used in determining short-span steel.

A similar procedure is used under the exterior column where the steel is spread over a width equal to 18 in. plus $d/2$, and not 18 in. plus $2(d/2)$, because sufficient room is not available on the property-line side of the column.

12.11 Footing Design for Equal Settlements

If three men are walking along a road carrying a log on their shoulders (a statically indeterminate situation) and one of them decides to lower his shoulder by 1 in., the result will be a drastic effect on the load supported by the other men. In the same way, if the footings of a building should settle by different amounts, the shears and moments throughout the structure will be greatly changed. In addition, there will be detrimental effects on the fitting of doors, windows, and partitions. Should all the footings settle by the same amount, however, these adverse effects will not occur. Thus, equal settlement is the goal of the designer.

The footings considered in preceding sections have had their areas selected by taking the total dead plus live loads and dividing the sum by the allowable soil pressure. It would seem that if such a procedure were followed for all the footings of an entire structure, the result would be uniform settlements throughout—*but geotechnical engineers have clearly shown that this assumption may be very much in error*.

A better way to handle the problem is to attempt to design the footings so that the *usual loads* on each footing will cause approximately the same pressures. The usual loads consist of the dead loads plus the average percentage of live loads normally present. The usual percentage of live loads present varies from building to building. For a church, it might be almost zero, perhaps 25% to 30% for an office building, and maybe 75% or more for some warehouses or libraries. Furthermore, the percentage in one part of a building may be entirely different from that in some other part (offices, storage, etc.).

One way to handle the problem is to design the footing that has the highest ratio of live to dead load, compute the usual soil pressure under that footing using dead load plus the estimated average percentage of live load, and then determine the areas required for the other footings so their usual soil pressures are all the same. It should be remembered that the dead load plus 100% of the live load must not cause a pressure greater than the allowable soil pressure under any of the footings.

A student of soil mechanics will realize that this method of determining usual pressures, although not a bad design procedure, will not ensure equal settlements. This approach at best will only lessen the amounts of differential settlements. The student will remember first that large footings tend to settle more than small footings, even though their soil pressures are the same, because the large

footings exert compression on a larger and deeper mass of soil. There are other items that can cause differential settlements. Different types of soils may be present at different parts of the building; part of the area may be in fill and part in cut: there may be mutual influence of one footing on another; and so forth.

Example 12.7 illustrates the usual load procedure for a group of five isolated footings.

Example 12.7

Determine the footing areas required for the loads given in Table 12.2 so that the usual soil pressures will be equal. Assume that the usual live load percentage is 30% for all the footings, $q_e = 4$ ksf.

SOLUTION

The largest percentage of live load to dead load occurs for footing *D*.

$$\text{Area required for footing } D = \frac{100\text{ k} + 150\text{ k}}{4\text{ ksf}} = 62.5\text{ ft}^2$$

$$\text{Usual soil pressure under footing } D = \frac{100\text{ k} + (0.30)\,(150\text{ k})}{62.5\text{ ft}^2} = 2.32\text{ ksf}$$

Computing the areas required for the other footings and determining their soil pressures under total service loads, we show the results in Table 12.3. Note from the last column in Table 12.3 that footing *D* is the only footing that will be stressed to its allowable bearing stress.

TABLE 12.2 Footings

Footing	Dead Load (k)	Live Load (k)
A	150	200
B	120	100
C	140	150
D	100	150
E	160	200

TABLE 12.3 Areas and Soil Pressures

Footing	Usual Load = $D + 0.30L$ (k)	Area Required = Usual Load ÷ 2.32 ksf (ft^2)	Total Soil Pressure (ksf)
A	210	90.5	3.87
B	150	64.7	3.40
C	185	79.7	3.64
D	145	62.5	4.00
E	220	94.8	3.80

12.12 Footings Subjected to Axial Loads and Moments

Walls or columns often transfer moments as well as vertical loads to their footings. These moments may be the result of gravity loads or lateral loads. Such a situation is represented by the vertical load P and the bending moment M shown in Figure 12.23.

Moment transfer from columns to footings depends on how the column–footing connection is constructed. Many designers treat the connection between columns and footings as a pinned connection. Others treat it as fixed, and still others treat it as somewhere in between. If it is truly pinned, no moment is transferred to the footing, and this section of the text is not applicable. If, however, it is treated as fixed or partially fixed, this section is applicable.

If a column–footing joint is to behave as a pin or hinge, it would have to be constructed accordingly. The reinforcing steel in the column might be terminated at the column base instead of continuing into the footing. Dowels would be provided, but these would not be adequate to provide a moment connection.

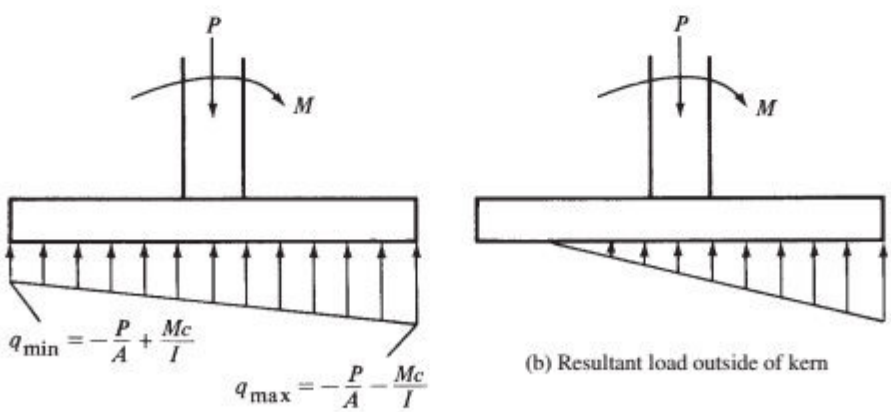
To provide continuity at the column–footing interface, the reinforcing steel would have to be continued into the footing. This is normally accomplished by embedding hooked bars into the footing and having them extend into the air where the columns will be located. The length they extend into the air must be at least the lap splice length; sometimes this can be a significant length. These bars are then lap spliced or mechanically spliced with the column bars, providing continuity of tension force in the reinforcing steel.

If there is a moment transfer from the column to the footing, the resultant force will not coincide with the centroid of the footing. Of course, if the moment is constant in magnitude and direction, it will be possible to place the center of the footing under the resultant load and avoid the eccentricity, but lateral forces such as wind and earthquake can come from any direction, and symmetrical footings will be needed.

The effect of the moment is to produce a linearly varying soil pressure, which can be determined at any point with the expression

$$q = -\frac{P}{A} \pm \frac{Mc}{I}$$

In this discussion, the term *kern* is used. If the resultant force strikes the footing base within the kern, the value of $-P/A$ is larger than $+Mc/I$ at every point, and the entire footing base is in compression, as shown in Figure 12.23(a). If the resultant force strikes the footing base outside the kern, the value of



(a) Resultant load in kern

(b) Resultant load outside of kern

FIGURE 12.23 Soil stress distributions under footings with overturning moments.

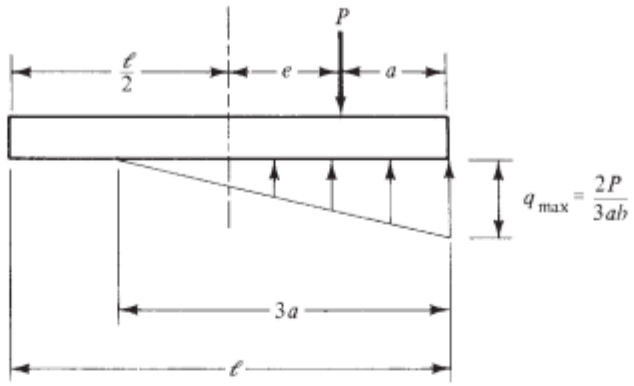


FIGURE 12.24 Soil stress distribution for eccentricity exceeding the kern eccentricity.

$+Mc/I$ will at some points be larger than $-P/A$, and there will be uplift or tension. The soil–footing interface cannot resist tension, and the pressure variation will be as shown in Figure 12.23(b). The location of the kern can be determined by replacing Mc/I with Pec/I, equating it to P/A, and solving for e.

Should the eccentricity be larger than this value, the method described for calculating soil pressures $[(-P/A) \pm (Mc/I)]$ is not correct. To compute the pressure for such a situation, it is necessary to realize that the centroid of the upward pressure must for equilibrium coincide with the centroid of the vertical component of the downward load. In Figure 12.24, it is assumed that the distance to this point from the right edge of the footing is a. Because the centroid of a triangle is located at one-third of its base, the soil pressure will be spread over the distance $3a$ as shown. For a rectangular footing with dimensions $\ell \times b$, the total upward soil pressure is equated to the downward load, and the resulting expression is solved for q_{max} as follows:

$$\left(\frac{1}{2}\right)(3ab)(q_{max}) = P$$

$$q_{max} = \frac{2P}{3ab}$$

Example 12.8 shows that the required area of a footing subjected to a vertical load and a lateral moment can be determined by trial and error. The procedure is to assume a size, calculate the maximum soil pressure, compare it with the allowable pressure, assume another size, and so on.

Once the area has been established, the remaining design will be handled as it was for other footings. Although the shears and moments are not uniform, the theory of design is unchanged. The factored loads are computed, the bearing pressures are determined, and the shears and moments are calculated. For strength design, the footing must be proportioned for the effects of these loads as required in ACI Code, Section 5.3.1.

Example 12.8

Determine the width needed for a wall footing to support loads: $D = 18$ k/ft and $L = 12$ k/ft. In addition, a moment of 39 ft-k must be transferred from the column to the footing. Assume that the footing is 18 in. thick, its base is 4 ft below the final grade, and $q_a = 4$ ksf.

SOLUTION

First Trial

$$\text{Neglecting moment } q_e = 4000 \text{ pcf} - \left(\frac{18 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) - \left(\frac{30 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 3525 \text{ psf}$$

$$\text{Width required} = \frac{18 \text{ k} + 12 \text{ k}}{3.525 \text{ ksf}} = 8.51 \text{ ft} \qquad \underline{\text{Try 9 ft 0 in.}}$$

$$A = (9 \text{ ft})(1 \text{ ft}) = 9 \text{ ft}^2$$

$$I = \left(\frac{1}{12}\right)(1 \text{ ft})(9 \text{ ft})^3 = 60.75 \text{ ft}^4$$

$$q_{max} = -\frac{P}{A} - \frac{Mc}{I} = -\frac{30 \text{ k}}{9 \text{ ft}^2} - \frac{(39 \text{ ft-k})(4.5 \text{ ft})}{60.75 \text{ ft}^4} \quad \text{(where minus indicates compression)}$$

$$= -6.22 \text{ ksf} > 3.525 \text{ ksf} \qquad \underline{\text{No good}}$$

$$q_{min} = -\frac{P}{A} + \frac{Mc}{I} = -\frac{30 \text{ k}}{9 \text{ ft}^2} + \frac{(39 \text{ ft-k})(4.5 \text{ ft})}{60.75 \text{ ft}^4} = -0.44 \text{ ksf}$$

Second Trial

Assume 14-ft-wide footing (after a check of a 13-ft-wide footing proves it to be insufficient)

$$A = (14 \text{ ft})(1 \text{ ft}) = 14 \text{ ft}^2$$

$$I = \left(\frac{1}{12}\right)(1 \text{ ft})(14 \text{ ft})^3 = 228.7 \text{ ft}^4$$

$$q_{max} = -\frac{30 \text{ k}}{14 \text{ ft}^2} - \frac{(39 \text{ ft-k})(7 \text{ ft})}{228.7 \text{ ft}^4} = -3.34 \text{ ksf} < 3.525 \text{ ksf} \quad \underline{\text{OK}}$$

$$q_{min} = -\frac{30 \text{ k}}{14 \text{ ft}^2} + \frac{(39 \text{ ft-k})(7 \text{ ft})}{228.7 \text{ ft}^4} = -0.95 \text{ ksf} \quad \underline{\text{OK}} \qquad \underline{\text{Use 14-ft-0-in. footing}}$$

Note that in both trials, the sign for q_{min} is negative, meaning that the soil–footing interface is in compression. Had the value been positive, the equations used to calculate stress would not have been valid. Instead, the designer would have to use the equation

$$q_{max} = \frac{2P}{3ab}$$

Footings must be designed to resist all applicable load combinations from ACI Code, Section 5.3.1. The experienced designer can often guess which will be most critical, design the footing accordingly, and check the others to see that the footing can resist them. Experience and computer programs help immensely in this process.

12.13 Transfer of Horizontal Forces

When it is necessary to transfer horizontal forces from walls or columns to footings, the shear friction method discussed in Section 8.12 of this text or other appropriate means should be used (ACI Code, Section 16.3.3.5). Sometimes shear keys (see Figure 13.1 in Chapter 13) are used between walls or columns and footings. This practice is of rather questionable value, however, because appreciable slipping has to occur to develop a shear key. A shear key may be thought of as providing an additional mechanical safety factor, but none of the lateral design force should be assigned to it.

The following example illustrates the consideration of lateral force transfer by the shear friction concept.

Example 12.9

A 14-in. × 14-in. column is supported by a 13-ft-0-in. × 13-ft-0-in. footing [f'_c = 3000 psi (normal weight) and f_y = 60,000 psi for both]. For vertical compression force transfer, six #6 dowels (2.65 in.^{2}) were selected extending 18 in. up into the column and 18 in. down into the footing. Design for a horizontal factored force V_u of 65 k acting at the base of the column. Assume that the footing concrete has not been intentionally roughened before the column concrete is placed. Thus, $\mu = 0.6\lambda = (0.6)\ (1.0)$ for normal-weight concrete = 0.6. (See ACI Code, Section 22.9.4.2.)

SOLUTION

Determine Minimum Shear Friction Reinforcement Required by ACI Code, Section 22.9.4

$$V_n = \frac{V_u}{\phi} = \frac{65\text{ k}}{0.75} = 86.7\text{ k}$$

$$V_n = A_{vf} f_y \mu \qquad \text{(ACI Equation 22.9.4.2)}$$

$$A_{vf} = \frac{86.7\text{ k}}{(60\text{ ksi})\ (0.6)} = 2.41\text{ in.}^2 < 2.65\text{ in.}^2 \qquad \underline{\text{OK}}$$

The six #6 dowels (2.65 in.^{2}) present may also be used as shear friction reinforcement. If their area had not been sufficient, it could have been increased and/or the value of μ could be increased significantly by intentionally roughening the concrete, as permitted in Section 22.9.4.2 of the ACI Code.

Check Tensile Development Lengths of These Dowels

Shear friction reinforcement acts in tension and thus must have tensile anchorage on both sides of the shear plane. It must also engage the main reinforcement in the footing to prevent cracks from occurring between the shear reinforcement and the body of the concrete.

$$\text{Assuming } \frac{c + K_{tr}}{d_b} = 2.5$$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left[\frac{60{,}000\text{ psi}}{(1.0)\sqrt{3000}\text{ psi}}\right]\frac{(1.0)\ (1.0)\ (0.8)}{2.5}\left(\frac{2.41\text{ in.}^2}{2.65\text{ in.}^2}\right) = 23.91\text{ diameters}$$

$$\ell_d = (23.91)\ (0.75\text{ in.}) = 17.93\text{ in.} \qquad \underline{\text{Say 18 in.}}$$

Compute Maximum Shear Transfer Strength Permitted by Section 22.9.4.4 of the ACI Code

$$V_u \le \phi 0.2 f'_c A_c, \text{ but not} > \phi(800 A_c)$$

$$\le (0.75)\ (0.2)\ (3\text{ ksi})\ (14\text{ in.} \times 14\text{ in.}) = 88.2\text{ k} > 65\text{ k}$$

$$\text{but not} > (0.75)\ (800\text{ lb/in.}^2)\ (14\text{ in.} \times 14\text{ in.}) = 117{,}600\text{ lb} = 117.6\text{ k} \qquad \underline{\text{OK}}$$

12.14 Plain Concrete Footings

Occasionally, plain concrete footings are used to support light loads if the supporting soil is of good quality. Very often the widths and thicknesses of such footings are determined by rules of thumb, such as the depth of a plain footing must be equal to no less than the projection beyond the edges of the wall. In this section, however, a plain concrete footing is designed in accordance with the requirements of the ACI.

Chapter 14 of the ACI Code is devoted to the design of structural plain concrete. Structural plain concrete is defined as concrete that is completely unreinforced or that contains less than the minimum required amounts of reinforcement previously specified here for reinforced concrete members. The minimum compressive strength permitted for such concrete is 2500 psi,[3] as given in ACI Code, Section 19.2.1.1.

Structural plain concrete may be used only for (1) members continuously supported by soil or by other structural members that are capable of providing continuous support, (2) walls and pedestals, and (3) structural members with arch action where compression occurs for all loading cases (ACI Code, Section 14.1.3).

The ACI Code states that when plain concrete footings are supported by soil, they cannot have an edge thickness less than 8 in. (Section 14.3.2.1) and they cannot be used on piles (Section 14.1.5). The critical sections for shear and moment for plain concrete footings are the same as for reinforced concrete footings.

In ACI Code, nominal bending (Section 14.5.2) and shear strengths (Section 14.5.5) are specified for structural plain concrete. The proportions of plain concrete members will nearly always be controlled by tensile strengths rather than shear strengths.

In the equations that follow, $\phi = 0.60$ (ACI Code, Section 21.2.1i) for all cases, S is the elastic section modulus of uncracked members, and β_c is the ratio of the long side to the short side of the column or loaded area. In computing the strengths, whether flexural or shear, for concrete cast against soil, *the overall thickness, h, is to be taken as 2 in. less than the actual thickness* (ACI Code, Section 14.5.1.7). This concrete is neglected to account for uneven excavation for the footing and for some loss of mixing water to the soil and other contamination.

Bending Strength

$$M_n = 5\lambda\sqrt{f'_c}S_m \qquad \text{(ACI Equation 14.5.2.1a)}$$

$$M_n = 0.85\sqrt{f'_c}S \qquad \text{(ACI Equation 14.5.2.1b)}$$

$$\phi M_n \geq M_u \qquad \text{(ACI Equation 14.5.1.1a)}$$

Shear Strength for One-Way or Beam Action

$$V_n = \frac{4}{3}\lambda\sqrt{f'_c}b_w h \qquad \text{(ACI Equation 14.5.5.1a)}$$

$$\phi V_n \geq V_u \qquad \text{(ACI Equation 14.5.1.1c)}$$

Shear Strength for Two-Way or Punching Action

$$V_n = \left(1 + \frac{2}{\beta}\right)\left(\frac{4}{3}\lambda\sqrt{f'_c}b_o h\right) \qquad \text{(ACI Equation 14.5.5.1b)}$$

$$\leq \frac{8}{3}\lambda\sqrt{f'_c}b_o h$$

In SI units

$$M_n = 0.42\lambda\sqrt{f'_c}S$$

$$V_n = 0.11\lambda\sqrt{f'_c}bh$$

$$V_n = 0.11\left(1 + \frac{2}{\beta}\right)\lambda\sqrt{f'_c}b_o h \leq 0.22\lambda\sqrt{f'_c}b_o h$$

[3] In SI units, it is 17 MPa.

On one hand, a plain concrete footing will obviously require considerably more concrete than will a reinforced one. On the other hand, the cost of purchasing reinforcing steel and placing it will be eliminated. Furthermore, the use of plain concrete footings will enable us to save construction time in that we don't have to order the reinforcing bars and place it before the concrete can be poured. Therefore, plain concrete footings may be economical on more occasions than one might realize.

Even though plain footings are designed in accordance with the ACI requirements, they should, at the very least, be reinforced in the longitudinal direction to keep temperature and shrinkage cracks within reason and to enable the footing to bridge over soft spots in the underlying soil. Example 12.10 presents the design of a plain concrete footing in accordance with the ACI Code.

Example 12.10

Design a plain concrete footing for a 12-in. reinforced concrete wall that supports a dead load of 12 k/ft, including the wall weight, and a 6-k/ft live load. The base of the footing is to be 5 ft below the final grade, $f'_c = 3000$ psi, and $q_a = 4000$ psf.

SOLUTION

Assume 24-in. Footing (see Figure 12.25)

$$q_e = 4000 \text{ psf} - \left(\frac{24 \text{ in.}}{12 \text{ in./ft}}\right)(145 \text{ pcf}) - \left(\frac{36 \text{ in.}}{12 \text{ in./ft}}\right)(100 \text{ pcf}) = 3410 \text{ psf}$$

$$\text{Width required} = \frac{18 \text{ k}}{3.41 \text{ k/ft}} = 5.28 \text{ ft} \qquad \text{Say 5 ft 6 in.}$$

Bearing Pressure for Strength Design

$$q_u = \frac{(1.2)(12 \text{ k/ft}) + (1.6)(6 \text{ k/ft})}{5.5 \text{ ft}} = 4.36 \text{ ksf}$$

Checking Bending Strength, Neglecting Bottom 2 in. of Footing (Figure 12.25)

$$M_u \text{ for 12 in. width of footing} = (4.36 \text{ klf})(2.25 \text{ ft})\left(\frac{2.25 \text{ ft}}{2}\right) = 11.04 \text{ ft-k}$$

$$S_m = \frac{bd^2}{6} = \frac{(12 \text{ in.})(22 \text{ in.})^2}{6} = 968 \text{ in.}^3$$

$$\phi M_n = \phi 5\sqrt{f'_c} S_m = (0.60)(5)(\sqrt{3000} \text{ psi})(968 \text{ in.}^3)$$

$$= 159{,}058 \text{ in.-lb} = 13.25 \text{ ft-k}$$

$$> 11.04 \text{ ft-k} \qquad \text{OK}$$

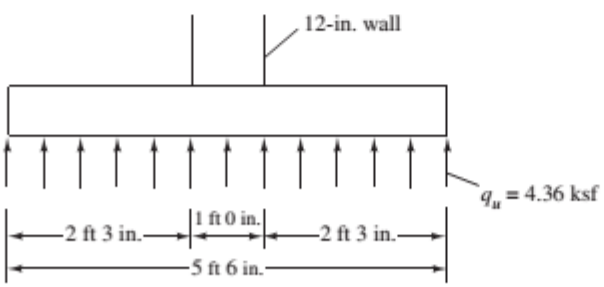


FIGURE 12.25 Plain concrete footing for Example 12.10.

Checking Shearing Strength at a Distance 22 in. from Face of Wall

$$V_u \text{ for 12 in. width of footing} = \left(2.25 \text{ ft} - \frac{22 \text{ in.}}{12 \text{ in./ft}}\right)(4.36 \text{ klf}) = 1.82 \text{ k}$$

$$V_u = \phi\left(\frac{4}{3}\right)\lambda\sqrt{f'_c}\,b_w h = (0.60)\left(\frac{4}{3}\right)(1.0)(\sqrt{3000} \text{ psi})(12 \text{ in.})(22 \text{ in.})$$

$$= 11{,}568 \text{ lb} = 11.57 \text{ k} > 1.82 \text{ k} \qquad \underline{\text{OK}}$$

Use 24-in. footing

Note: A 23-in.-deep footing will work in this case, and a 22-in.-deep footing will almost work (within 1% moment capacity). The authors prefer to use the 24-in. depth for simplicity in this case.

12.15 SI Example

Example 12.11

Design a reinforced concrete wall footing to support a 300-mm-wide reinforced concrete wall with a dead load $D = 300$ kN/m and a live load $L = 200$ kN/m. The bottom of the footing is to be 1 m below the final grade, the soil weight is 16 kN/m^3, the concrete weight is 24 kN/m^3, the allowable soil pressure, q_u, is 190 MPa/m^2, $f_y = 420$ MPa, and $f'_c = 28$ MPa.

SOLUTION

Assume 450-mm-Deep Footing ($d = 360$ mm)

$$q_e = 190 \text{ MPa/m}^2 - \left(\frac{450 \text{ mm}}{1000}\right)(24 \text{ kN/m}^3) - \left(\frac{550 \text{ mm}}{1000}\right)(16 \text{ kN/m}^3)$$

$$= 170.4 \text{ kN/m}^2$$

$$\text{Width required} = \frac{300 \text{ kN/m} + 200 \text{ kN/m}}{170.4 \text{ kN/m}^2} = 2.93 \text{ m} \qquad \underline{\text{Say 3.00 m}}$$

Bearing Pressure for Strength Design

$$q_u = \frac{(1.2)(300 \text{ kN/m}) + (1.6)(200 \text{ kN/m})}{3.00 \text{ m}} = 226.7 \text{ kN/m}^2$$

Depth Required for Shear at a Distance *d* from Face of Wall for a One Meter Width (See Figure 12.26)

$$V_u = \left(\frac{3.00\text{ m}}{2} - \frac{150\text{ mm}}{1000} - \frac{360\text{ mm}}{1000}\right)(226.7\text{ kN/m}) = 224.4\text{ kN}$$

$$d = \frac{6V_u}{(0.75)\,\lambda\,(\sqrt{f'_c})\,(b_w)} = \frac{(6)\,(224.4\text{ kN})\,(10)^3}{(0.75)\,(1.0)\,(\sqrt{28}\text{ MPa})\,(1000\text{ mm})}$$

$$= 339\text{ mm} < 360\text{ mm} \qquad \underline{\text{OK}}$$

Steel Area ($d = 360$ mm) Taking Moments at Face of Wall

$$\text{Cantilever length} = \frac{3.00\text{ m}}{2} - \frac{150\text{ mm}}{1000} = 1.35\text{ m}$$

$$M_u = (1.35\text{ m})\,(226.7\text{ kN/m})\left(\frac{1.35\text{ m}}{2}\right) = 206.58\text{ kN/m}$$

$$\frac{M_u}{\phi bd^2} = \frac{206.58\text{ kN-m} \times 10^6}{(0.9)\,(1000\text{ mm})(360\text{ mm})^2} = 1.771$$

$$\rho = 0.00439 \text{ (from Appendix B, Table B.9)}$$

$$A_s = (0.00439)\,(1000\text{ mm})\,(360\text{ mm}) = 1580\text{ mm}^2/\text{m}$$

$$\underline{\text{Use \#22 bars @ 225 mm o.c. (1720 mm}^2/\text{m)}}$$

Development Length

From Table 7.1 in Chapter 7

$$\psi_t = \psi_e = \psi_s = \lambda = 1.0$$

$$\text{Assume } c_b = \text{cover} = 90\text{ mm} \leftarrow$$

$$c = \text{one-half c. to c. of bars} = 112.5\text{ mm}$$

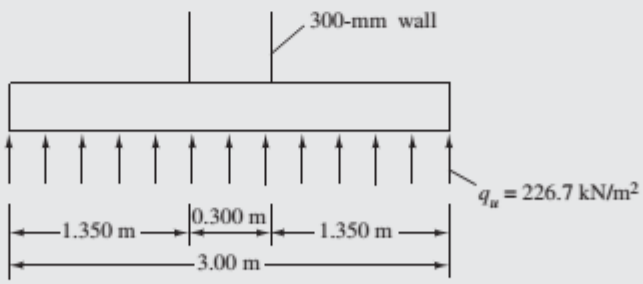


FIGURE 12.26 Wall footing for Example 12.11.

Letting $K_{tr} = 0$

$$\frac{c_b + K_{tr}}{d_b} = \frac{90 \text{ mm} + 0}{22.2 \text{ mm}} = 4.05 > 2.50 \qquad \therefore \text{Use } 2.5$$

$$\frac{\ell_d}{d_b} = \frac{f_y \psi_1 \psi_2 \psi_3}{1.1 \lambda \sqrt{f'_c} \left(\frac{c_b + K_{tr}}{d_b} \right)}$$

$$= \frac{(420 \text{ MPa})\,(1.0)\,(1.0)\,(1.0)}{[1.1(1.0)\sqrt{28} \text{ MPa}]\,(2.5)} = 28.57 \text{ diameters}$$

$$\frac{\ell_d}{d_b} \frac{A_{s\,\text{reqd}}}{A_{s\,\text{furn}}} = (28.57) \left(\frac{1580 \text{ mm}^2}{1720 \text{ mm}^2} \right) = 26.24 \text{ diameters}$$

$$\ell_d = (26.24)\,(22.2 \text{ mm}) = 583 \text{ mm} < 1350 \text{ mm} - 100 \text{ mm}$$

$$= 1250 \text{ mm available} \qquad \text{OK}$$

Use 450-mm footing 3 m wide with #22 bars @ 225 mm.

12.16 Computer Examples

Example 12.12

Repeat Example 12.1 using the Excel spreadsheets provided for Chapter 12.

SOLUTION

Open the Chapter 12 spreadsheet and the Wall Footing worksheet. Enter values only for the cells highlighted in yellow (only in Excel spreadsheets, not the printed example), beginning on the left side of the worksheet. When entering "*trial h*," estimate a reasonable value. In this case, $h = 12$ in. is the first try. Based on this assumption, d, q_e, and ℓ_{min} are calculated. Look at the footing width ℓ_{min} (9.86 ft), and enter a value that is more practical and slightly larger for the actual width in the next cell for ℓ(10.0 ft). Now observe the values of d_{shear} and h_{shear} a few cells below. The correct theoretical answer for h_{shear} lies between the *trial h* and h_{shear}. So it is somewhere between 12 in. and 21.96 in. Split the difference, and go back to *trial h* with a value of *trial h* = 16 in. The footing width can remain 10 ft, and h_{shear} is now 20.34 in. Split the difference again, and enter *trial h* = 19 in. Now h_{shear} = 19.12 in. Trial and error can be avoided by using the Goal Seek feature. In the first cycle, after trying $h = 12$ in., go to the cell called trial $h - h_{shear}$. Theoretically, this should be zero. Highlight this cell, go to Tools on the menu bar, and select Goal Seek from the drop-down menu. In "To value," enter 0, and in "By changing cell," enter C14. Click OK, and the status window says there is a solution. Select OK, and observe that cells C14 and C22 are now both 19.09 in. This may be faster than the trial-and-error method, but there is no need for the precision of the answer provided. Either method leads to the thickness of the footing of 20 in.

Now go to the right side of the worksheet, and enter "select $h = 20$ in." The required area of reinforcing steel is calculated, and a list of possible choices of bar size and theoretical spacing is displayed. Pick the one you prefer, being sure to round the spacing down to a practical value. If #7 bars are selected, the theoretical spacing of 10.55 in. is rounded down to 10 in. (the same answer as Example 12.1).

A screenshot of the software for the first cycle with trial $h = 12$ in. is shown here.

Wall Footing Design

P_D =	20	k/ft
P_L =	15	k/ft
P_u =	48	k/ft
t =	12	in.
cover =	3	in.
f'_c =	3000	psi
γ_c =	145	pcf
γ_s =	100	pcf
f_y =	60,000	psi
λ =	1.00	
q_a =	4.00	ksf
d_{grade} =	4.00	ft
trial h =	12.00	in.
d =	8.50	in.
q_e =	3.55	ksf
ℓ_{min} =	9.86	ft
ℓ =	10	ft
q_u =	4.8	ksf
V_u =	18.20	k/ft
d_{shear} =	18.46	in.
h_{shear} =	21.96	in.
trial $h - h_{shear}$ =	9.96	in.

select h =	20	in.
d =	16.50	in.
M_u =	48.6	ft-k
R_n =	198.3471	psi
ρ =	0.00345	
$A_{s\ flexure}$ =	0.682	in.^{2}/ft
$A_{s\ t\&s}$ =	0.432	in.^{2}/ft
$A_{s\ min}$ =	0.660	in.
A_s =	0.682	in.^{2}/ft

	s_{theor}	
#3	1.93	in.
#4	3.52	in.
#5	5.45	in.
#6	7.74	in.
#7	10.55	in.
#8	13.90	in.
#9	17.59	in.

Select a bar size and spacing from the table above.

Example 12.13

Repeat Example 12.2 using the Excel spreadsheets provided for Chapter 12.

SOLUTION

Open the Chapter 12 spreadsheet and the Square Footing worksheet. Enter values only for the cells highlighted in yellow, beginning on the left side of the worksheet. When entering "trial h," estimate a reasonable value. In Example 12.2, an initial value of $h = 24$ in. was used. This turned out to be the correct answer, which is not usually the case. Let's pick a value of $h = 18$ in. to illustrate the use of the spreadsheet's ability to converge on the correct answer. Note that ℓ_{min} is 9.02 ft, and a value of $\ell = 9.00$ ft was selected. This slightly nonconservative choice is consistent with what was done in

Example 12.2. Now look at the upper right side of the worksheet under Two-way shear. A value of $h_2 = 28.17$ in. means the trial h of 18 in. isn't enough. A quick look at the One-way shear results shows $h_1 = 21.32$ in. Because h_2 is larger, two-way shear is more critical. The correct value of h is somewhere between 18 in. and 28.17 in. Go back to trial h, and split the difference by entering 23 in. A quick look at the required h_2 and h_1 shows this choice is acceptable for one-way shear, but not two-way, which needs $h_2 = 24.14$ in. Now enter a trial h of 24 in., and both h_1 and h_2 are exceeded, indicating $h = 24$ in. is enough for shear. You can also do this by using Goal Seek as described in Example 12.12.

Go to the lower part of the worksheet, and enter 24 in. under select h. The required A_s in both directions is computed, and a table of possible choices is displayed. One of the choices is nine #8 bars, the same as was selected in Example 12.2.

A screenshot of the software for the last cycle with trial $h = 24$ in. is shown here.

Square Single-Column Footing Design

P_D =	200	k
P_L =	160	k
P_u =	496	k
a =	16	in.
b =	16	in.
cover =	3	in.
f'_c =	3000	psi
γ_c =	145	pcf
γ_s =	100	pcf
f_y =	60,000	psi
λ =	1.00	
q_a =	5.00	ksf
d_{grade} =	5.00	ft
trial h =	24.00	in.
d =	19.50	in.
q_e =	4.40	ksf
ℓ_{min} =	9.05	ksf
ℓ =	9	ft
q_u =	6.12	ksf

Two-Way Shear			
	V_{u2} =	442.4	k
	b_o =	142	in.
	α_s =	40.00	
	β_c =	1	
$d_{2\ shear}$ =		18.96	in.
$d_{2\ shear}$ =		12.6405	in.
$d_{2\ shear}$ =		10.12	in.
	d_2 =	18.96	in.
	h_2 =	23.46	in.
trial $h - h_{2\ shear}$ =		0.54	

One-Way Shear			
	V_{u1} =	121.70	in.^{2}
$d_{1\ shear}$ =		13.72	in.
$h_{1\ shear}$ =		18.22	in.
trial $h - h_{1\ shear}$ =		5.78	in.

select h =	24	in.
d =	19.50	in.

M_u =	404.914	ft-k
R_n =	131.464	psi
ρ =	0.00225	
$A_{S\ flexure}$ =	4.74	in.^{2}
$A_{s\ t\&s}$ =	4.67	in.^{2}
$A_{s\ min}$ =	7.02	iin.^{2}
A_s =	7.02	in.^{2}

	Theoretical	No. of Bars	Spacing, in.
#4	35.10	36	3.03
#5	22.65	23	4.67
#6	15.95	16	6.56
#7	11.70	12	8.85
#8	8.89	9	11.51
#9	7.02	8	14.36
#10	5.53	6	17.92

Select a bar size and number of bars from the table above, rounding up to the next integer.

PROBLEMS

For Problems 12.1 to 12.30, assume that the reinforced concrete weighs 150 lb/ft^3, plain concrete weighs 145 lb/ft^3, and soil weighs 100 lb/ft^3.

Wall Footings

For Problems 12.1 to 12.5, design wall footings for the values given. The walls are to consist of reinforced concrete.

Problem no.	Wall thickness (in.)	D (k/ft)	L (k/ft)	f'_c(ksi)	f_y (ksi)	q_a (ksf)	Distance from bottom of footing to final grade (ft)
12.1	14	15	12	4	60	3	3
12.2	14	21	20	4	60	4	4
12.3	15	15	18	5	60	5	5
12.4	15	24	30	4	60	4	4
12.5	12	10	8	4	60	5	4

(*Answer to Problem 12.1:* 15-in. footing, 10 ft 3 in. wide with #7 bars @ 9-in. main steel)
(*Answer to Problem 12.3:* 15-in. footing, 7 ft 6 in. wide with #5 bars @ 6-in. main steel)
(*Answer to Problem 12.5:* 10-in. footing, 4 ft 0 in. wide with #5 bars @ 15-in. main steel)

Problem 12.6 Repeat Problem 12.1 if a masonry wall is used.

Column Footings

For Problems 12.7 to 12.12, design square single-column footings for the values given. All columns are interior columns.

Problem no.	Column size (in.)	D (k)	L (k)	f'_c(ksi)	f_y (ksi)	q_a (ksf)	Distance from bottom of footing to final grade (ft)
12.7	12 × 12	100	160	4	60	4	4
12.8	12 × 12	100	80	3	60	5	5
12.9	15 × 15	160	150	4	60	4	4
12.10	15 × 15	150	120	3	60	4	4
12.11	16 × 16	110	100	3	60	5	5
12.12	Round, 18 dia.	240	140	4	60	5	6

(*Answer to Problem 12.7:* 21-in. footing, 9 ft 0 in. × 9 ft 0 in. with 8 #8 bars both directions)
(*Answer to Problem 12.9:* 22-in. footing, 9 ft 6 in. × 9 ft 6 in. with 12 #7 bars both directions)
(*Answer to Problem 12.11:* 18-in. footing, 7 ft 0 in. × 7 ft 0 in. with 7 #7 bars both directions)

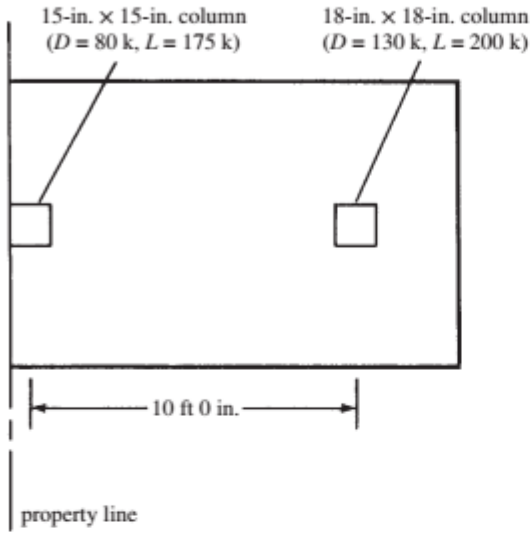
Problem 12.13 Design for load transfer from a 16-in. × 16-in. column with six #8 bars (D = 200 k, L = 350 k) to an 8-ft-0-in. × 8-ft-0-in. footing. f'_c = 4 ksi for footing, 5 ksi for column, and f_y = 60 ksi. (*Ans*. 4 #6, 13.5 in. into footing, 13.5 in. into column)

Problem 12.14 Repeat Problem 12.7 if a rectangular footing with one side of the footing is limited to 7 ft.

Problem 12.15 Design a footing with one side limited to 7 ft for the following: 12-in. × 12-in. edge column, D = 130 k, L = 155 k, f'_c = 3000 psi, f_y = 60,000 psi, q_a = 4 ksf, and a distance from top of backfill to bottom of footing = 4 ft. (*Ans*. 7-ft-0-in. × 11-ft-8-in. footing, 24 in. deep, 8 #8 bars in long direction)

Problem 12.16 Design a footing limited to a maximum width of 7 ft 0 in. for the following: 15-in. × 15-in. interior column, D = 180 k, L = 160 k, f'_c = 4000 psi, f_y = 60,000 psi, q_a = 4 ksf, and a distance from top of backfill to bottom of footing = 5 ft.

Problem 12.17 Design a rectangular combined footing for the two columns shown. The bottom of the footing is to be 5 ft below the final grade, $f'_c = 3.5$ ksi, $f_y = 50$ ksi, and $q_a = 5$ ksf. (*Ans.* 10 ft 6 in. × 12 ft 9 in., 26 in. deep, 11 #9 bars long direction)



Problem 12.18 Determine the footing areas required for the loads given in the accompanying table so that the usual soil pressures are equal. Assume $q_e = 5$ ksf and a usual live load percentage of 30 for all of the footings.

Footing	Dead load (k)	Live load (k)
A	120	200
B	130	170
C	120	200
D	150	200
E	140	180
F	140	200

Footings with Moments

For Problems 12.19 and 12.20, determine the width required for the wall footings. Assume that the footings have total thicknesses of 24 in.

Problem no.	Reinforced concrete wall thickness (in.)	D (k/ft)	L (k/ft)	f'_c(ksi)	f_y (ksi)	q_a (ksf)	Moment (ft-k)	Footing thickness, h (in.)	Distance from bottom of footing to final grade (ft)
12.19	12	14	18	4	60	4	32	24	4
12.20	14	16	24	3	60	4	50	24	5

(*Answer to Problem 12.19:* 15 ft 0 in.)

Problem 12.21 Repeat Problem 12.13 if a lateral force $V_u = 100$ k acts at the base of the column. Use the shear friction concept. Assume that the footing concrete is not intentionally roughened before column concrete is placed and that normal-weight concrete is used ($\mu = 0.6\lambda$). (*Ans.* 6 #8 dowels, 23 in. into footing, 20 in. into column)

Problem 12.22 Repeat Problem 12.21 using intentionally roughened concrete ($\mu = 1.0\lambda$).

Plain Concrete Footings

For Problems 12.23 and 12.24, design plain concrete wall footings of uniform thickness.

Problem no.	Reinforced concrete wall thickness (in.)	D (k/ft)	L (k/ft)	f'_c(ksi)	q_a (ksf)	Distance from bottom of footing to final grade (ft)
12.23	12	16	14	4	5	5
12.24	14	12	10	3	4	4

(*Answer to Problem 12.23:* 30-in.-deep footing, 7 ft 0 in. wide)

For Problems 12.25 and 12.26, design square plain concrete column footings of uniform thickness.

Problem no.	Reinforced concrete column size (in.)	D (k)	L (k)	f'_c(ksi)	q_a (ksf)	Distance from bottom of footing to final grade (ft)
12.25	12 × 12	70	60	3	4	5
12.26	14 × 14	90	75	3.5	4	5

(*Answer to Problem 12.25:* 6-ft-3-in. × 6-ft-3-in. footing, 28 in. deep)

Computer Problems

For Problems 12.27 to 12.30, use the Chapter 12 spreadsheet.

Problem 12.27 Repeat Problem 12.2. (*Ans*. 11 ft 9 in. width, 21 in. depth with #8 @ 10 in.)

Problem 12.28 Repeat Problem 12.8.

Problem 12.29 Repeat Problem 12.10. (*Ans*. 8 ft 10 in. × 8 ft 10 in., 21 in. depth with 8 #8 each way)

Problem 12.30 Repeat Problem 12.16.

Problems in SI Units

For Problems 12.31 and 12.32, design wall footings for the values given. The walls are to consist of reinforced concrete. Concrete weight = 24 kN/m^3 and soil weight = 16 kN/m^3.

Problem no.	Wall thickness (mm)	D (kN/m)	L (kN/m)	f'_c (MPa)	f_y (MPa)	q_a (kN/m^2)	Distance from bottom of footing to final grade (m)
12.31	300	150	200	21	420	170	1.500
12.32	400	180	250	28	420	210	1.200

(*Answer to Problem 12.31:* 370-mm footing, width = 2.5 m, and #16 bars @ 300 mm main steel)

For Problems 12.33 to 12.35, design square single-column footings for the values given. Concrete weight = 24 kN/m^3, soil weight = 16 kN/m^3, and all columns are interior ones.

Problem no.	Column size (mm)	D (kN)	L (kN)	f'_c (MPa)	f_y (MPa)	q_a (kN/m^2)	Distance from bottom of footing to final grade (m)
12.33	350 × 350	400	500	21	420	170	1.200
12.34	400 × 400	650	800	28	420	170	1.200
12.35	450 × 450	750	1000	28	420	210	1.600

(*Answer to Problem 12.33:* 480-mm footing, 2.5 m × 2.5 m with 11 #19 bars both directions)
(*Answer to Problem 12.35:* 600-mm footing, 3.2 m × 3.2 m with 10 #25 bars both directions)

Problem 12.36 Design a plain concrete wall footing for a 300-mm-thick reinforced concrete wall that supports a 100-kN/m dead load (including its own weight) and a 120-kN/m live load. f'_c = 21 MPa and q_a = 170 kN/m^2. The base of the footing is to be 1.250 m below the final grade. Concrete weight = 24 kN/m^3 and soil weight = 16 kN/m^3.

Problem 12.37 Design a square plain concrete column footing to support a 300-mm × 300-mm reinforced concrete column that in turn supports a 130-kN dead load and a 200-kN live load. f'_c = 28 MPa and q_a = 210 kN/m^2. The base of the footing is to be 1.500 m below the final grade. Concrete weight = 24 kN/m^3 and soil weight = 16 kN/m^3. (*Ans*. 1.4 m × 1.4 m, 520 mm thick)

Retaining Walls

CHAPTER 13

13.1 Introduction

A retaining wall is a structure built for the purpose of holding back, or retaining, or providing one-sided lateral confinement of soil or other loose material. The loose material being retained pushes against the wall, tending to overturn and slide it. Retaining walls are used in many design situations where there are abrupt changes in the ground slope. Perhaps the most obvious examples to the reader occur along highway or railroad cuts and fills. Often retaining walls are used in these locations to reduce the quantities of cut and fill as well as to reduce the right-of-way width required if the soils were allowed to assume their natural slopes. Retaining walls are used in many other locations as well, such as for bridge abutments, basement walls, and culverts.

Several different types of retaining walls are discussed in the next section, but whichever type is used, there will be three forces involved that must be brought into equilibrium: (1) the gravity loads of the concrete wall and any soil on top of the footing (the so-called *developed weight*), (2) the lateral pressure from the soil, and (3) the bearing resistance of the soil. In addition, the stresses within the structure have to be within permissible values, and the loads must be supported in a manner such that undue settlements do not occur. A retaining wall must be designed in such a way that the concrete elements that make up the wall comply with the ACI Code using, for the most part, principles already discussed in this text. In addition, the overall stability of the wall must be ensured. The wall may slide or tip over due to global instability without failure of the concrete elements.

13.2 Types of Retaining Walls

Retaining walls are generally classed as being gravity or cantilever types, with several variations possible. These are described in the paragraphs to follow, with reference being made to Figure 13.1.

The *gravity retaining wall*, shown in Figure 13.1(a), is used for walls of up to about 10 ft to 12 ft in height. It is usually constructed with plain concrete and depends completely on its own weight for stability against sliding and overturning. It is usually so massive that it is unreinforced. Tensile stresses calculated by the working-stress method are usually kept below $1.6\sqrt{f'_c}$. Gravity walls may also be constructed with stone or block masonry.

Semigravity retaining walls, shown in Figure 13.1(b), fall between the gravity and cantilever types (to be discussed in the next paragraph). They depend on their own weights plus the weight of some soil behind the wall to provide stability. Semigravity walls are used for approximately the same range of heights as the gravity walls and usually have some light reinforcement.

The *cantilever retaining wall* or one of its variations is the most common type of retaining wall. Such walls are generally used for heights from about 10 ft to 25 ft. In discussing retaining walls, the vertical wall is referred to as the *stem*. The outside part of the footing that is pressed down into the soil is called the *toe*, while the part that tends to be lifted is called the *heel*. These parts are indicated for the cantilever retaining wall in Figure 13.1(c). The concrete and its reinforcement are so arranged that part of the material behind the wall is used along with the concrete weight to produce the necessary resisting moment against overturning. This resisting moment is generally referred to as the *righting moment*.

When it is necessary to construct retaining walls of greater heights than approximately 20 ft to 25 ft, the bending moments at the junction of the stem and footing become so large that the designer

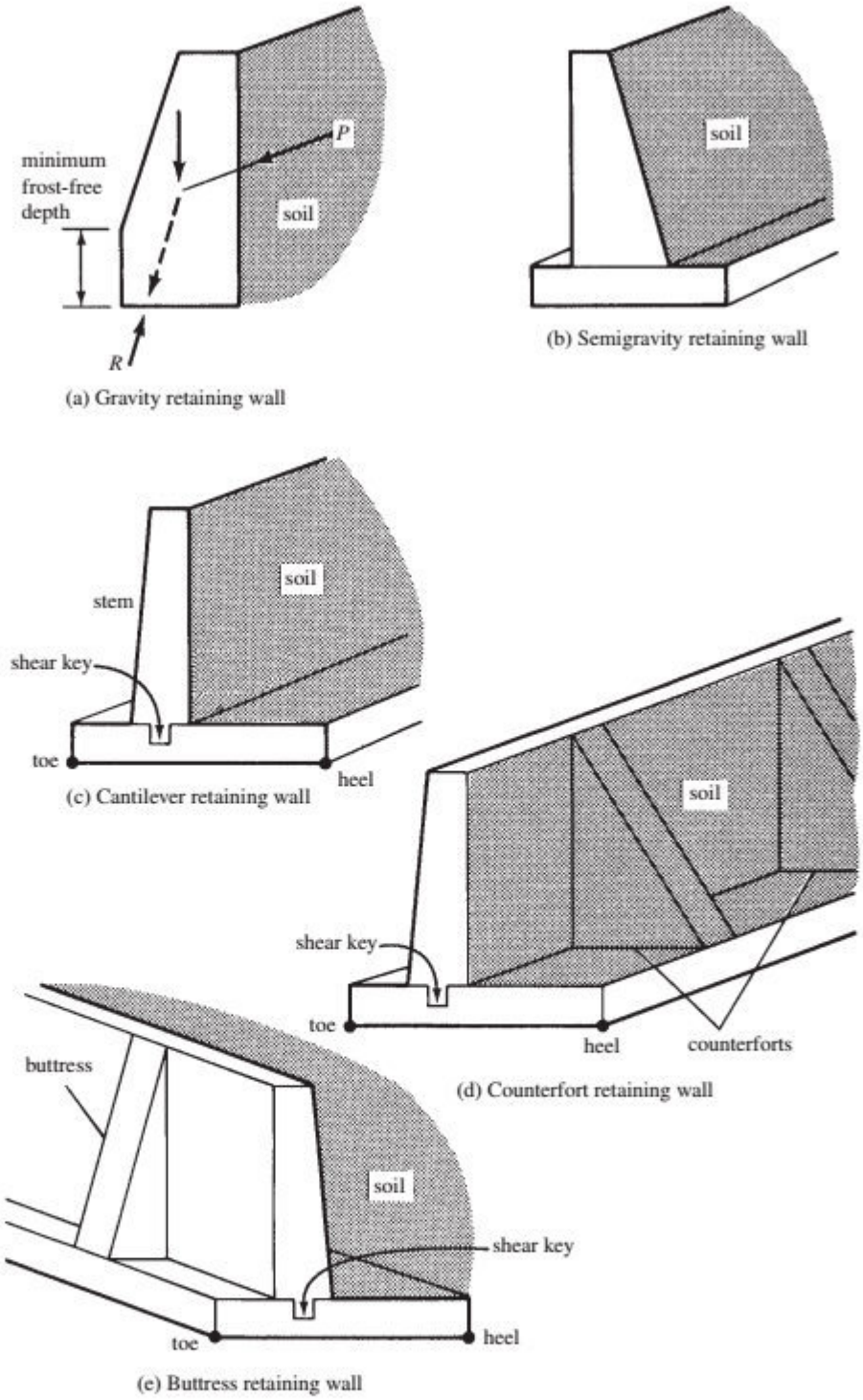


FIGURE 13.1 Retaining wall.

will, from economic necessity, have to consider other types of walls to handle the moments. This can be done by introducing vertical cross walls on the front or back of the stem. If the cross walls are behind the stem (i.e., inside the soil) and not visible, the retaining walls are called *counterfort walls*. If the cross walls are visible (i.e., on the toe side), the retaining walls are called *buttress walls*. These walls are illustrated in parts (d) and (e) of Figure 13.1. The stems for these walls are continuous members

supported at intervals by the buttresses or counterforts. Counterforts or buttresses are usually spaced at distances approximately equal to one-half (or a little more) of the retaining wall heights.

The counterfort type is more commonly used because it is normally thought to be more attractive, because the cross walls or counterforts are not visible. Not only are the buttresses visible on the toe side, but their protrusion on the outside or toe side of the wall will use up valuable space. Nevertheless, buttresses are somewhat more efficient than counterforts because they consist of concrete that is put in compression by the overturning moments, whereas counterforts are concrete members used in a tension situation, and they need to be tied to the wall with stirrups. Occasionally, high walls are designed with both buttresses and counterforts.

Figure 13.2 presents a few other retaining wall variations. When a retaining wall is placed at a property boundary or next to an existing building, it may be necessary to use a wall without a toe, as shown in part (a) of the figure, or without a heel, as shown in part (b). Another type of retaining wall very often encountered is the bridge abutment shown in part (c) of the figure. Abutments may very well have wing wall extensions on the sides to retain the soil in the approach area. The abutment, in addition to other loads, will have to support the end reactions from the bridge.

The use of precast retaining walls is becoming more common each year. The walls are built with some type of precast units, and the footings are probably poured in place. The results are very attractive, and the units are high-quality concrete members made under "plant-controlled" conditions. Less site preparation is required, and the erection of the walls is much quicker than cast-in-place ones. The precast

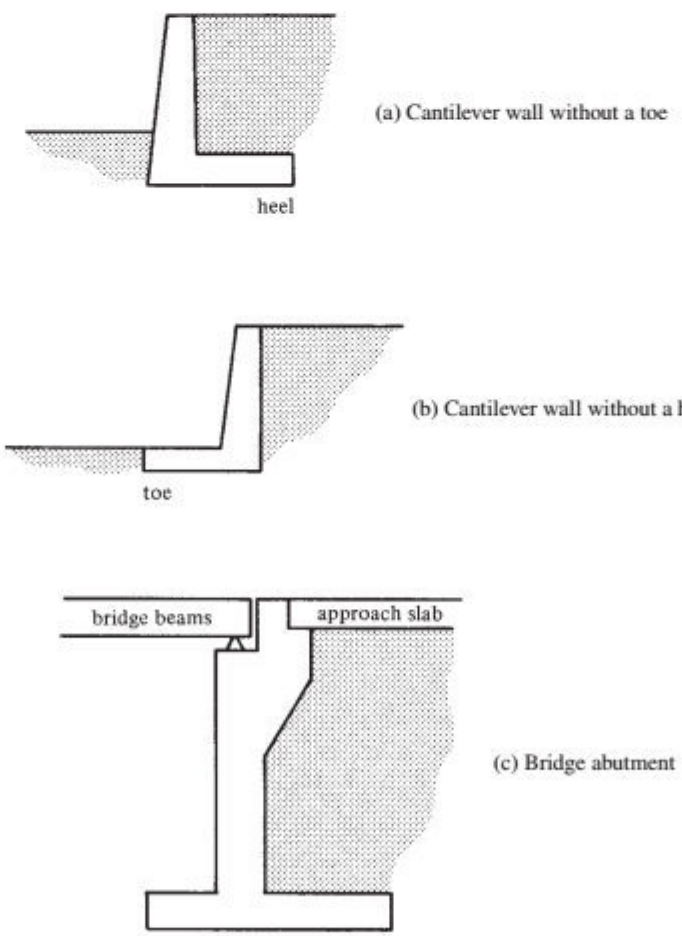


FIGURE 13.2 More retaining walls.

units can later be disassembled and the units used again. Other types of precast retaining walls consist of walls or sheeting actually driven into the ground before excavation. Also showing promise are *gabions*, or wire baskets of stone, used in conjunction with geotextile-reinforced embankments.

13.3 Drainage

One of the most important items in designing and constructing successful retaining walls is the prevention of water accumulation behind the walls. If water is allowed to build up there, the result can be great lateral water pressure against the wall and perhaps an even worse situation in cold climates due to frost action.

The best possible backfill for a retaining wall is a well-drained and cohesionless soil. This is the condition for which the designer normally plans and designs. In addition to a granular backfill material, weep holes of 4 in. or more in diameter (the large sizes are used for easy cleaning) are placed in the walls approximately 5 ft to 10 ft on center, horizontally and vertically, as shown in Figure 13.3(a). If the backfill consists of a coarse sand, it is desirable to put a few shovels of pea gravel around the weep holes to try to prevent the sand from stopping up the holes.

Weep holes have the disadvantages that the water draining through the wall is somewhat unsightly and also may cause a softening of the soil in the area of the highest soil pressure (under the footing toe). A better method includes the use of a 6-in. or 8-in. perforated pipe in a bed of gravel running along the base of the wall, as shown in Figure 13.3(b). Unfortunately, both weep holes and drainage pipes can become clogged, as a result of which increased water pressure can occur. Manufactured drainage blankets or porous mats placed between the wall and the soil allow moisture to migrate freely to drainage systems, as shown in Figure 13.3(b).

The drainage methods described in the preceding paragraphs are also quite effective for reducing frost action in colder areas. Frost action can cause very large movements of walls, not just in terms of inches but perhaps even in terms of a foot or two, and over a period of time can lead to failures. Frost action, however, can be greatly reduced if coarse, properly drained materials are placed behind the walls. The thickness of the fill material perpendicular to a wall should equal at least the depth of frost penetration in the ground in that area.

The best situation of all would be to keep the water out of the backfill altogether. Such a goal is normally impossible, but sometimes the surface of the backfill can be paved with asphalt or some other material, or perhaps a surface drain can be provided to remove the water, or it may be possible in some other manner to divert the water before it can get to the backfill.

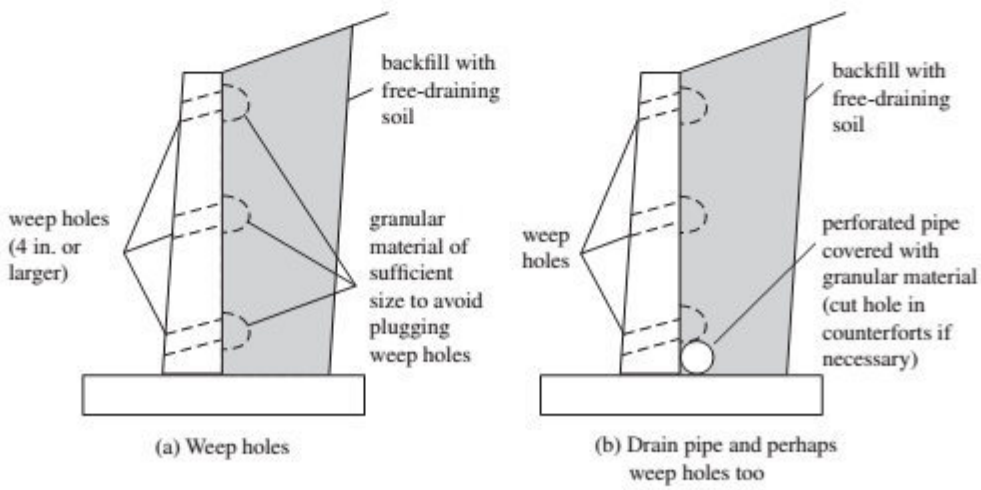
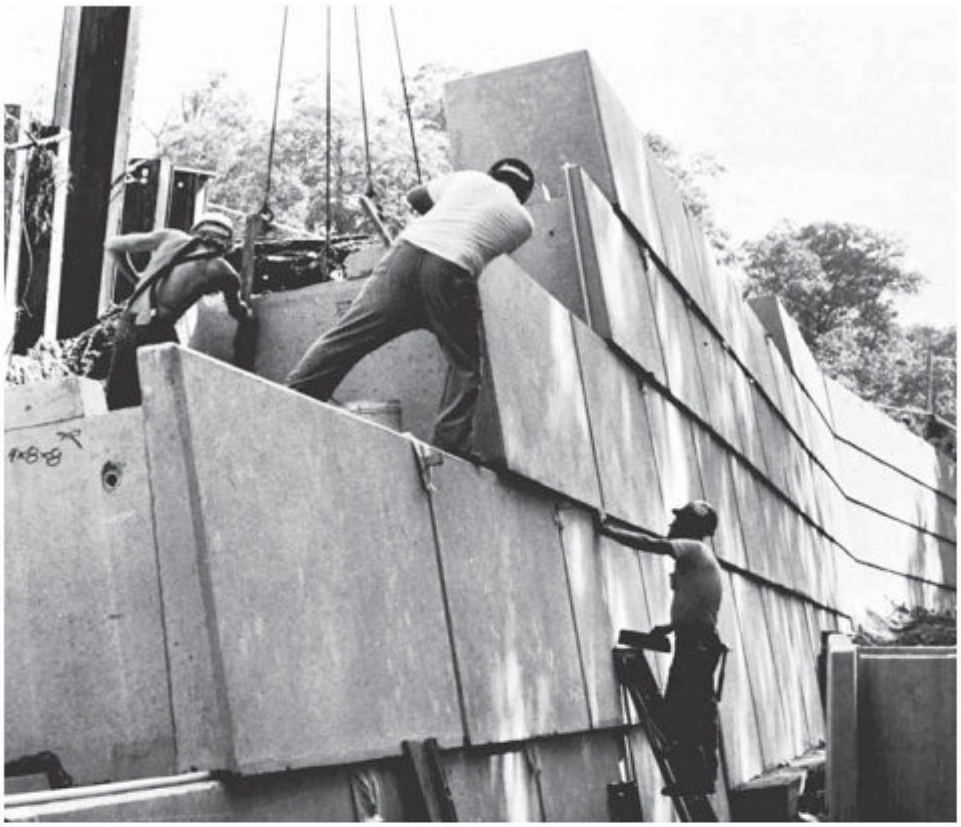

backfill with free-draining soil
weep holes (4 in. or larger)
granular material of sufficient size to avoid plugging weep holes
(a) Weep holes
backfill with free-draining soil
weep holes
perforated pipe covered with granular material (cut hole in counterforts if necessary)
(b) Drain pipe and perhaps weep holes too

FIGURE 13.3 Retaining wall drainage.



Courtesy of Doublewal—Division of United Concrete.

Retaining wall for Long Island Railroad, Huntington, New York. Constructed with precast interlocking reinforced concrete modules.

13.4 Failures of Retaining Walls

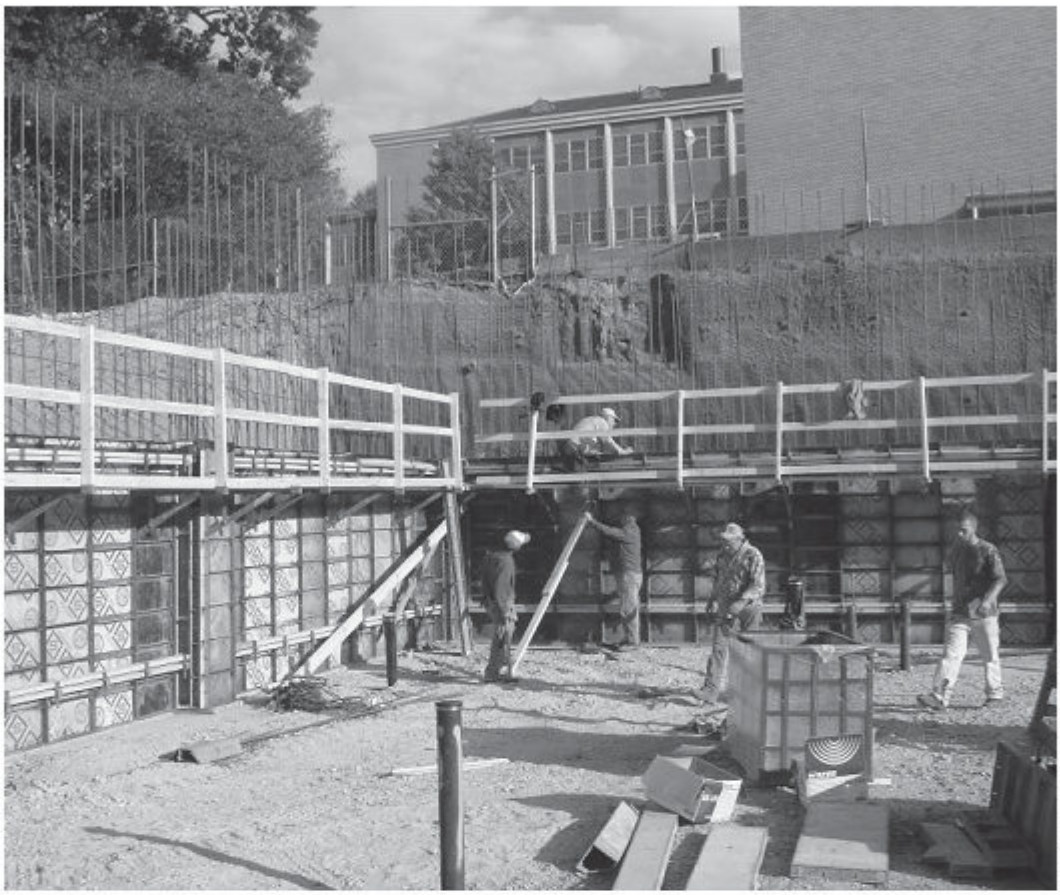
The number of failures or partial failures of retaining walls is rather alarming. The truth of the matter is that if large safety factors were not used, the situation would be even more severe. One reason for the large number of failures is the fact that designs are so often based on methods that are suitable only for certain special situations. For instance, if a wall that has a saturated clay behind it (never a good idea) is designed by a method that is suitable for a dry granular material, future trouble will be the result.

13.5 Lateral Pressure on Retaining Walls

The actual pressures that occur behind retaining walls are quite difficult to estimate because of the large number of variables present. These include the kinds of backfill materials and their compactions and moisture contents, the types of materials beneath the footings, the presence or absence of surcharge, and other variables. As a result, the detailed estimation of the lateral forces applied to various retaining walls is clearly a problem in theoretical soil mechanics. For this reason, the discussion to follow is limited to a rather narrow range of cases.

If a retaining wall is constructed against a solid rock face, there will be no pressure applied to the wall by the rock. If the wall is built to retain a body of water, however, hydrostatic pressure will be applied to the wall. At any point, the pressure, p, will equal wh, where w is the unit weight of the water and h is the vertical distance from the surface of the water to the point in question.

If a wall is built to retain the soil, the soil's behavior will generally be somewhere between that of rock and water (*but as you will learn, the pressure caused by some soils is much higher than that caused*



Courtesy of Russell H. Brown.

Retaining wall showing formwork under construction and reinforcing steel projecting from top (Rhodes Annex, Clemson University)

by water). The pressure exerted against the wall will increase, as did the water pressure, with depth but usually not as rapidly. This pressure at any depth can be estimated with the following expression:

$$p = Cwh$$

In this equation, w is the unit weight of the soil, h is the distance from the surface to the point in question, and C is a constant that is dependent on the characteristics of the backfill. Unfortunately, the value of C can vary quite a bit, being perhaps as low as 0.3 or 0.4 for loose granular soils and perhaps as high as 0.9 or even 1.0 or more for some clayey soils. Figure 13.4 presents charts that are sometimes used for estimating the vertical and horizontal pressures applied by soil backfills of up to 20 ft heights. Several different types of backfill materials are considered in the figure. Unit weights of soils will vary roughly as follows: 90 lb/ft^3 to 100 lb/ft^3 for soft clays, 100 lb/ft^3 to 120 lb/ft^3 for stiff clays, 110 lb/ft^3 to 120 lb/ft^3 for sands, and 120 lb/ft^3 to 130 lb/ft^3 for sand and gravel mixes.

If you carefully study the second chart of Figure 13.4, you will probably be amazed to see how high lateral pressures can be, particularly for clays and silts. As an illustration, a 1-ft-wide vertical strip is considered for a 15-ft-high retaining wall backfilled with soil number (4) with an assumed δ of 10° (6 : 1 slope). The total estimated horizontal pressure on the strip is

$$P_h = \frac{1}{2}k_h h^2 = \left(\frac{1}{2}\right)(102 \text{ psf})\,(15 \text{ ft})^2 = 11{,}475 \text{ lb}$$

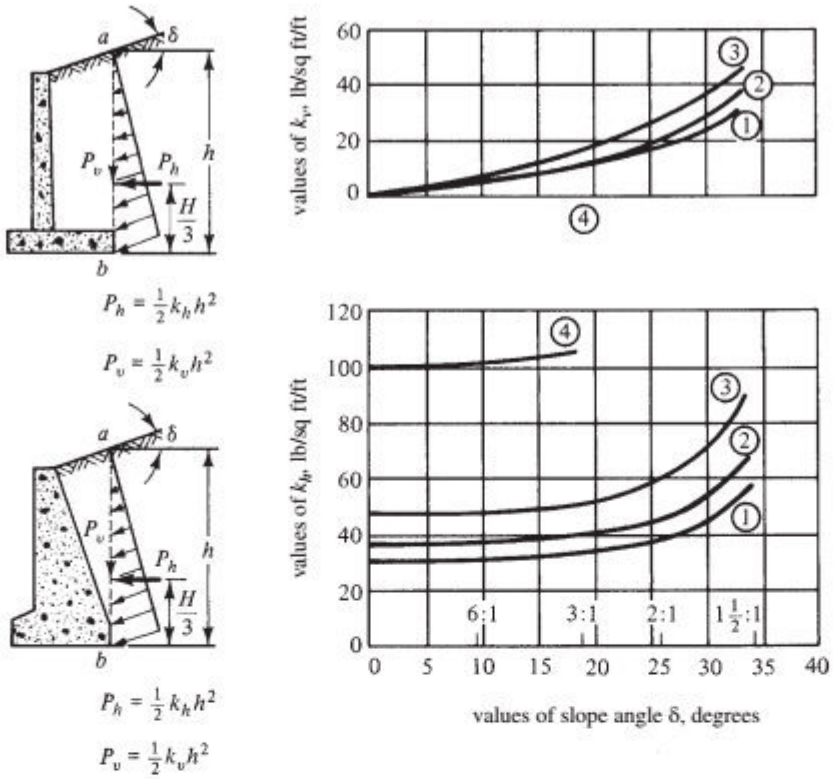


FIGURE 13.4 Chart for estimating pressure of backfill against retaining walls supporting backfills with plane surface. Use of this chart is limited to walls not over about 20 ft high. (1) Backfill of coarse-grained soil without admixture of fine particles, very permeable, such as clean sand or gravel. (2) Backfill of coarse-grained soil of low permeability because of admixture of particles of silt size. (3) Backfill of fine silty sand, granular materials with conspicuous clay content, and residual soil with stones. (4) Backfill of soft or very soft clay, organic silt, or silty clay.[1]

If a 15-ft-deep lake is assumed to be behind the same wall, the total horizontal pressure on the strip will be

$$P_h = \left(\frac{1}{2}\right)(15\text{ ft})\,(15\text{ ft})\,(62.4\text{ lb/ft}^3) = 7020\text{ lb}$$

(only 61% as large as the estimated pressure for the soil)

For this introductory discussion, a retaining wall supporting a sloping earth fill is shown in Figure 13.5. Part of the earth behind the wall (shown by the hatched area) tends to slide along a curved surface (represented by the dashed line) and push against the retaining wall. The tendency of this soil to slide is resisted by friction along the soil underneath (called *internal friction*) and by friction along the vertical face of the retaining wall.

Internal friction is greater for a cohesive soil than for a noncohesive one, but the wetter such a soil becomes, the smaller will be its cohesiveness and, thus, the flatter the plane of rupture. The flatter the plane of rupture, the greater is the volume of earth tending to slide and push against the wall. Once again, it is clear that good drainage is of the utmost importance. Usually the designer assumes that a cohesionless granular backfill will be placed behind the walls.

[1] Peck, R. B., Hanson, W. E., and Thornburn, T. H., 1974, *Foundation Engineering*, 2nd ed. (Hoboken, NJ: John Wiley & Sons), p. 425.

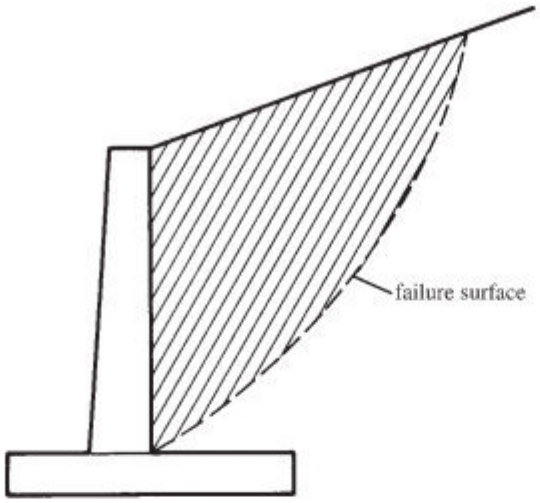


FIGURE 13.5 Possible sliding failure surface for a retaining wall supporting a sloping earth fill.

Due to lateral pressure, the usual retaining wall will give or deflect a little because it is constructed of elastic materials. Furthermore, unless the wall rests on a rock foundation, it will tilt or lean a small distance away from the soil due to the compressible nature of the supporting soils. For these reasons, retaining walls are frequently constructed with a slight batter, or inclination, toward the backfill so that the deformations described are not obvious to passersby.

Under the lateral pressures described, the usual retaining wall will move a little distance and *active soil pressure* will develop, as shown in Figure 13.6. Among the many factors that affect the pressure applied to a particular wall are the kind of backfill material used, the drainage situation, the level of the water table, the seasonal conditions such as dry or wet or frozen, and the presence of trucks or other equipment on the backfill.

For design purposes, it is usually satisfactory to assume that the active pressure varies linearly with the depth of the backfill. In other words, it is just as though (so far as lateral pressure is concerned) there is a liquid of some weight behind the wall that can vary from considerably less than the weight of water to considerably more. The chart of Figure 13.4 shows this large variation in possible lateral pressures. The assumed lateral pressures are often referred to as *equivalent fluid pressures. Values from 30 pcf to 50 pcf are normally assumed but may be much too low for clay and silt materials.*

If the wall moves away from the backfill and against the soil at the toe, a passive soil pressure will be the result. Passive pressure, which is also assumed to vary linearly with depth, is illustrated

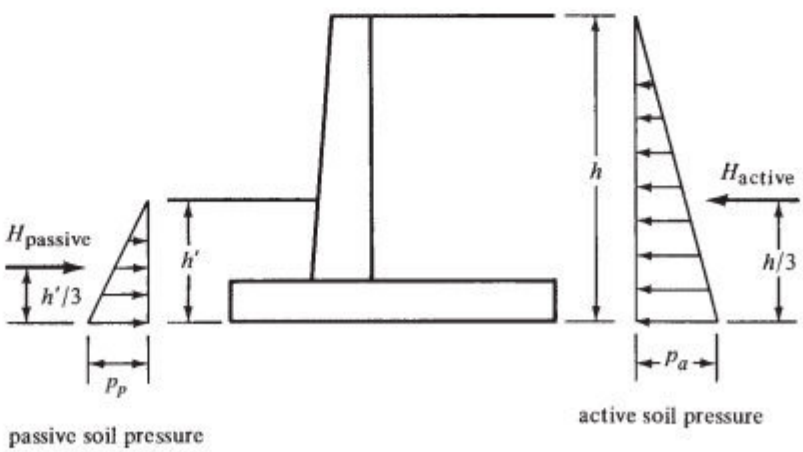


FIGURE 13.6 Active and passive soil pressures.

in Figure 13.6. The inclusion or noninclusion of passive pressure in the design calculations is a matter of judgment on the designer's part. For effective passive pressure to be developed at the toe, the toe concrete must be placed against undisturbed earth without the use of vertical forms. Even if this procedure is followed, the designer will probably reduce the height of the undisturbed soil (h' in Figure 13.6) used in the calculations to account for some disturbance of the earth during construction operations.

As long as the backfills are granular, noncohesive, and dry, the assumption of an equivalent liquid pressure is fairly satisfactory. Formulas based on an assumption of dry sand or gravel backfills are not satisfactory for soft clays or saturated sands. Actually, clays should not be used for backfills because their shear characteristics change easily and they may tend to creep against the wall, increasing pressures as time goes by.

If a linear pressure variation is assumed, the active pressure at any depth can be determined as

$$p_a = k_a w h$$

or, for passive pressure,

$$p_p = k_p w h'$$

In these expressions, k_a and k_p are the approximate coefficients of active and passive pressures, respectively. These coefficients can be calculated by theoretical equations such as those of Rankine or Coulomb.[2] For a granular material, typical values of k_a and k_p are 0.3 and 3.3. The Rankine equation (published in 1857) neglects the friction of the soil on the wall, whereas the Coulomb formula (published in 1776) takes it into consideration. These two equations were developed for cohesionless soils. For cohesive soils containing clays and/or silts, it is necessary to use empirical values determined from field measurements (such as those given in Figure 13.4).

It has been estimated that the cost of constructing retaining walls varies directly with the square of their heights. Thus, as retaining walls become higher, the accuracy of the computed lateral pressures becomes more and more important in providing economical designs. Because the Coulomb equation does take into account friction on the wall, it is thought to be the more accurate one and is often used for walls of over 20 ft. The Rankine equation is commonly used for ordinary retaining walls of 20 ft or less in height. It is interesting to note that the two methods give identical results if the friction of the soil on the wall is neglected.

The Rankine expressions for the active and passive pressure coefficients are given at the end of this paragraph with reference to Figure 13.7. In these expressions, δ is the angle the backfill makes with the horizontal, while ϕ is the angle of internal friction of the soil. For well-drained sand or gravel backfills, the angle of internal friction is often taken as the angle of repose of the slope. One common slope used is 1 vertically to $1\frac{1}{2}$ horizontally (33°40′).

$$k_a = \cos\delta\left(\frac{\cos\delta - \sqrt{\cos^2\delta - \cos^2\phi}}{\cos\delta + \sqrt{\cos^2\delta - \cos^2\phi}}\right)$$

$$k_p = \cos\delta\left(\frac{\cos\delta + \sqrt{\cos^2\delta - \cos^2\phi}}{\cos\delta - \sqrt{\cos^2\delta - \cos^2\phi}}\right)$$

Should the backfill be horizontal—that is, should δ be equal to zero—the expressions become

$$k_a = \frac{1 - \sin\phi}{1 + \sin\phi}$$

$$k_p = \frac{1 + \sin\phi}{1 - \sin\phi}$$

[2] Terzaghi, K. and Peck, R. B., 1948, *Soil Mechanics in Engineering Practice* (Hoboken, NJ: John Wiley & Sons), pp. 138–166.

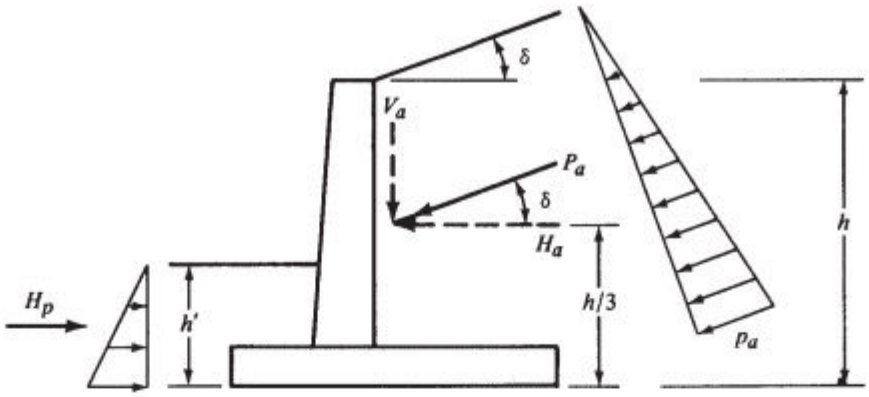


FIGURE 13.7 Active and passive soil pressures with sloping backfill.

One trouble with using these expressions is in the determination of ϕ. It can be as small as 0° to 10° for soft clays and as high as 30° or 40° for some granular materials. As a result, the values of k_a can vary from perhaps 0.30 for some granular materials up to about 1.0 for some wet clays.

Once the values of k_a and k_p are determined, the total horizontal pressures, H_a and H_p, can be calculated as being equal to the areas of the respective triangular pressure diagrams. For instance, with reference to Figure 13.7, the value of the active pressure is

$$H_a = \left(\frac{1}{2}\right)(p_a)\,(h) = \left(\frac{1}{2}\right)(k_a wh)\,(h)$$

$$H_a = \frac{k_a wh^2}{2}$$

and, similarly,

$$H_p = \frac{k_p wh'^2}{2}$$

In addition to these lateral pressures applied to the retaining wall, it is considered necessary in many parts of the country to add the effect of frost action at the top of the stem—perhaps as much as 600 lb or 700 lb per linear foot in areas experiencing extreme weather conditions.

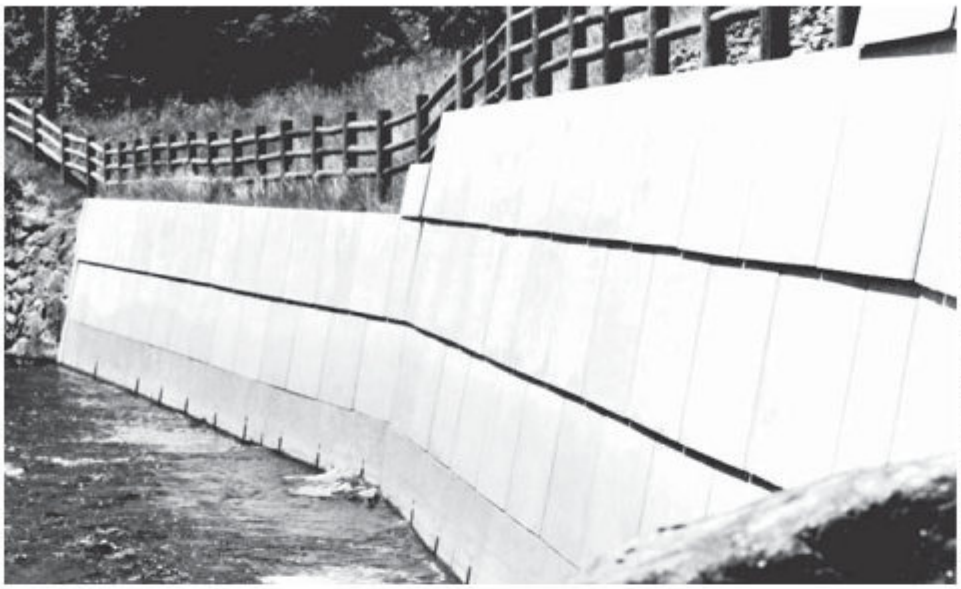
13.6 Footing Soil Pressures

Because of lateral forces, the resultant of the horizontal and vertical forces, R, intersects the soil underneath the footing as an eccentric load, causing greater pressure at the toe. This toe pressure should be less than the permissible value, q_a, of the particular soil. It is also desirable to keep the resultant force within the kern or the middle third of the footing base.

If the resultant force intersects the soil within the middle third of the footing, the soil pressure at any point can be calculated with the formula to follow exactly as the stresses are determined in an eccentrically loaded column.

$$q = -\frac{R_v}{A} \pm \frac{R_v ec}{I}$$

In this expression, R_v is the vertical component of R or the total vertical load, e is the eccentricity of the load from the center of the footing, A is the area of a 1-ft-wide strip of soil of length equal to the width of the footing base, and I is the moment of inertia of the same area about its centroid. This expression is correct only if R_v falls within the kern.



Retaining wall for United States Army Corps of Engineers, Colchester, Connecticut. Constructed with precast interlocking reinforced concrete modules.

This expression can be reduced to the following expression, in which L is the width of the footing from heel to toe.

$$q = -\frac{R_v}{L} \pm \frac{R_v e(L/2)}{L^3/12} = -\frac{R_v}{L}\left(1 \pm \frac{6e}{L}\right)$$

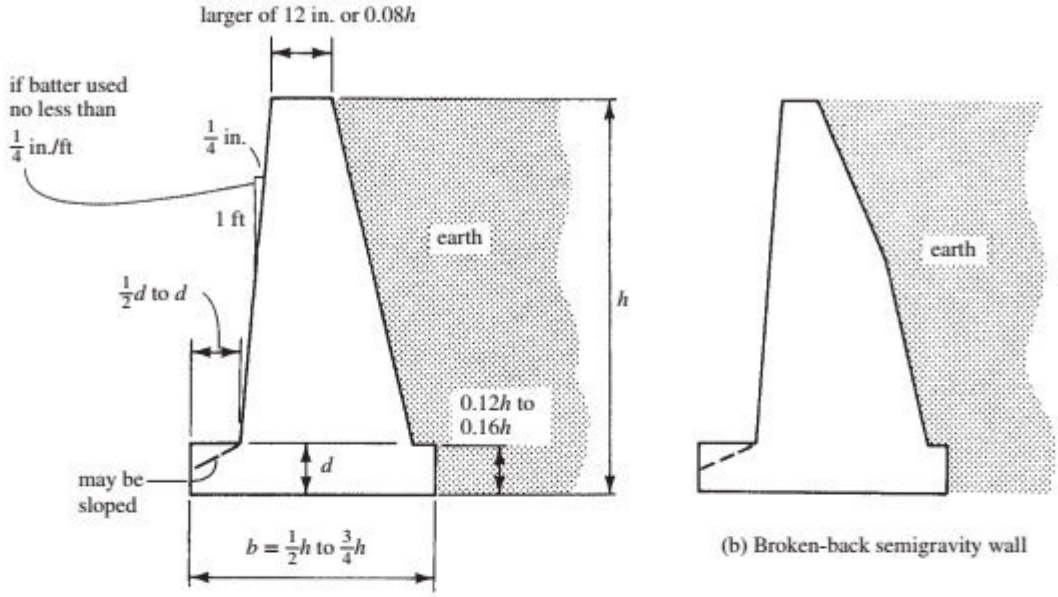
If the resultant force falls outside of the middle third of the footing, the preceding expressions are not applicable because they indicate a tensile stress on one side of the footing—a stress the soil cannot supply. For such cases, the soil pressures can be determined as previously described in Section 12.12 and illustrated in Figure 12.24 of Chapter 12. Such a situation should not be permitted in a retaining wall and is not considered further.

The soil pressures computed in this manner are only rough estimates of the real values and thus should not be valued too highly. The true pressures are appreciably affected by quite a few items other than the retaining wall weight. Included are drainage conditions, temperature, settlement, pore water, and so on.

13.7 Design of Semigravity Retaining Walls

As previously mentioned, semigravity retaining walls are designed to resist earth pressure by means of their own weight plus some developed soil weight. Because they are normally constructed with plain concrete, stone, or perhaps some other type of masonry, their design is based on the assumption that only very little tension or none at all can be permitted in the structure. If the resultant of the earth pressure and the wall weight (including any developed soil weight) falls within the middle third of the wall base, tensile stresses will probably be negligible.

A wall size is assumed, safety factors against sliding and overturning are calculated, the point where the resultant force strikes the base is determined, and the soil pressures are calculated. It is normally felt that safety factors against sliding should be at least 1.5 for cohesionless backfills and 2.0 for cohesive ones. Safety factors of 2.0 for overturning are normally specified. A suitable wall is probably obtained after two or three trial sizes. Example 13.1 illustrates the calculations that need to be made for each trial.



(a) Some approximate dimensions for semigravity walls

(b) Broken-back semigravity wall

FIGURE 13.8 Semigravity retaining walls.

Figure 13.8(a) shows a set of approximate dimensions that are often used for sizing semigravity walls. Dimensions may be assumed to be approximately equal to the values given and the safety factors against overturning and sliding computed. If the values are not suitable, the dimensions are adjusted and the safety factors are recalculated, and so on. Semigravity walls are normally trapezoidal in shape, as shown in Figure 13.8(a), but sometimes they may have broken backs, as illustrated in Figure 13.8(b).

Example 13.1

A semigravity retaining wall consisting of plain concrete (weight = 145 lb/ft^3) is shown in Figure 13.9. The bank of supported earth is assumed to weigh 110 lb/ft^3, to have a ϕ of 30°, and to have a coefficient of friction against sliding on soil of 0.5. Determine the safety factors against overturning and sliding and determine the bearing pressure underneath the toe of the footing. Use the Rankine expression for calculating the horizontal pressures.

SOLUTION

Computing the Soil Pressure Coefficients

$$k_a = \frac{1 - \sin\phi}{1 + \sin\phi} = \frac{1 - 0.5}{1 + 0.5} = 0.333$$

$$k_p = \frac{1 + \sin\phi}{1 - \sin\phi} = \frac{1 + 0.5}{1 - 0.5} = 3.00$$

Value of H_a

$$H_a = \frac{k_a w h^2}{2} = \frac{(0.333)\,(110\text{ pcf})\,(12\text{ ft})^2}{2} = 2637\text{ lb/ft}$$

Overturning Moment

$$\text{O.T.M.} = (2637\text{ lb/ft})\left(\frac{12\text{ ft}}{3}\right) = 10{,}548\text{ ft-lb/ft}$$

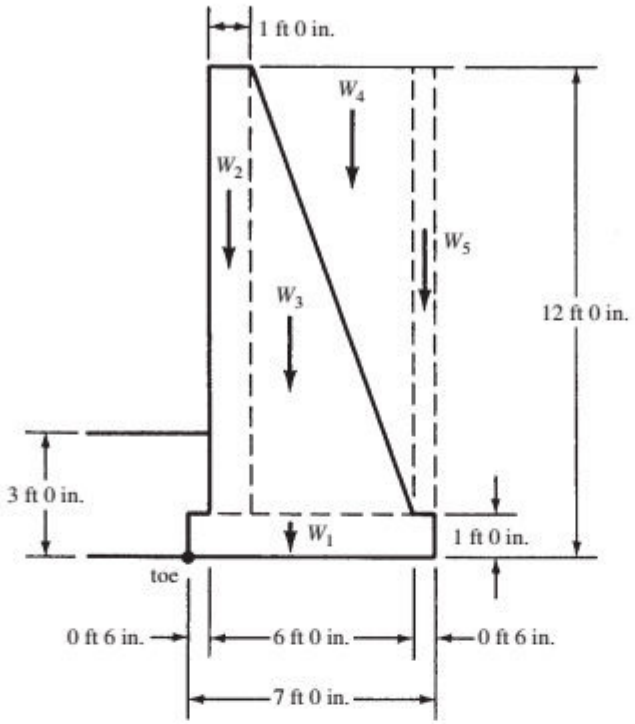


FIGURE 13.9 Semigravity retaining wall for Example 13.1.

Righting Moments (Taken about Toe)

Force		Moment arm		Moment
$W_1 = (7)\,(1)\,(145\text{ pcf})$	=	1015 lb × 3.5 ft	=	3552 ft-lb
$W_2 = (1)\,(11)\,(145\text{ pcf})$	=	1595 lb × 1.0 ft	=	1595 ft-lb
$W_3 = \left(\frac{1}{2}\right)(5)\,(11)\,(145\text{ pcf})$	=	3988 lb × 3.17 ft	=	12,642 ft-lb
$W_4 = \left(\frac{1}{2}\right)(5)\,(11)\,(110\text{ pcf})$	=	3025 lb × 4.83 ft	=	14,611 ft-lb
$W_5 = (0.5)\,(11)\,(110\text{ pcf})$	=	605 lb × 6.75 ft	=	4084 ft-lb
		R_v = 10,228 lb		M = 36,484 ft-lb

Safety Factor against Overturning (to Be Discussed at Some Length in Section 13.10)

$$\text{Safety factor} = \frac{36{,}484\text{ ft-lb}}{10{,}548\text{ ft-lb}} = 3.46 > 2.00 \quad \underline{\underline{\text{OK}}}$$

Safety Factor against Sliding (Also Discussed at Length in Section 13.10)

Assuming soil above the footing toe has eroded, and thus the passive pressure is due only to soil of a depth equal to footing thickness,

$$H_p = \frac{k_p w h'^2}{2} = \frac{(3.0)\,(110\text{ pcf})\,(1\text{ ft})^2}{2} = 165\text{ lb}$$

$$\text{Safety factor against sliding} = \frac{(0.5)\,(10{,}228\text{ lb}) + 165\text{ lb}}{2637\text{ lb}} = 2.00 > 1.50 \quad \underline{\underline{\text{OK}}}$$

Distance of Resultant from Toe

$$\text{Distance} = \frac{36{,}484 \text{ ft-lb} - 10{,}548 \text{ ft-lb}}{10{,}228 \text{ lb}} = 2.54 \text{ ft} > 2.33 \text{ ft} \qquad \therefore \text{ Inside middle third}$$

Soil Pressure under Heel and Toe

$$A = (1 \text{ ft})\,(7.0 \text{ ft}) = 7.0 \text{ ft}^2$$

$$I = \left(\frac{1}{12}\right)(1 \text{ ft})\,(7 \text{ ft})^3 = 28.58 \text{ ft}^4$$

$$f_{toe} = -\frac{R_v}{A} - \frac{R_v ec}{I} = -\frac{10{,}228 \text{ lb}}{7.0 \text{ ft}^2} - \frac{(10{,}228 \text{ lb})\,(3.50 \text{ ft} - 2.54 \text{ ft})\,(3.50 \text{ ft})}{28.58 \text{ ft}^4}$$

$$= -1461 \text{ psf} - 1202 \text{ psf} = -2663 \text{ psf}$$

$$f_{heel} = -\frac{R_v}{A} + \frac{R_v ec}{I} = -1461 \text{ psf} + 1202 \text{ psf} = -259 \text{ psf}$$

13.8 Effect of Surcharge

Should there be earth or other loads on the surface of the backfill, as shown in Figure 13.10, the horizontal pressure applied to the wall will be increased. If the surcharge is uniform over the sliding area behind the wall, the resulting pressure is assumed to equal the pressure that would be caused by an increased backfill height having the same total weight as the surcharge. It is usually easy to handle this situation by adding a uniform pressure to the triangular soil pressure for a wall without surcharge, as shown in the figure.

If the surcharge does not cover the area entirely behind the wall, some rather complex soil theories are available to consider the resulting horizontal pressures developed. As a consequence, the designer usually uses a rule of thumb to cover the case, a procedure that works reasonably well.

He or she may assume, as shown in Figure 13.11, that the surcharge cannot affect the pressure above the intersection of a 45° line from the edge of the surcharge to the wall. The lateral pressure is increased, as by a full surcharge, below the intersection point. This is shown on the right side of the figure.

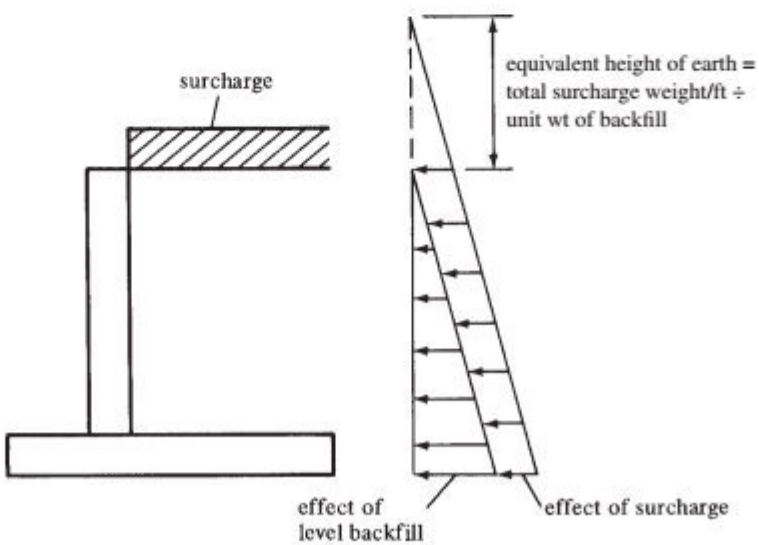


FIGURE 13.10 Equivalent height for surcharge.

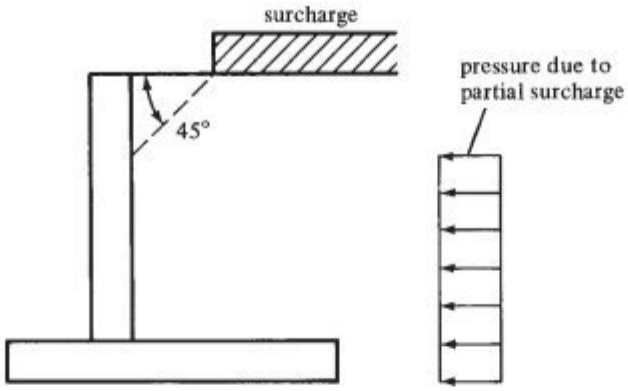


FIGURE 13.11 Effect of partial surcharge.

13.9 Estimating the Sizes of Cantilever Retaining Walls

The statical analysis of retaining walls and consideration of their stability as to overturning and sliding are based on service-load conditions. In other words, the length of the footing and the position of the stem on the footing are based entirely on the actual soil backfill, estimated lateral pressure, coefficient of sliding friction of the soil, and so on.

On the other hand, the detailed designs of the stem and footing and their reinforcement are determined by the strength design method. To carry out these calculations, it is necessary to multiply the service loads and pressures by the appropriate load factors. From these factored loads, the bearing pressures, moments, and shears are determined for use in the design.

Thus, the initial part of the design consists of an approximate sizing of the retaining wall. Although this is actually a trial-and-error procedure, the values obtained are not too sensitive to slightly incorrect values, and usually one or two trials are sufficient.

Various guidelines are available with which excellent initial size estimates can be made. In addition, various handbooks present the final sizes of retaining walls that have been designed for certain specific cases. This information will enable the designer to estimate very well the proportions of a wall to be designed. The *CRSI Design Handbook* is one such useful reference.[3] In the next few paragraphs, suggested methods are presented for estimating sizes without the use of a handbook. These approximate methods are very satisfactory as long as the conditions are not too much out of the ordinary.

Height of Wall

The necessary elevation at the top of the wall is normally obvious from the conditions of the problem. The elevation at the base of the footing should be selected so that it is below frost penetration in the particular area—about 3 ft to 6 ft below ground level in the northern part of the United States. From these elevations, the overall height of the wall can be determined.

Stem Thickness

Stems are theoretically thickest at their bases because the shears and moments are greatest there. They will ordinarily have total thicknesses somewhere in the range of 7% to 12% of the overall heights of the retaining walls. The shears and moments in the stem decrease from the bottom to the top; as a result,

[3] Concrete Reinforcing Steel Institute, 2008, *CRSI Design Handbook*, 10th ed. (Chicago, IL: CRSI), pp. 14-1 to 14-46.

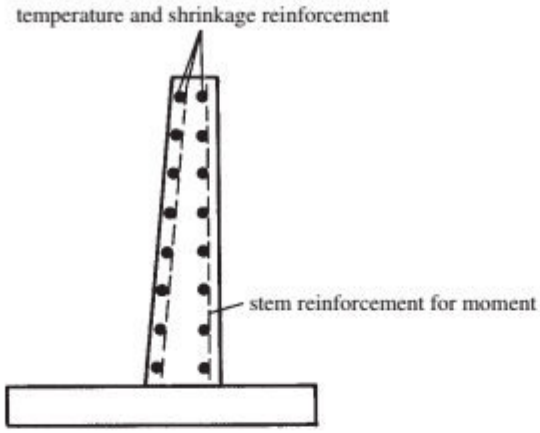


FIGURE 13.12 Cantilever retaining wall with tapered stem.

thicknesses and reinforcement can be reduced proportionately. Stems are normally tapered, as shown in Figure 13.12. The minimum thickness at the top of the stem is 8 in., with 12 in. preferable. As will be shown in Section 13.10, it is necessary to have a mat of reinforcing in the inside face of the stem and another mat in the outside face. To provide room for these two mats of reinforcing, for cover, and for spacing between the mats, a minimum total thickness of at least 8 in. is required.

The use of the minimum thickness possible for walls that are primarily reinforced in one direction (here it's the vertical bars) doesn't necessarily provide the best economy. The reason is that the reinforcing steel is a major part of the total cost. Making the walls as thin as possible will save some concrete but will substantially increase the amount of reinforcing steel needed. For fairly high and heavily loaded walls, greater thicknesses of concrete may be economical.

If ρ in the stem is limited to a maximum value of approximately $(0.18f'_c/f_y)$, the stem thickness required for moment will probably provide sufficient shear resistance without using stirrups. Furthermore, it will probably be sufficiently thick to limit lateral deflections to reasonable values.

For heights up to about 12 ft, the stems of cantilever retaining walls are normally made of constant thickness because the extra cost of setting the tapered formwork is usually not offset by the savings in concrete. Above 12 ft heights, concrete savings are usually sufficiently large to make tapering economical.

Actually, the sloping face of the wall can be either the front or the back, but if the outside face is tapered, it will tend to counteract somewhat the deflection and tilting of the wall because of lateral pressures. A taper or batter of $\frac{1}{4}$ in. per foot of height is often recommended to offset deflection or the forward tilting of the wall.

Base Thickness

The final thickness of the base will be determined on the basis of shears and moments. For estimating, however, its total thickness will probably fall somewhere between 7% and 10% of the overall wall height. Minimum thicknesses of at least 10 in. to 12 in. are used.

Base Length

For preliminary estimates, the base length can be taken to be about 40% to 60% of the overall wall height. A little better estimate, however, can be made by using the method described by the late Professor Ferguson in his reinforced concrete text.[4] For this discussion, reference is made to Figure 13.13.

[4] Ferguson, P. M., 1979, *Reinforced Concrete Fundamentals*, 4th ed. (Hoboken, NJ: John Wiley & Sons), p. 256.

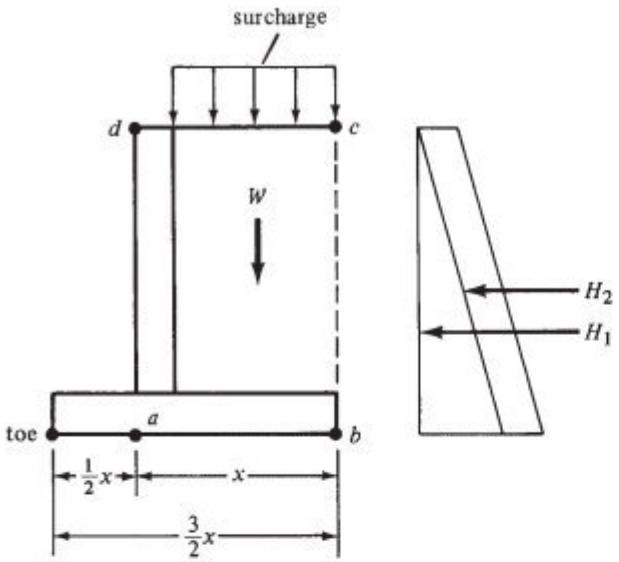


FIGURE 13.13 Forces acting on a cantilever retaining wall.

In this figure, W is assumed to equal the weight of all the material within area *abcd*. This area contains both concrete and soil, but the authors assume here that it is all soil. This means that a slightly larger safety factor will be developed against overturning than assumed. When surcharge is present, it will be included as an additional depth of soil, as shown in the figure.

If the sum of moments about point a due to W and the lateral forces H_1 and H_2 equal zero, the resultant force, R, will pass through point a. Such a moment equation can be written, equated to zero, and solved for x. Should the distance from the footing toe to point a be equal to one-half of the distance x in the figure and the resultant force, R, pass through point a, the footing pressure diagram will be triangular. In addition, if moments are taken about the toe of all the loads and forces for the conditions described, the safety factor against overturning will be approximately two.

A summary of the preceding approximate first trial sizes for cantilever retaining walls is shown in Figure 13.14. These sizes are based on the dimensions of walls successfully constructed in the past. They often will be on the conservative side.

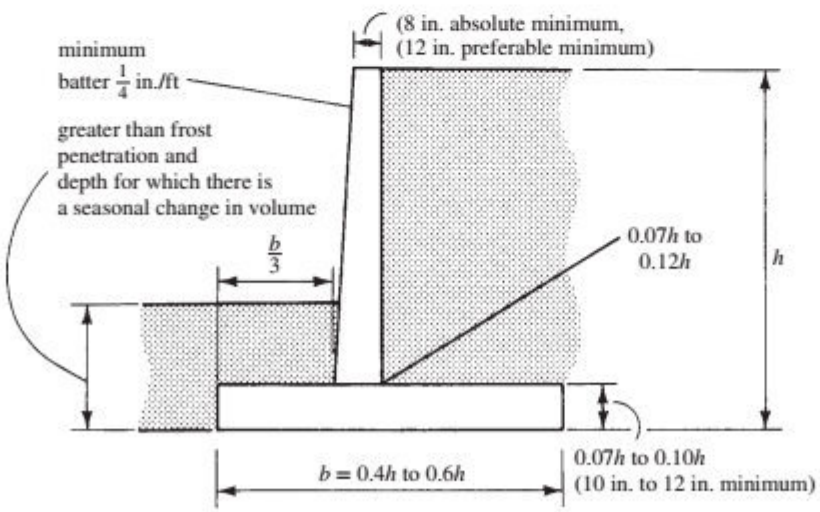


FIGURE 13.14 Guidelines for proportioning cantilever retaining walls.

Example 13.2

Using the approximate rules presented in this section, estimate the sizes of the parts of the retaining wall shown in Figure 13.15. The soil weighs 100 lb/ft^3, and a surcharge of 300 psf is present. Assume $k_a = 0.32$. (For many practical soils such as clays or silts, k_a will be two or more times this large.)

SOLUTION

Stem Thickness

Assume 12 in. thickness at top.

$$\text{Assume bottom thickness} = 0.07h = (0.07)\,(21\text{ ft}) = 1.47\text{ ft} \qquad \underline{\text{Say 1 ft 6 in.}}$$

Base Thickness

Assume base $t = 7\%$ to 10% of overall wall height.

$$t = (0.07)\,(21\text{ ft}) = 1.47\text{ ft} \qquad \underline{\text{Say 1 ft 6 in.}}$$

$$\text{Height of stem} = 21\text{ ft } 0\text{ in.} - 1\text{ ft } 6\text{ in.} = \underline{19\text{ ft } 6\text{ in.}}$$

Base Length and Position of Stem

Calculating horizontal forces without load factors, as shown in Figure 13.16.

$$p_a = k_a wh = (0.32)\,(100\text{ pcf})\,(21\text{ ft}) = 672\text{ lb/ft}^2$$

$$H_1 = \left(\frac{1}{2}\right)(21\text{ ft})\,(672\text{ lb/ft}^2) = 7056\text{ lb/ft}$$

$$H_2 = (21\text{ ft})\,(96\text{ psf}) = 2016\text{ lb/ft}$$

$$W = (x)\,(24\text{ ft})\,(100\text{ psf}) = 2400x$$

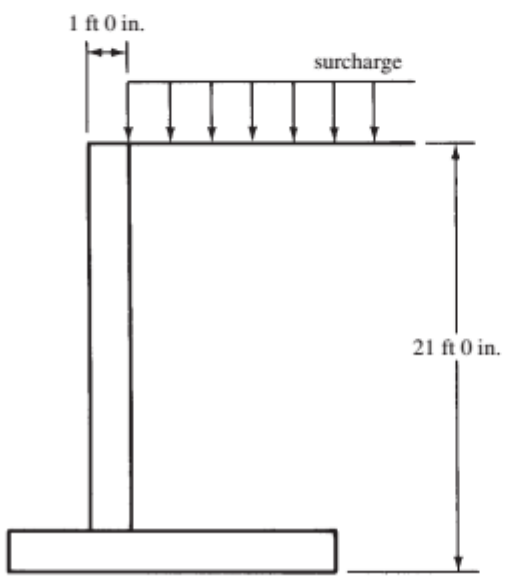


FIGURE 13.15 Cantilever retaining wall for Example 13.2.

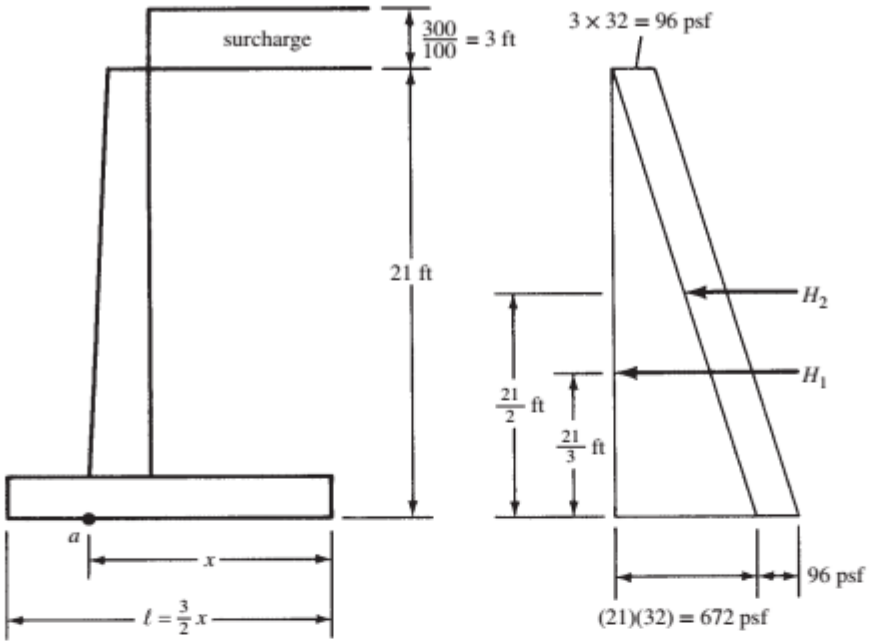


FIGURE 13.16 Forces acting on retaining wall for Example 13.2.

$\Sigma M_a = 0$

$$-(7056 \text{ lb/ft})\,(7.00 \text{ ft}) - (2016 \text{ lb/ft})\,(10.5 \text{ ft}) + (2400x)\left(\frac{x}{2}\right) = 0$$

$$x = 7.67 \text{ ft}$$

$$b = \left(\frac{3}{2}\right)(7.67 \text{ ft}) = 11.505 \text{ ft} \qquad \text{Say 11 ft 6 in.}$$

The final trial dimensions are shown in Figure 13.22.

13.10 Design Procedure for Cantilever Retaining Walls

This section is presented to describe in some detail the procedure used for designing a cantilever retaining wall. At the end of this section, the complete design of such a wall is presented. Once the approximate size of the wall has been established, the stem, toe, and heel can be designed in detail. Each of these parts will be designed individually as a cantilever sticking out of a central mass, as shown in Figure 13.17.

Stem

The values of shear and moment at the base of the stem resulting from lateral earth pressures are computed and used to determine the stem thickness and necessary reinforcement. Because the lateral pressures are considered to be live load forces, a load factor of 1.6 is used.

It will be noted that the bending moment requires the use of vertical reinforcing bars on the soil side of the stem. In addition, temperature and shrinkage reinforceing must be provided. In Section 11.6.1 of the ACI Code, a minimum value of horizontal reinforcement equal to 0.0025 of the area of the wall, bt, is required as well as a minimum amount of vertical reinforcement (0.0015). These values may be reduced to 0.0020 and 0.0012 if the reinforcement is $\frac{5}{8}$ in. or less in diameter and if it consists of bars or welded wire fabric (not larger than $W31$ or $D31$), with f_y equal to or greater than 60,000 psi.

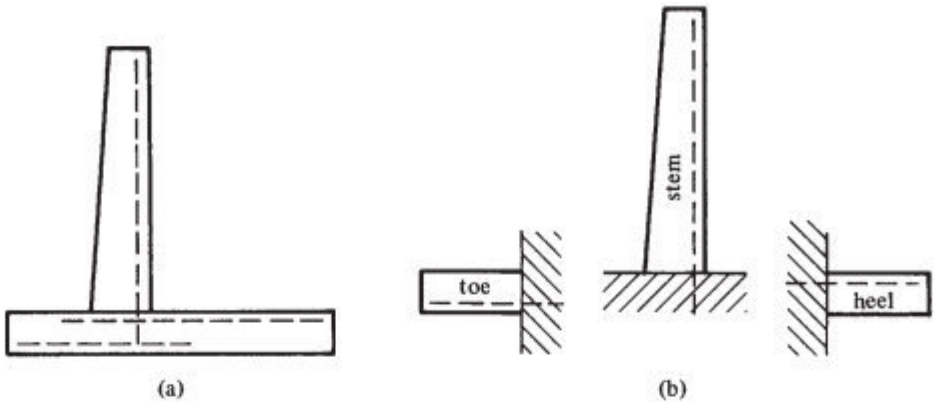


FIGURE 13.17 Cantilever beam model used to design retaining wall stem, heel, and toe.

The major changes in temperature occur on the front or exposed face of the stem. For this reason, most of the horizontal reinforcement (perhaps two-thirds) should be placed on that face with just enough vertical steel used to support the horizontal bars. The concrete for a retaining wall should be placed in fairly short lengths—not greater than 20-ft or 30-ft sections—to reduce shrinkage stresses.

Factor of Safety against Overturning

Moments are taken about the toe of the unfactored overturning and righting forces. Traditionally, it has been felt that the safety factor against overturning should be at least equal to 2. In making these calculations, backfill on the toe is usually neglected because it may very well be eroded. Of course, there are cases where there is a slab (e.g., a highway pavement on top of the toe backfill) that holds the backfill in place over the toe. For such situations, it may be reasonable to include the loads on the toe.

Factor of Safety against Sliding

Consideration of sliding for retaining walls is a most important topic because a very large percentage of retaining wall failures occur because of sliding. To calculate the factor of safety against sliding, the estimated sliding resistance (equal to the coefficient of friction for concrete on soil times the resultant vertical force, μR_v) is divided by the total horizontal force. The passive pressure against the wall is neglected, and the unfactored loads are used.

Typical design values of μ, the coefficient of friction between the footing concrete and the supporting soil, are as follows: 0.45 to 0.55 for coarse-grained soils, with the lower value applying if some silt is present, and 0.6 if the footing is supported on sound rock with a rough surface. Values of 0.3 to 0.35 are used if the supporting material is silt.

It is usually felt that the factor of safety against sliding should be at least equal to 1.5. When retaining walls are initially designed, the calculated factor of safety against sliding is very often considerably less than this value. To correct the situation, the most common practice is to widen the footing on the heel side. Another practice is to use a lug or key, as shown in Figure 13.18, with the front face cast directly against undisturbed soil. (Many designers feel that the construction of keys disturbs the soil so much that they are not worthwhile.) Keys are thought to be particularly necessary for moist clayey soils. The purpose of a key is to cause the development of passive pressure in front of and below the base of the footing, as shown by P_p in the figure. The actual theory involved, and thus the design of keys, is still a question among geotechnical engineers. As a result, many designers select the sizes of keys by rules of thumb. One common practice is to give them a depth between two-thirds and the full depth of the footing. They are usually made approximately square in cross section and have no reinforcement provided other than perhaps the dowels mentioned in the next paragraph.

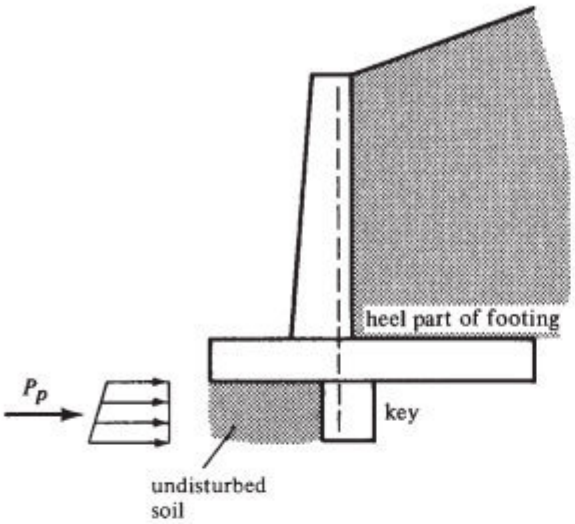


FIGURE 13.18 Passive pressure in a lug.

Courtesy of EFCO Corp.

El Teniente Copper Mine, Rancagua, Chile.

Keys are often located below the stem so that some dowels or extended vertical reinforcing bars may be extended into them. If this procedure is used, the front face of the key needs to be at least 5 in. or 6 in. in front of the back face of the stem to allow room for the dowels. From a soil mechanics view, keys may be a little more effective if they are placed a little farther toward the heel.

If the key can be extended down into a very firm soil or even rock, the result will be a greatly increased sliding resistance—that resistance being equal to the force necessary to shear the key off from the footing, that is, a shear friction calculated as described in Sections 8.12 and 12.13 of this text.

Heel Design

Lateral earth pressure tends to cause the retaining wall to rotate about its toe. This action tends to pick up the heel into the backfill. The backfill pushes down on the heel cantilever, causing tension in its top. The major force applied to the heel of a retaining wall is the downward weight of the backfill behind the wall. Although it is true that there is some upward soil pressure, many designers choose to neglect it because it is relatively small. The downward loads tend to push the heel of the footing down, and the necessary upward reaction to hold it attached to the stem is provided by the vertical tensile steel in the stem, which is extended down into the footing.

Because the reaction in the direction of the shear does not introduce compression into the heel part of the footing in the region of the stem, it is not permissible to determine V_u at a distance d from the face of the stem, as provided in Section 7.4.3.2 of the ACI Code. The value of V_u is determined instead at the face of the stem because of the downward loads. This shear is often of such magnitude as to control the thickness, but the moment at the face of the stem should be checked also. Because the load here consists of soil and concrete, a load factor of 1.2 is used for making the calculations.

It will be noted that the bars in the heel will be in the top of the footing. As a result, the required development length of these "top bars" may be rather large.

The percentage of flexural steel required for the heel frequently is less than the $\rho_{\min}$ of $200/f_y$ and $3\sqrt{f'_c}/f_y$. Despite the fact that the ACI Code, Section 7.6.1.1 exempts slabs of uniform from these $\rho_{\min}$ values, the authors recommend that these be used because the retaining wall is a major beamlike structure.

Toe Design

The toe is assumed to be a beam cantilevered from the front face of the stem. The loads it must support include the weight of the cantilever slab and the upward soil pressure beneath. Usually any earth fill on top of the toe is neglected (as though it has been eroded). Obviously, such a fill would increase the upward soil pressure beneath the footing, but because it acts downward and cancels out the upward pressure, it produces no appreciable changes in the shears and moments in the toe.

A study of Figure 13.19 shows that the upward soil pressure is the major force applied to the toe. Because this pressure is primarily caused by the lateral force H, a load factor of 1.6 is used for the calculations. (Section 4.1 of this text shows that all load combinations including soil loads have a load factor of 1.6 associated with H.) The maximum moment for design is taken at the face of the stem, whereas the maximum shear for design is assumed to occur at a distance d from the face of the stem because the reaction in the direction of the shear does introduce compression into the toe of the footing. The average designer makes the thickness of the toe the same as the thickness of the heel, although such a practice is not essential.

It is a common practice in retaining wall construction to provide a shear keyway between the base of the stem and the footing. This practice, although definitely not detrimental, is of questionable value. The keyway is normally formed by pushing a beveled timber of 2 in. × 4 in. or 2 in. × 6 in. into the top of the footing, as shown in Figure 13.20. After the concrete hardens, the wood member is removed, and when the stem is cast in place above, a keyway is formed. It is becoming more and more common

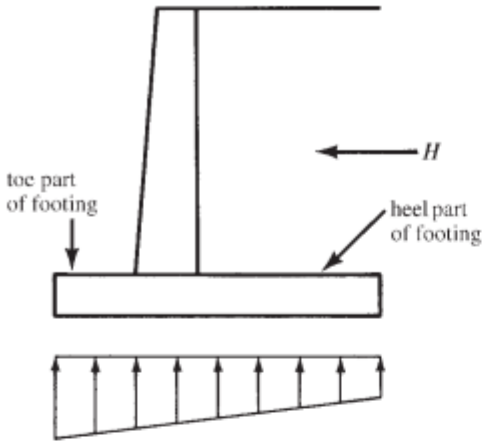


FIGURE 13.19 Assumed soil stress distribution at the base.

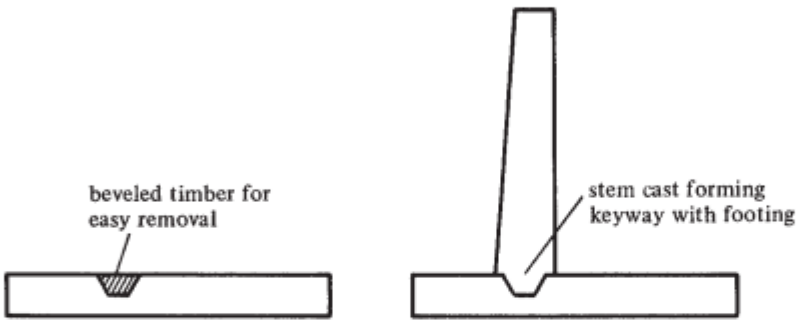


FIGURE 13.20 Keyway for improved shear capacity.

simply to use a roughened surface on the top of the footing where the stem will be placed. This practice seems to be just as satisfactory as the use of a keyway.

In Example 13.3, #8 bars 6 in. on center are selected for the vertical steel at the base of the stem. These bars need to be embedded into the footing for development purposes, or dowels equal to the stem steel need to be used for the transfer. This latter practice is quite common because it is rather difficult to hold the stem steel in position while the base concrete is placed.

The required development length of the #8 bars down into the footing or for #8 dowels is 33 in. when $f_y = 60{,}000$ psi and $f'_c = 3000$ psi. This length cannot be obtained vertically in the 1-ft-6-in. footing used unless the bars or dowels are either bent as shown in Figure 13.21(a) or extended through the

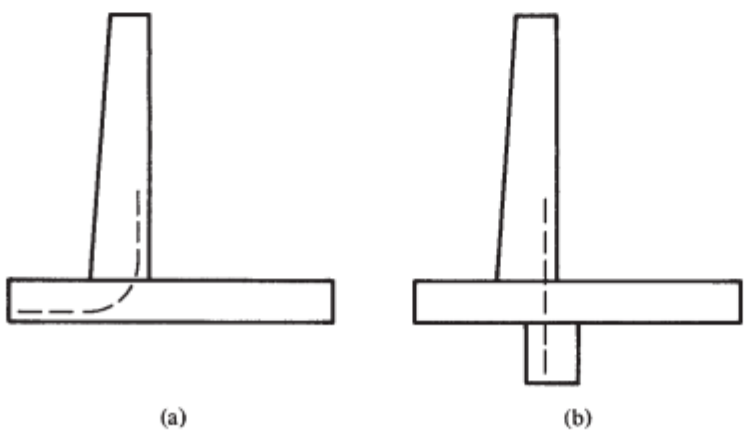


FIGURE 13.21 Bar development options.

footing and into the base key as shown in Figure 13.21(b). Actually, the required development length can be reduced if more but smaller dowels are used. For example, if #6 dowels are used, ℓ_d is 20 in.

If instead of dowels the vertical stem bars are embedded into the footing, they should not extend up into the wall more than 8 ft or 10 ft before they are spliced because they are difficult to handle in construction and may easily be bent out of place or even broken. Actually, after examining Figure 13.21(a), you can see that such an arrangement of stem steel can sometimes be very advantageous economically.

The bending moment in the stem decreases rapidly above the base; as a result, the amount of reinforcement can be similarly reduced. It is to be remembered that these bars can be cut off only in accordance with the ACI Code development length requirements.

Example 13.3 illustrates the detailed design of a cantilever retaining wall. Several important descriptive remarks are presented in the solution, and these should be carefully read.

Example 13.3

Complete the design of the cantilever retaining wall whose dimensions were estimated in Example 13.2 and are shown in Figure 13.22 if $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, $q_a = 4000$ psf, and the coefficient of sliding friction equals 0.50 for concrete on soil. Use ρ approximately equal to $0.18f'_c/f_y$ to maintain reasonable deflection control.

SOLUTION

The safety factors against overturning and sliding and the soil pressures under the heel and toe are computed using the actual unfactored loads.

$$\text{Safety factor against overturning} = \frac{149{,}456 \text{ ft-lb}}{70{,}560 \text{ ft-lb}} = 2.12 > 2.00 \quad \underline{\underline{\text{OK}}}$$

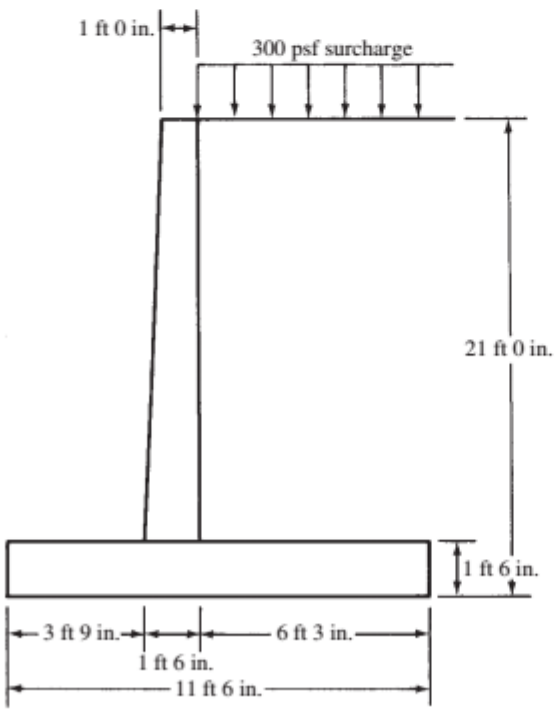


FIGURE 13.22 Dimensions of retaining wall for Example 13.3.

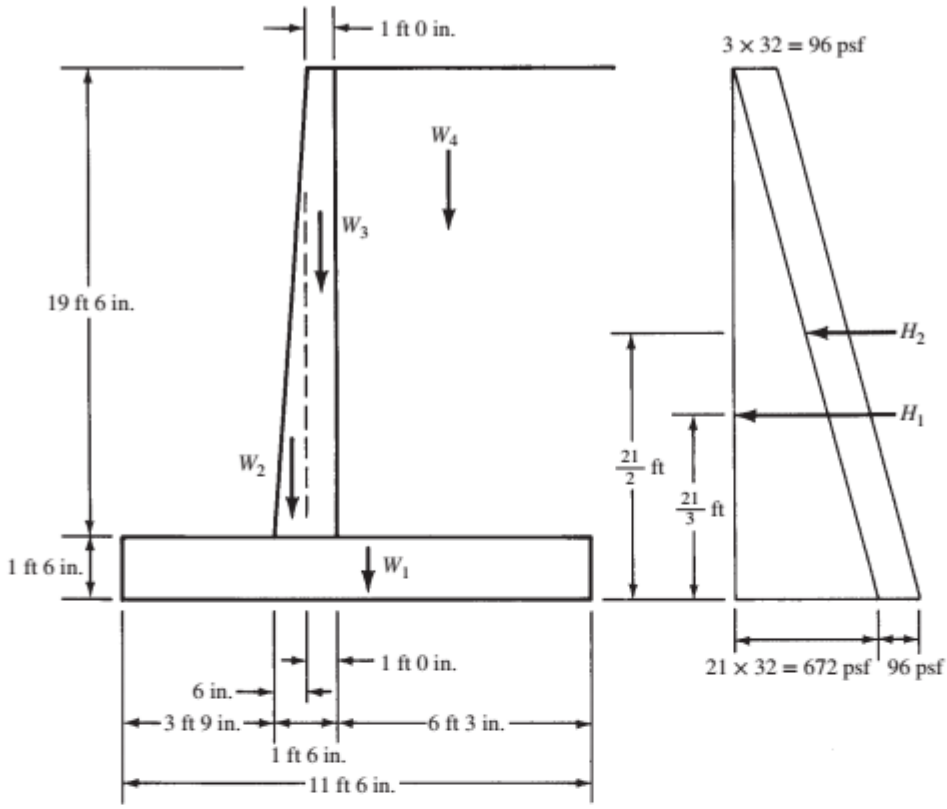


FIGURE 13.23 Forces acting on retaining wall for Example 13.3.

Safety Factor against Overturning (with Reference to Figure 13.23)

Overturning moment		
Force	Moment arm	Moment
$H_1 = \left(\frac{1}{2}\right)$ (21 ft) (672 psf) = 7056 lb ×	7.00 ft =	49,392 ft-lb
H_2 = (21 ft) (96 psf) = 2016 lb ×	10.50 ft =	21,168 ft-lb
Total		70,560 ft-lb

Righting moment			
Force		Moment arm	Moment
W_1 = (1.5 ft) (11.5 ft) (150 pcf)	= 2588 lb ×	5.75 ft =	14,881 ft-lb
$W_2 = \left(\frac{1}{2}\right)$ (19.5 ft) $\left(\frac{6}{12}\right.$ ft$\left.\right)$ (150 pcf) =	731 lb ×	4.08 ft =	2982 ft-lb
$W_3 = (19.5 \text{ ft}) \left(\frac{12}{12} \text{ ft}\right)$ (150 pcf)	= 2925 lb ×	4.75 ft =	13,894 ft-lb
W_4 = (22.5 ft) (6.25 ft) (100 pcf)	= 14,062 lb ×	8.37 ft =	117,699 ft-lb*
	R_v = 20,306 lb	M =	149,456 ft-lb

* Includes surcharge.

Factor of Safety against Sliding

Here the passive pressure against the wall is neglected. Normally it is felt that the factor of safety should be at least 1.5. If it is not satisfactory, a little wider footing on the heel side will easily take care of the situation. In addition to or instead of this solution, a key, perhaps 1 ft 6 in. × 1 ft 6 in. (size selected to provide sufficient development length for the dowels selected later in this design) can be used. Space is not taken here to improve this safety factor.

$$\text{Force causing sliding} = H_1 + H_2 = 9072 \text{ lb}$$

$$\text{Resisting force} = \mu R_v = (0.50)\,(20{,}306 \text{ lb}) = 10{,}153 \text{ lb}$$

$$\text{Safety factor} = \frac{10{,}153 \text{ lb}}{9072 \text{ lb}} = 1.12 < 1.50 \quad \text{No good}$$

Footing Soil Pressures

$R_v = 20{,}306$ lb and is located a distance $\bar{x}$ from the toe of the footing

$$\bar{x} = \frac{149{,}456 \text{ ft-lb} - 70{,}560 \text{ ft-lb}}{20{,}306 \text{ lb}} = \frac{78{,}896 \text{ ft-lb}}{20{,}306 \text{ lb}} = 3.89 \text{ ft} \quad \text{Just inside middle third}$$

$$\text{Soil pressure} = -\frac{R_v}{A} \pm \frac{Mc}{I}$$

$$A = (1 \text{ ft})\,(11.5 \text{ ft}) = 11.5 \text{ ft}^2$$

$$I = \left(\frac{1}{12}\right)(1 \text{ ft})\,(11.5 \text{ ft})^3 = 126.74 \text{ ft}^4$$

$$f_{toe} = -\frac{20{,}306 \text{ lb}}{11.5 \text{ ft}^2} - \frac{(20{,}306 \text{ lb})\,(5.75 \text{ ft} - 3.89 \text{ ft})\,(5.75 \text{ ft})}{126.74 \text{ ft}^4}$$

$$= -1766 \text{ psf} - 1714 \text{ psf} = -3480 \text{ psf}$$

$$f_{heel} = -1766 \text{ psf} + 1714 \text{ psf} = -52 \text{ psf}$$

Design of Stem

The lateral forces applied to the stem are calculated using a load factor of 1.6, as shown in Figure 13.24.

Design of Stem for Moment

$$M_u = (H_1)\,(6.50 \text{ ft}) + (H_2)\,(9.75 \text{ ft}) = (9734 \text{ lb})\,(6.50 \text{ ft}) + (2995 \text{ lb})\,(9.75 \text{ ft})$$

$$M_u = 92{,}472 \text{ ft-lb}$$

Use

$$\rho = \text{approximately } \frac{0.18 f'_c}{f_y} = \frac{(0.18)\,(3000 \text{ psi})}{60{,}000 \text{ psi}} = 0.009$$

$$\frac{M_u}{\phi b d^2} = 482.6 \text{ psi (from Appendix A, Table A.12)}$$

$$bd^2 = \frac{(12 \text{ in./ft})(92{,}472 \text{ ft-lb})}{(0.9)(482.6 \text{ psi})} = 2555 \text{ in.}^3$$

$$d = \sqrt{\frac{2555 \text{ in.}^3}{12 \text{ in.}}} = 14.59 \text{ in.}$$

$$h = 14.59 \text{ in.} + 2 \text{ in.} + \frac{1 \text{ in.}}{2} = 17.09 \text{ in.} \quad \text{Say 18 in. } (d = 15.50 \text{ in.})$$

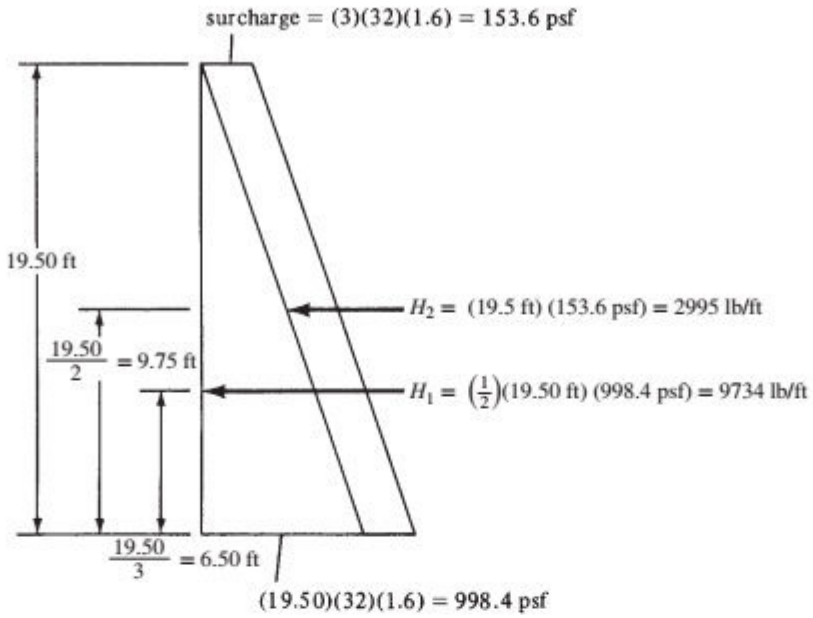


FIGURE 13.24 Lateral forces from backfill and surcharge.

$$\frac{M_u}{\phi bd^2} = \frac{(12 \text{ in./ft})\,(92{,}472 \text{ ft-lb})}{(0.90)\,(12 \text{ in.})\,(15.5 \text{ in.})^2} = 427.7 \text{ psi}$$

$\rho = 0.00786$ (from Appendix A, Table A.12)

$A_s = (0.00786)\,(12 \text{ in.})\,(15.5 \text{ in.}) = 1.46 \text{ in.}^2$ Use #8 @ 6 in. (1.57 in.^{2}/ft)

Minimum vertical ρ by ACI Code, Section 11.6.1 = 0.0015 $< \dfrac{1.57 \text{ in.}^2}{(12 \text{ in.})\,(15.5 \text{ in.})} = 0.0084$ OK

Minimum horizontal $A_s = (0.0025)\,(12 \text{ in.})$ (average stem t)

$$= (0.0025)\,(12 \text{ in.})\left(\frac{12 \text{ in.} + 18 \text{ in.}}{2}\right) = 0.450 \text{ in.}^2$$

(say one-third inside face and two-thirds outside face)

Use #4 @ $7\frac{1}{2}$ in. outside face and #4 @ 15 in. inside face

Checking Shear Stress in Stem

Actually, V_u at a distance d from the top of the footing can be used, but for simplicity:

$$V_u = H_1 + H_2 = 9734 \text{ lb} + 2995 \text{ lb} = 12{,}729 \text{ lb}$$

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}bd = (0.75)\,(2)\,(1.0)\,(\sqrt{3000} \text{ psi})\,(12 \text{ in.})\,(15.5 \text{ in.})$$

$= 15{,}281 \text{ lb} > 12{,}729 \text{ lb}$ OK

Design of Heel

The upward soil pressure is conservatively neglected, and a load factor of 1.2 is used for calculating the shear and moment because soil and concrete make up the load.

$$V_u = (22.5 \text{ ft})\,(6.25 \text{ ft})\,(100 \text{ pcf})\,(1.2) + (1.5 \text{ ft})\,(6.25 \text{ ft})\,(150 \text{ pcf})\,(1.2) = 18{,}563 \text{ lb/ft}$$

$\phi V_c = (0.75)\,(2)\,(1.0)\,(\sqrt{3000} \text{ psi})\,(12 \text{ in.})\,(14.5 \text{ in.}) = 14{,}295 \text{ lb} < 18{,}563 \text{ lb}$ No good

Try 24 in. Depth (d = 20.5 in.)

Neglecting slight change in V_u with different depth

$$\phi V_c = (0.75)\,(2)\,(1.0)\,(\sqrt{3000}\text{ psi})\,(12\text{ in.})\,(20.5\text{ in.})$$

$$= 20{,}211\text{ lb} > 18{,}563\text{ lb} \quad \underline{\text{OK}}$$

$$M_u \text{ at face of stem} = (18{,}563\text{ lb})\left(\frac{6.25\text{ ft}}{2}\right) = 58{,}009\text{ ft-lb}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12\text{ in./ft})\,(58{,}009\text{ ft-lb})}{(0.9)\,(12\text{ in.})\,(20.5\text{ in.})^2} = 153\text{ psi}$$

$$\rho = \rho_{\min}$$

Using $\rho = 0.00333$,

$$A_s = (0.00333)\,(12\text{ in.})\,(20.5\text{ in.}) = 0.82\text{ in.}^2\text{/ft} \qquad \underline{\text{Use \#8 @ 11 in.}}$$

ℓ_d required calculated with ACI Equation 25.4.2.3a for #8 top bars with $c_b = 2.50$ in. and $K_{tr} = 0$ is 43 in. < 72 in. available. <u>OK</u>

Heel reinforcement is shown in Figure 13.25.

Note: Temperature and shrinkage steel is normally considered unnecessary in the heel and toe. However, the authors have placed #4 bars at 18 in. on center in the long direction, as shown in Figures 13.25 and 13.27, to serve as spacers for the flexural steel and to form mats out of the reinforcing.

Design of Toe

For service loads, the soil pressures previously determined are multiplied by a load factor of 1.6 because they are primarily caused by the lateral forces, as shown in Figure 13.26.

$$V_u = 10{,}440\text{ lb} + 7086\text{ lb} = 17{,}526\text{ lb}$$

(The shear can be calculated a distance d from the face of the stem because the reaction in the direction of the shear does introduce compression into the toe of the slab, but this advantage is neglected because 17,526 lb is already less than the 19,125 lb shear in the heel, which was satisfactory.)

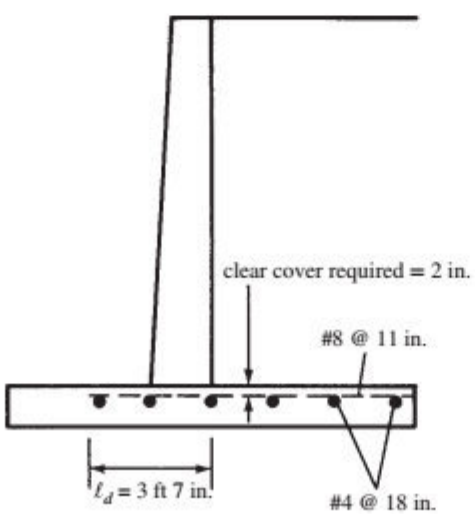


FIGURE 13.25 Heel reinforcement.

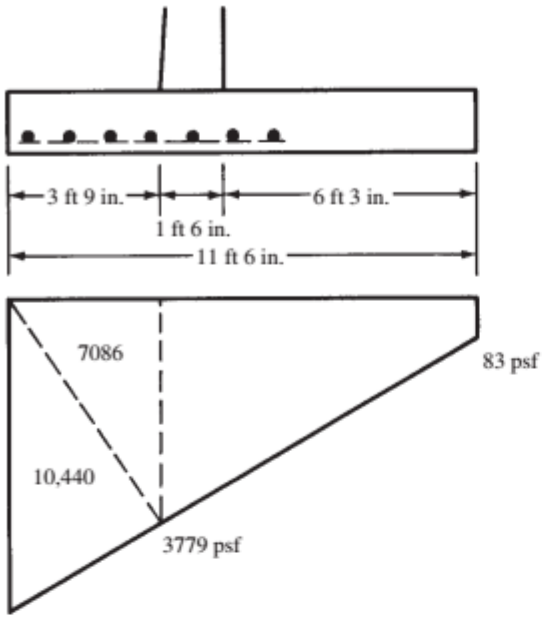


FIGURE 13.26 Soil reactions.

$$M_u \text{ at face of stem} = (7086 \text{ lb})\left(\frac{3.75 \text{ ft}}{3}\right) + (10{,}440 \text{ lb})\left(\frac{2}{3} \times 3.75 \text{ ft}\right) = 34{,}958 \text{ ft-lb}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})\,(34{,}958 \text{ ft-lb})}{(0.9)\,(12 \text{ in.})\,(20.5 \text{ in.})^2} = 92 \text{ psi}$$

$$\rho = \text{less than } \rho_{min}$$

Therefore, use

$$\frac{200}{60{,}000 \text{ psi}} = 0.00333$$

$$A_s = (0.00333)\,(12 \text{ in.})\,(20.5 \text{ in.}) = 0.82 \text{ in.}^2\text{/ft} \qquad \underline{\text{Use \#8 @ 11 in.}}$$

ℓ_d required calculated with ACI Equation 25.4.2.3a for #8 bottom bars with $c_b = 2.50$ in. and $K_{tr} = 0$ equals 33 in. < 42 in. available. OK

Toe reinforcement is shown in Figure 13.27.

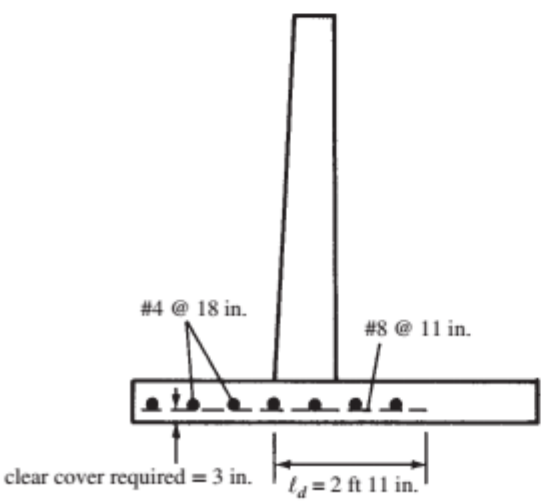


FIGURE 13.27 Toe reinforcement.

TABLE 13.1 Stem Design for Example 13.3

Distance from Top of Stem (ft)	M_u (ft-lb)	Effective Stem d (in.)	ρ	$A_{s\ \text{required}}$ (in.^{2}/ft)	Bars Needed
5	2987	11.04	Use $\rho_{min} = 0.00333$	0.44	#8 @ 18 in.
10	16,213	12.58	Use $\rho_{min} = 0.00333$	0.50	#8 @ 18 in.
15	46,080	14.12	0.00452	0.77	#8 @ 12 in.
19.5	92,472	15.50	0.00786	1.46	#8 @ 6 in.

Selection of Dowels and Lengths of Vertical Stem Reinforcing Bars

The detailed selection of vertical bar lengths in the stem is omitted here to save space, and only a few general comments are presented. Table 13.1 shows the reduced bending moments up in the stem and the corresponding reductions in reinforcing required.

After considering the possible arrangements of the steel in Figure 13.21 and the required areas of steel at different elevations in Table 13.1, the authors decided to use dowels for load transfer at the stem base.

Use #8 dowels at 6 in. extending 33 in. down into footing and key.

If these dowels are spliced to the vertical stem reinforcing with no more than one-half the bars being spliced within the required lap length, the splices will fall into the class B category (ACI Code, Section 25.5.2.1), and their lap length should at least equal $1.3\ell_d = (1.3)\ (33) = 43$ in. Therefore, two dowel lengths are used—half 3 ft 7 in. up into the stem and the other half 7 ft 2 in.—and the #7 bars are lapped over them, half running to the top of the wall and the other half to middepth. Actually, a much more refined design can be made that involves more cutting of bars. For such a design, a diagram comparing the theoretical steel area required at various elevations in the stem and the actual steel furnished is very useful. It is to be remembered (ACI Code, Sections 7.7.3.3 and 11.1.4) that the bars cut off must run at least a distance d or 12 diameters beyond their theoretical cutoff points and must also meet the necessary development length requirements.

13.11 Cracks and Wall Joints

Objectionable horizontal cracks are rare in retaining walls because the compression faces are the ones that are visible. When they do occur, it is usually a sign of an unsatisfactory structural design and not shrinkage. In Chapter 6 of this book, the ACI procedure (Section 24.3.1) for limiting crack sizes in tensile zones of one-way beams and slabs was presented. These provisions may be applied to vertical retaining wall steel. However, they are usually thought unnecessary because the vertical steel is on the earth side of the wall.

Vertical cracks in walls, however, are quite common unless sufficient construction joints are used. Vertical cracks are related to the relief of tension stresses because of shrinkage, with the resulting tensile forces exceeding the longitudinal steel capacity.

Construction joints may be used both horizontally and vertically between successive pours of concrete. The surface of the hardened concrete can be cleaned and roughened, or keys can be used as shown in Figure 13.28(a) to form horizontal construction joints.

If concrete is restrained from free movement when shrinking—for example, by being attached to more rigid parts of the structure—it will crack at points of weakness. Contraction joints are weakened places constructed so that shrinkage failures will occur at prepared locations. When the shrinkage tensile

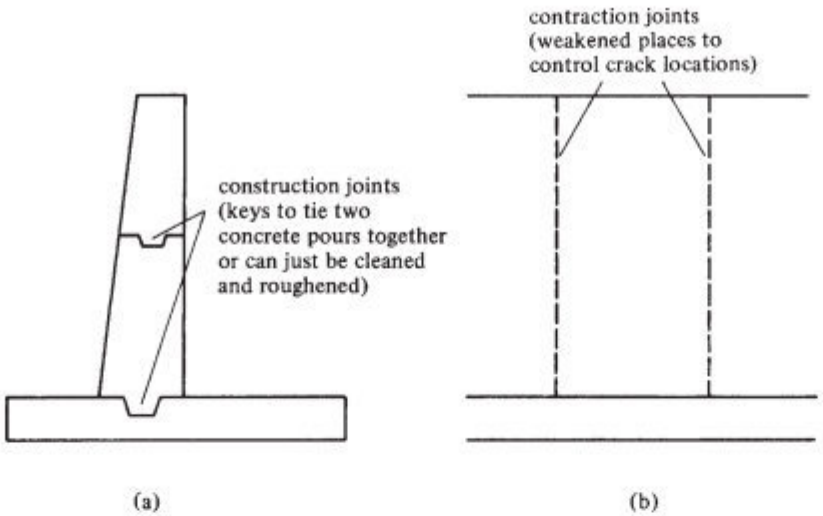


FIGURE 13.28 Examples of construction joints.

stresses become too large, they will pull these contraction joints apart and form neat cracks rather than the crooked, unsightly ones that might otherwise occur. In addition to handling shrinkage problems, contraction joints are useful in handling differential settlements. They need to be spaced at intervals about 25 ft on center. The joints are usually constructed with rubber strips that are left in place or with wood strips that are later removed and replaced with caulking.

Expansion joints are vertical joints that completely separate the different parts of a wall. They are placed approximately 50 ft to 100 ft on centers. Reinforcing bars are generally run through all joints so that vertical and horizontal alignment is maintained. When the bars do run through a joint, one end of the bars on one side of the joint is either greased or sheathed so that the desired expansion can take place.

Courtesy of EFCO Corp.

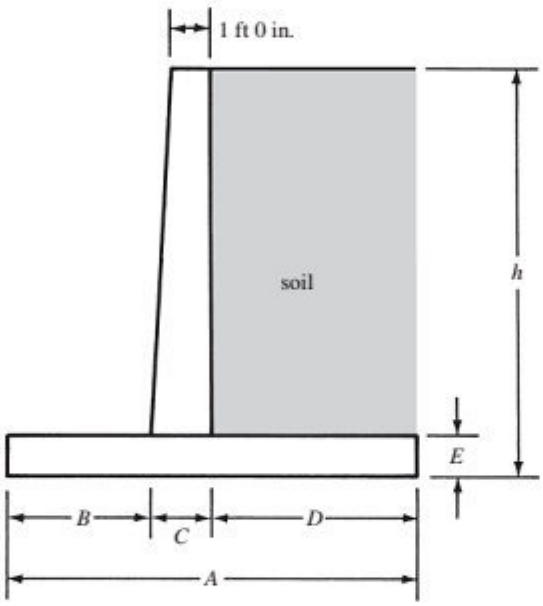
One World Trade Center, New York City, New York.

It is difficult to estimate the amount of shrinkage or expansion of a particular wall because the wall must slide on the soil beneath, and the resulting frictional resistance may be sufficient so that movement will be greatly reduced or even prevented. A rough value for the width of an expansion joint can be determined from the following expression, in which ΔL is the change in length, L is the distance between joints, ΔT is the estimated temperature change, and 0.000005 per unit length per degree Fahrenheit is the estimated coefficient of contraction of the wall.

$$\Delta L = (0.000005L)\,(\Delta T)$$

PROBLEMS

For Problems 13.1 to 13.4, use the Rankine equation to calculate the total horizontal active force and the overturning moment for the wall shown. Assume $\phi = 30°$ and the soil weighs 100 lb/ft^3. Neglect the fill on the toe for each wall.



Problem no.	*A*	*B*	*C*	*D*	*E*	*h*
13.1	8 ft 0 in.	2 ft 0 in.	1 ft 6 in.	4 ft 6 in.	1 ft 6 in.	14 ft 0 in.
13.2	10 ft 6 in.	2 ft 6 in.	1 ft 9 in.	6 ft 3 in.	1 ft 8 in.	18 ft 0 in.
13.3	11 ft 0 in.	3 ft 6 in.	1 ft 6 in.	6 ft 0 in.	1 ft 6 in.	20 ft 0 in.
13.4	12 ft 6 in.	4 ft 0 in.	1 ft 6 in.	7 ft 0 in.	2 ft 0 in.	22 ft 0 in.

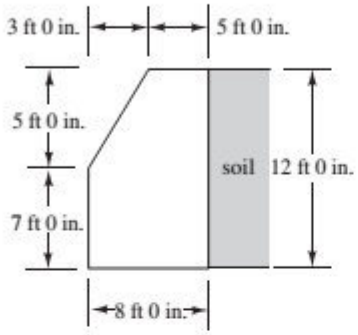
(*Answer to Problem 13.1:* 3266 lb, 15,242 ft-lb)
(*Answer to Problem 13.3:* 6666 lb, 44,440 ft-lb)

Problem 13.5 Repeat Problem 13.1 if δ is 20°. (*Ans.* 4059 lb, 18,943 ft-lb)

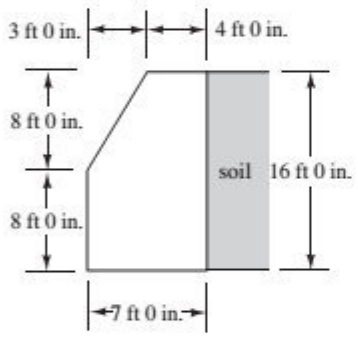
Problem 13.6 Repeat Problem 13.3 if δ is 23°40′.

For Problems 13.7 to 13.9, determine the safety factors against overturning and sliding for the gravity and semigravity walls shown if $\phi = 30°$ and the coefficient of friction (concrete on soil) is 0.5. Compute also the soil pressure under the toe and heel of each footing. The soil weighs 100 lb/ft^3 and the plain concrete used in the footing weighs 145 lb/ft^3. Determine horizontal pressures using the Rankine equation.

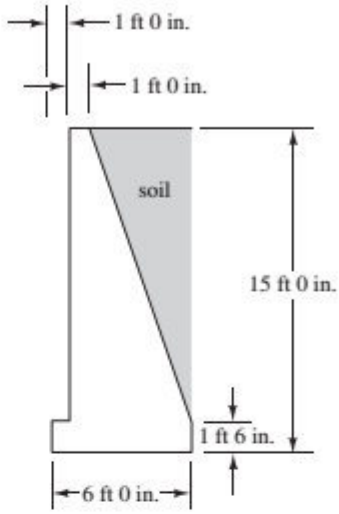
Problem 13.7 (*Ans.* 5.69, 2.67, −2193 psf, −1015 psf)



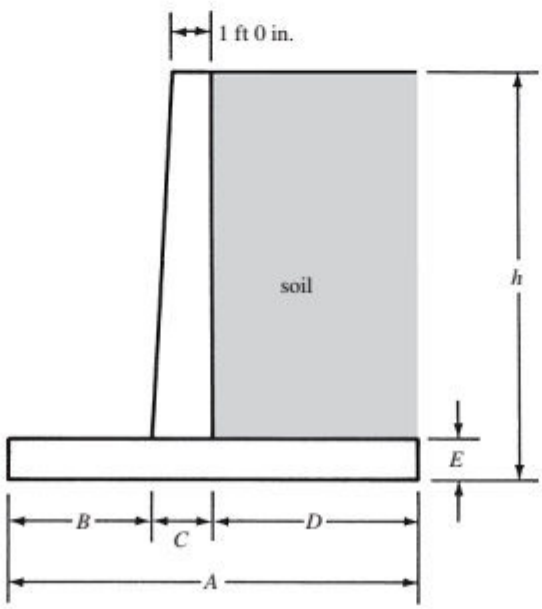
Problem 13.8



Problem 13.9 (*Ans.* 1.73, 1.32, −4739 psf, 0 psf)



For Problems 13.10 to 13.13, if Rankine's coefficient k_a is 0.75, the soil weighs 110 lb/ft^3, the concrete weighs 150 lb/ft^3, and the coefficient of friction (concrete on soil) is 0.55, determine the safety factors against overturning and sliding for the wall shown.



Problem no.	*A*	*B*	*C*	*D*	*E*	*h*
13.10	8 ft 0 in.	2 ft 0 in.	1 ft 0 in.	5 ft 0 in.	1 ft 3 in.	14 ft 0 in.
13.11	11 ft 0 in.	2 ft 6 in.	1 ft 6 in.	7 ft 0 in.	1 ft 6 in.	15 ft 0 in.
13.12	13 ft 6 in.	4 ft 0 in.	1 ft 6 in.	8 ft 0 in.	1 ft 6 in.	18 ft 0 in.
13.13	14 ft 0 in.	3 ft 6 in.	1 ft 6 in.	9 ft 0 in.	1 ft 9 in.	20 ft 0 in.

(*Answer to Problem 13.11:* 2.70, 0.93)
(*Answer to Problem 13.13:* 2.41, 0.86)

Problem 13.14 Repeat Problem 13.4 assuming a surcharge of 200 psf. Calculate overturning moment.

Problem 13.15 Repeat Problem 13.9 assuming a surcharge of 200 psf. (*Ans.* 1.36, 1.12, −8089 psf, 0 psf)

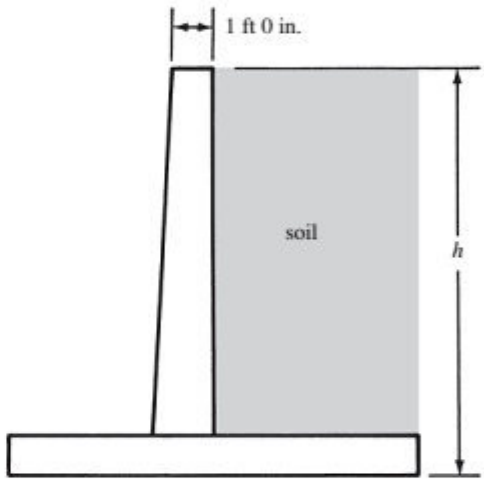
Problem 13.16 Repeat Problem 13.12 assuming a surcharge of 330 psf. Also determine toe and heel soil pressures.

For Problems 13.17 to 13.20, determine approximate dimensions of retaining walls, check safety factors against overturning and sliding, and calculate soil pressures for the wall shown. Also determine the required stem thickness at their bases and select vertical reinforcing there, using $f_y = 60{,}000$ psi, $f'_c = 3000$ psi, $q_a = 5000$ psf, ρ = approximately $0.18f'_c/f_y$, angle of internal friction = 33°40′, and coefficient of sliding friction (concrete on soil) = 0.45. Soil weight = 100 lb/ft^3 and concrete weight = 150 lb/ft^3.

Problem no.	*h*	Surcharge
13.17	12 ft 0 in.	None
13.18	14 ft 9 in.	None
13.19	18 ft 0 in.	None
13.20	22 ft 0 in.	200 psf

(*Answer to Problem 13.17:* 6 ft 0 in. wide, O.T. safety factor = 2.62)
(*Answer to Problem 13.19:* 8 ft 6 in. wide, O.T. safety factor = 2.30)

For Problems 13.21 to 13.23, determine the same information required for Problems 13.17 to 13.20 with same data, but design heels instead of stems.

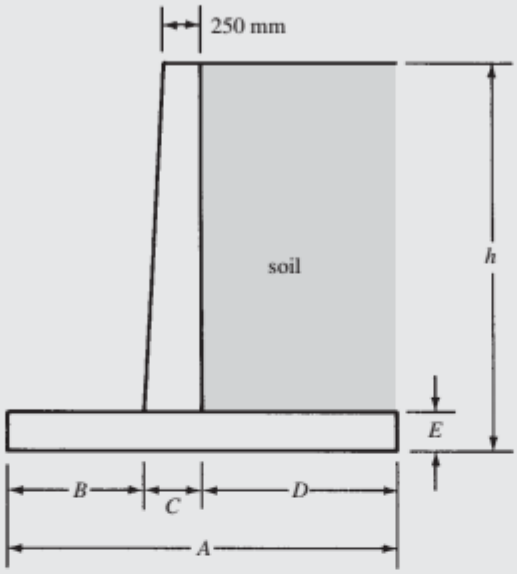


Problem no.	*h*	Surcharge
13.21	14 ft 0 in.	None
13.22	18 ft 0 in.	300 psf
13.23	20 ft 0 in.	300 psf starting 4 ft 0 in. from inside face of wall

(*Answer to Problem 13.21:* 6 ft 6 in. wide, O.T. safety factor = 2.24)
(*Answer to Problem 13.23:* 10 ft 3 in. wide, O.T. safety factor = 2.16)

Problems in SI Units

For Problems 13.24 to 13.26, use the Rankine equation to calculate the total horizontal force and the overturning moment for the wall shown. Assume $\sin \phi = 0.5$ and the soil weighs 16 kN/m^3.

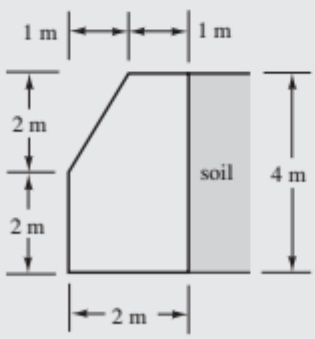


Problem no.	A (m)	B (mm)	C (mm)	D (m)	E (mm)	h (m)
13.24	2.700	700	500	1.500	500	6
13.25	2.400	600	500	1.300	450	4
13.26	3.150	800	550	1.800	500	8

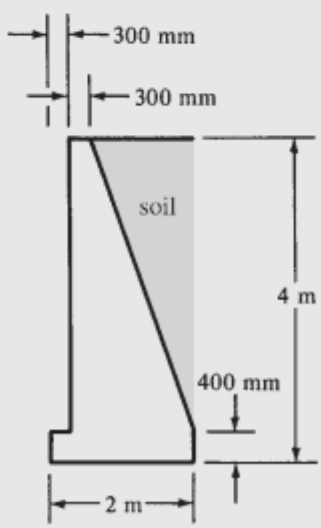
(*Answer to Problem 13.25:* 42,624 kN, 56,832 kN-m)

For Problems 13.27 and 13.28, determine the safety factors against overturning and sliding for the gravity and semigravity walls shown if $\phi = 30°$ and the coefficient of sliding (concrete on soil) is 0.45. Compute also the soil pressure under the toe and heel of each footing. The soil weighs 16 kN/m^3 and the plain concrete used in the footing weighs 22.7 kN/m^3.

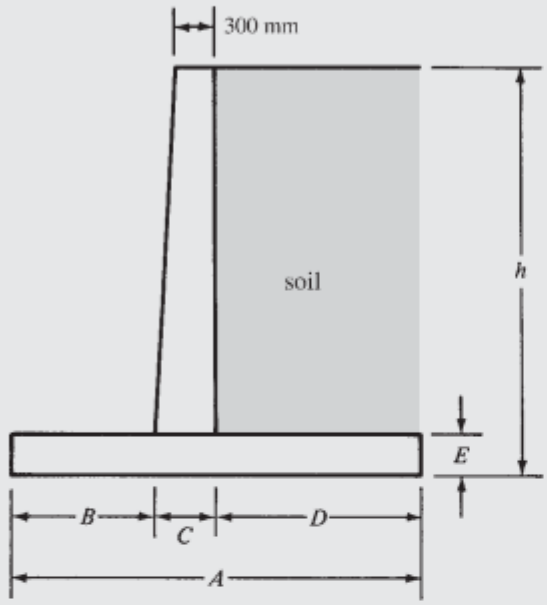
Problem 13.27 (*Ans.* 3.06, 1.68, −141.87 kN/m^2, −17.03 kN/m^2)



Problem 13.28



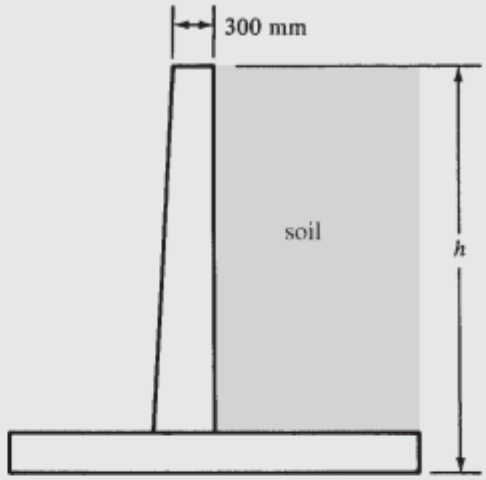
For Problems 13.29 and 13.30, if Rankine's coefficient is 0.35, the soil weight is 16 kN/m^3, the concrete weight is 23.5 kN/m^3, and the coefficient of friction (concrete on soil) is 0.50, determine the safety factors against overturning and sliding for the wall shown.



Problem no.	A (m)	B (m)	C (mm)	D (m)	E (mm)	h (m)
13.29	5	1.5	500	3.0	800	7
13.30	4	1.5	300	2.2	700	5

(*Answer to Problem 13.29:* 4.31, 1.64)

For Problems 13.31 to 13.33, select approximate dimensions for the cantilever retaining wall shown and determine reinforcement required at base of stem, using those dimensions and the following data: $f'_c = 21$ MPa, $f_y = 420$ MPa, $\rho =$ approximately $\frac{3}{8}\rho_{bal}$, angle of internal friction = 33°40′, soil weight = 16 kN/m^3, and reinforced concrete weight = 23.5 kN/m^3.



Problem no.	h (m)	Surcharge
13.31	4	none
13.32	6	none
13.33	7	4 kN/m

(*Answer to Problem 13.31:* Use 320-mm stem at base with $d = 250$ mm and #16 bars @ 225-mm vertical steel)
(*Answer to Problem 13.33:* Use 560-mm stem at base with $d = 490$ mm and #25 bars @ 225-mm vertical steel)

Continuous Reinforced Concrete Structures

CHAPTER 14

14.1 Introduction

During the construction of reinforced concrete structures, as much concrete as possible is placed in each pour. For instance, the concrete for a whole floor or for a large part of it, including the supporting beams and girders and parts of the columns, may be placed at the same time. The reinforcing bars extend from member to member, as from one span of a beam into the next. When there are construction joints, the reinforcing bars are left protruding from the older concrete, so they may be lapped or spliced to the bars in the newer concrete. In addition, the old concrete is cleaned so that the newer concrete will bond to it as well as possible. The result of all these facts is that reinforced concrete structures are generally monolithic or continuous and, thus, statically indeterminate.

A load placed in one span of a continuous structure will cause shears, moments, and deflections in the other spans of that structure. Not only are the beams of a reinforced concrete structure continuous, but the entire structure is also continuous. In other words, loads applied to a column affect the beams, slabs, and other columns, and vice versa.

The result is that more economical structures are obtained because the bending moments are smaller, and thus member sizes are smaller. Although the analyses and designs of continuous structures are more complicated than they are for statically determinate structures, this fact has become less important because of the constantly increasing availability of good software.

14.2 General Discussion of Analysis Methods

In reinforced concrete design today, we use elastic methods to analyze structures loaded with factored or ultimate loads. Such a procedure probably doesn't seem quite correct to the reader, but it does yield satisfactory results. The reader might very well ask, "Why don't we use ultimate or inelastic analyses for reinforced concrete structures?" The answer is that our theory and tests are just not sufficiently advanced.

It is true that under certain circumstances, some modifications of moments are permitted to recognize ultimate or inelastic behavior as described in Section 14.5 of this chapter. In general, however, we will discuss elastic analyses for reinforced concrete structures. Actually, no method of analysis, elastic or inelastic, will give exact results because of the unknown effects of creep, settlement, shrinkage, workmanship, and so on.

14.3 Qualitative Influence Lines

Many methods might be used to analyze continuous structures. The most common hand calculation method is moment distribution, but other methods are frequently used, such as matrix methods, computer solutions, and others. Whichever method is used, you should understand that to determine maximum shears and moments at different sections in the structure, it is necessary to consider different positions of the live loads. As a background for this material, a brief review of *qualitative influence lines* is presented.

Qualitative influence lines are based on the principle introduced by the German professor Heinrich Müller-Breslau. This principle is as follows: *The deflected shape of a structure represents to some scale*

Courtesy of EFCO Corp.

Confinazas Financial Center, Caracas, Venezuela.

the influence line for a function such as reaction, shear, or moment if the function in question is allowed to act through a small distance. In other words, the structure draws its own influence line when the proper displacement is made.

The shape of the usual influence line needed for continuous structures is so simple to obtain with the Müller-Breslau principle that, in many situations, it is unnecessary to compute the numerical values of the coordinates. It is possible to sketch the diagram roughly with sufficient accuracy to locate the critical positions for live loads for various functions of the structure. These diagrams are referred to as *qualitative* influence lines, whereas those with numerical values are referred to as *quantitative* influence lines.[1]

If the influence line is desired for the left reaction of the continuous beam of Figure 14.1(a), its general shape can be determined by letting the reaction act upward through a unit distance, as shown in Figure 14.1(b). If the left end of the beam is pushed up, the beam will take the shape shown. This distorted shape can be sketched easily by remembering that the other supports are considered to be unyielding. The influence line for V_c, drawn in a similar manner, is shown in Figure 14.1(c).

Figure 14.1(d) shows the influence line for positive moment at point x near the center of the left-hand span. The beam is assumed to have a pin or hinge inserted at x and a couple is applied adjacent to each side of the pin, which will cause compression in the top fibers. Bending the beam on each side of the pin causes the left span to take the shape indicated, and the deflected shape of the remainder of the beam may be roughly sketched. A similar procedure is used to draw the influence line for negative moment at point y in the third span, except that a moment couple is applied at the assumed pins, which will tend to cause compression in the bottom beam fibers, corresponding with negative moment.

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[1] McCormac, J. C., 2007, *Structural Analysis: Using Classical and Matrix Methods*, 4th ed. (Hoboken, NJ: John Wiley & Sons), pp. 189–194.

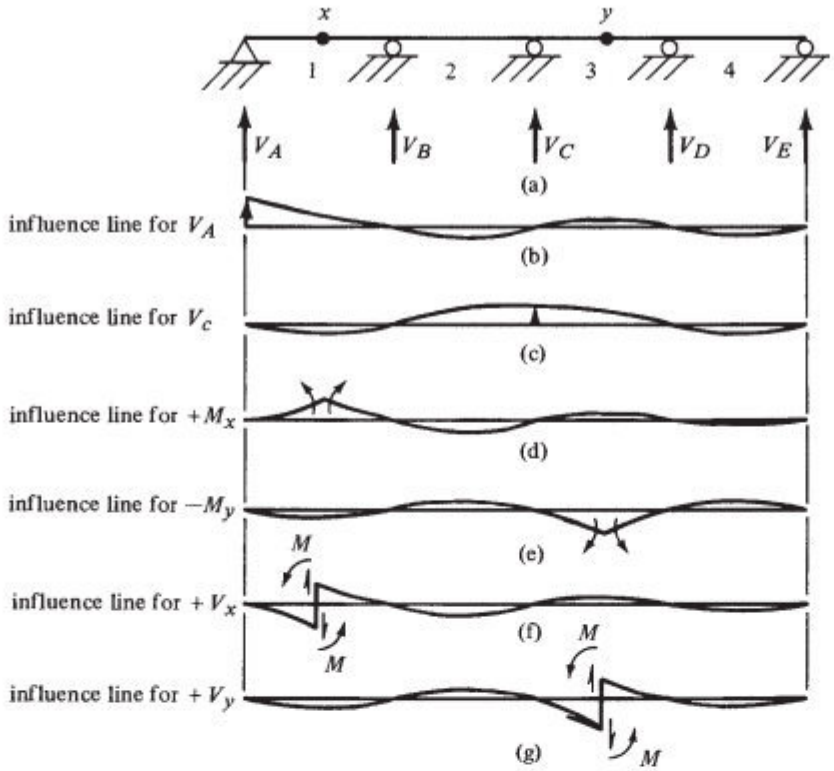

FIGURE 14.1 Qualitative influence lines.

Finally, qualitative influence lines are drawn for positive shear at points x and y. At point x, the beam is assumed to be cut, and the two vertical forces of the nature required to give positive shear are applied to the beam on the sides of the cut section. The beam will take the shape shown in Figure 14.1(f). The same procedure is used in Figure 14.1(g) to draw a diagram for positive shear at point y. (Theoretically, for qualitative shear influence lines, it is necessary to have a moment on each side of the cut section sufficient to maintain equal slopes. Such moments are indicated in parts (f) and (g) of the figure by the letter M.)

From these diagrams, considerable information is available concerning critical live loading conditions. If a maximum positive value of V_A were desired for a uniform live load, the load would be placed in spans 1 and 3, where the diagram has positive ordinates; if maximum negative moment were required at point y, spans 2 and 4 would be loaded, and so on.

Qualitative influence lines are particularly valuable for determining critical load positions for buildings, as illustrated by the moment influence line for the building of Figure 14.2. In drawing diagrams for an entire frame, the joints are assumed to be free to rotate, but the members at each joint are assumed to be rigidly connected to each other so that the angles between them do not change during rotation. The influence line shown in the figure is for positive moment at the center of beam AB.

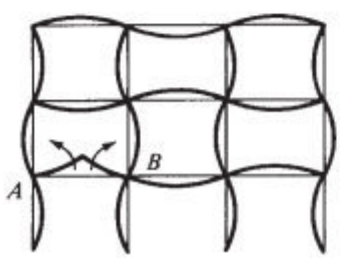

FIGURE 14.2 Qualitative influence line for moment in a frame.

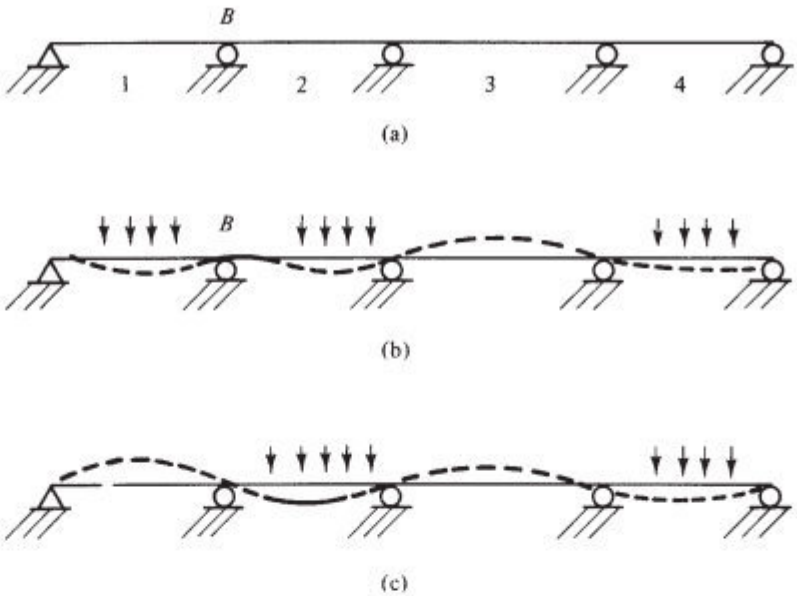

FIGURE 14.3 Load positions for maximum moments.

The spans that should be loaded to cause maximum positive moment are obvious from the diagram. It should be realized that loads on a member more than approximately three spans away have little effect on the function under consideration.

In the last few paragraphs, influence lines have been used to determine the critical positions for placing live loads to cause maximum moments. The same results can be obtained (and perhaps more easily) by considering the deflected shape or curvature of a member under load. If the live loads are placed so that they cause the greatest curvature at a particular point, they will have bent the structure the greatest amount at that point, which means that the greatest moment will have been obtained.

For the continuous beam of Figure 14.3(a), it is desired to cause the maximum negative moment at support B by the proper placement of a uniform live load. In part (b) of the figure, the deflected shape of the beam is sketched as it would be when a negative moment occurs at B, and the rest of the beam's deflected shape is drawn as shown by the dashed line. The live uniform load is then placed in the locations that would exaggerate that deflected shape. This is done by placing the load in spans 1, 2, and 4.

A similar situation is shown in Figure 14.3(c), where it is desired to obtain maximum positive moment at the middle of the second span. The deflected shape of the beam is sketched as it would be when a positive moment occurs in that span, and the rest of the beam's deflected shape is drawn in. To exaggerate this positive or downward bending in the second span, it can be seen that the live load should be placed in spans 2 and 4.

14.4 Limit Design

It can be clearly shown that a statically indeterminate beam or frame normally will not collapse when its ultimate moment capacity is reached at just one section. Instead, there is a redistribution of the moments in the structure. Its behavior is rather similar to the case where three men are walking along with a log on their shoulders and one of the men gets tired and lowers his shoulder just a little. The result is a redistribution of loads to the other men and, thus, changes in the shears and moments throughout the log.

It might be well at this point to attempt to distinguish between the terms *plastic design* as used in structural steel and *limit design* as used in reinforced concrete. In structural steel, plastic design involves both (a) the increased resisting moment of a member after the extreme fiber of the member is stressed to its yield point and (b) the redistribution or change in the moment pattern in the member. (Load and

resistance factor design [LRFD] is a steel design method that incorporates much of the theory associated with plastic design.) In reinforced concrete, the increase in resisting moment of a section after part of the section has been stressed to its yield point has already been accounted for in the strength design procedure. Therefore, limit design for reinforced concrete structures is concerned only with the change in the moment pattern after the steel reinforcing at some cross section is stressed to its yield point.

The basic assumption used for limit design of reinforced concrete structures and for plastic design of steel structures is the ability of these materials to resist a so-called yield moment while an appreciable increase in local curvature occurs. In effect, if one section of a statically indeterminate member reaches this moment, it begins to yield but does not fail. Rather, it acts like a hinge (called a *plastic hinge*) and throws the excess load off to sections of the members that have lesser stresses. The resulting behavior is much like that of the log supported by three men when one man lowered his shoulder.

To apply the limit design or plastic theory to a particular structure, it is necessary for that structure to behave plastically. For this initial discussion, it is assumed that an ideal plastic material, such as a ductile structural steel, is involved. Figure 14.4 shows the relationship of moment to the resulting curvature of a short length of a ductile steel member. The theoretical ultimate or nominal resisting moment of a section is referred to in this text as M_n (it's the same as the plastic moment M_p).

Although the moment–curvature relationship for reinforced concrete is quite different from the ideal one pictured in Figure 14.4, the actual curve can be approximated reasonably well by the ideal one, as shown in Figure 14.5. The dashed line in the figure represents the ideal curve, whereas the solid line is a typical one for reinforced concrete. Tests have shown that the lower the reinforcing percentage

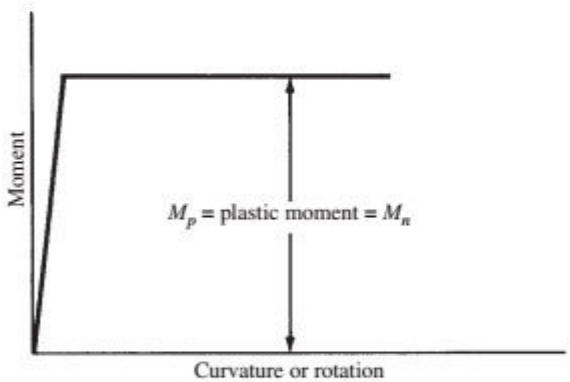

FIGURE 14.4 Moment–curvature relationship for an ideal plastic material.

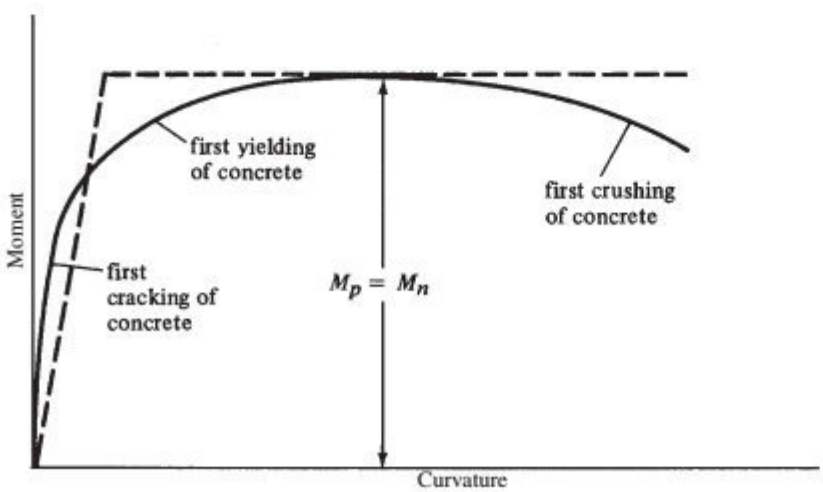

FIGURE 14.5 Typical moment–curvature relationship for a reinforced concrete member.

in the concrete ρ or $\rho - \rho'$ (where ρ' is the percentage of compressive reinforcement), the closer will the concrete curve approach the ideal curve. This is particularly true when very ductile reinforcing steels, such as Grade 40, are used. Should a large percentage of steel be present in a reinforced concrete member, the yielding that actually occurs before failure will be so limited that the ultimate or limit behavior of the member will not be greatly affected by yielding.

The Collapse Mechanism

To understand moment redistribution in steel or reinforced concrete structures, it is necessary first to consider the location and number of plastic hinges required to cause a structure to collapse. A statically determinate beam will fail if one plastic hinge develops. To illustrate this fact, the simple beam of constant cross section loaded with a concentrated load at midspan shown in Figure 14.6(a) is considered. Should the load be increased until a plastic hinge is developed at the point of maximum moment (underneath the load in this case), an unstable structure will have been created, as shown in Figure 14.6(b). Any further increase in load will cause collapse.

The plastic theory is of little advantage for statically determinate beams and frames, but it may be of decided advantage for statically indeterminate beams and frames. For a statically indeterminate structure to fail, it is necessary for more than one plastic hinge to form. The number of plastic hinges required for failure of statically indeterminate structures will be shown to vary from structure to structure but can never be less than two. The fixed-end beam of Figure 14.7 cannot fail unless the three plastic hinges shown in the figure are developed.

Although a plastic hinge may be formed in a statically indeterminate structure, the load can still be increased without causing failure if the geometry of the structure permits. The plastic hinge will act like a real hinge insofar as increased loading is concerned. As the load is increased, there is a redistribution of moment because the plastic hinge can resist no more moment. As more plastic hinges are formed in the structure, there will eventually be a sufficient number of them to cause collapse.

The propped beam of Figure 14.8 is an example of a structure that will fail after two plastic hinges develop. Three hinges are required for collapse, but there is a real hinge at the right end. In this beam, the largest elastic moment caused by the design-concentrated load is at the fixed end. As the magnitude of the load is increased, a plastic hinge will form at that point.

The load may be further increased until the moment at some other point (here it will be at the concentrated load) reaches the plastic moment. Additional load will cause the beam to collapse. The arrangement of plastic hinges, and perhaps real hinges that permit collapse in a structure, is called the *mechanism*. Part (b) of Figures 14.6 to 14.8 show mechanisms for various beams.

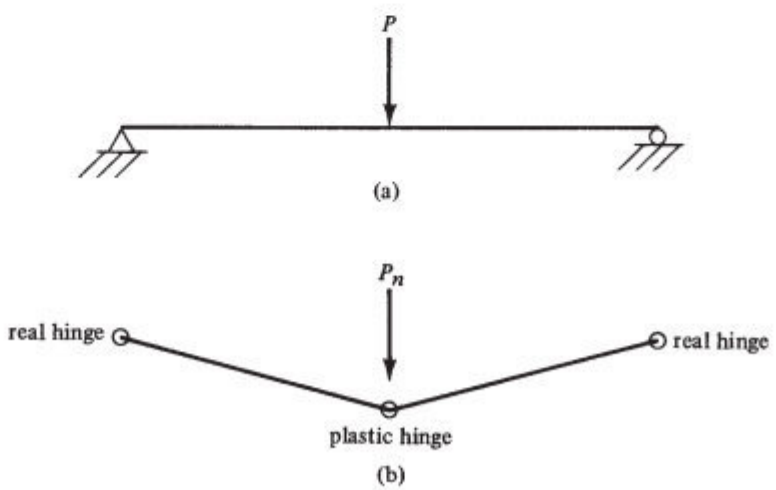

FIGURE 14.6 Plastic hinge formation in a statically determinate beam.

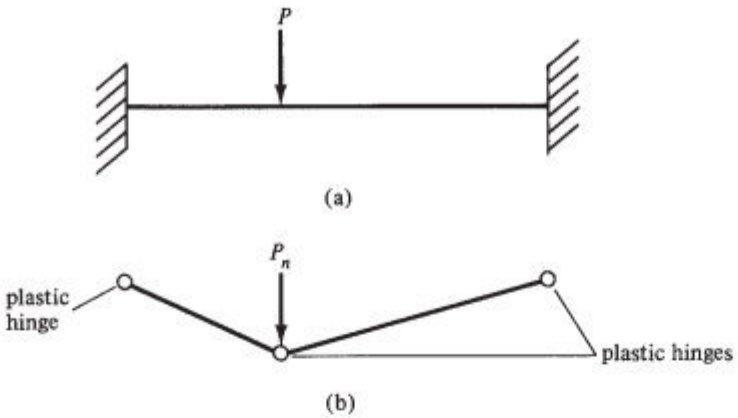

FIGURE 14.7 Plastic hinge formation in a fixed-fixed statically indeterminate beam.

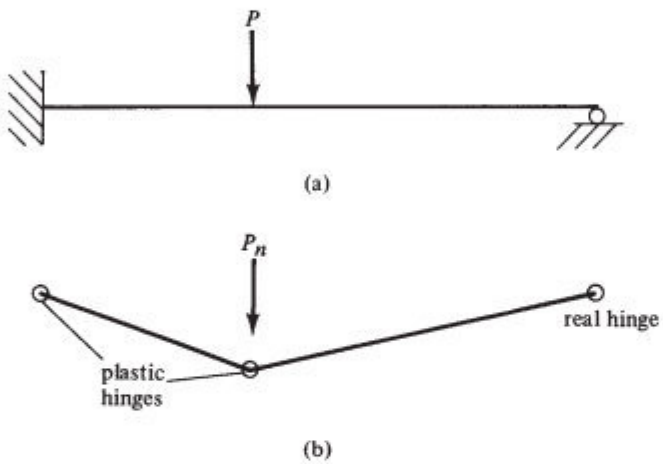

FIGURE 14.8 Plastic hinge formation in a propped cantilever statically indeterminate beam.

Plastic Analysis by the Equilibrium Method

To analyze a structure plastically, it is necessary to compute the plastic or ultimate moments of the sections, to consider the moment redistribution after the ultimate moments develop, and finally to determine the ultimate loads that exist when the collapse mechanism is created. The method of plastic analysis known as the *equilibrium method* will be illustrated in this section.

The fixed-end beam of Figure 14.9 is considered first. It is desired to determine the value of w_n, the theoretical ultimate load the beam can support. The maximum moments in a uniformly loaded fixed-end beam in the elastic range occur at the fixed ends, as shown in the figure.

If the magnitude of the uniform load is increased, the moments in the beam will be increased proportionately until a plastic moment eventually develops at some point. Because of symmetry, plastic moments will be developed at the beam ends, as shown in Figure 14.10(b). Should the loads be further increased, the beam will be unable to resist moments larger than M_n at its ends. Those points will rotate through large angles, and thus the beam will be permitted to deflect more and allow the moments to increase out in the span. Although the plastic moment has been reached at the ends and plastic hinges are formed, the beam cannot fail because it has, in effect, become a simple end-supported beam for further load increases, as shown in Figure 14.10(c).

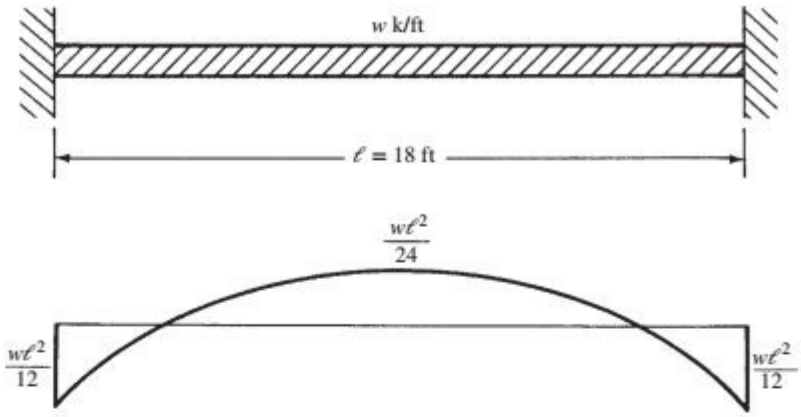

FIGURE 14.9 Elastic moment diagram in a fixed-fixed statically indeterminate beam.

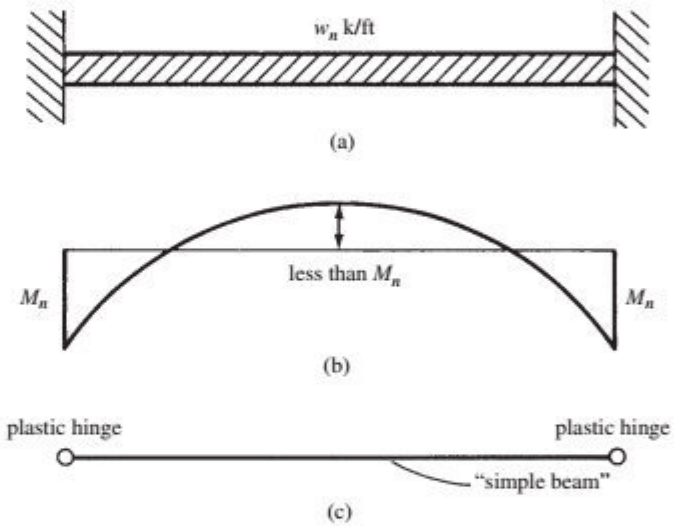

FIGURE 14.10 Moment diagram after hinge formation at supports in a fixed-fixed beam.

The load can now be increased on this "simple" beam, and the moments at the ends will remain constant; however, the moment out in the span will increase as it would in a uniformly loaded simple beam. This increase is shown by the dashed line in Figure 14.11(b). The load can be increased until the moment at some other point (here the beam centerline) reaches the plastic moment. When this happens, a third plastic hinge will have developed and a mechanism will have been created, permitting collapse.

One method of determining the value of w_n is to take moments at the centerline of the beam (knowing the moment there is M_n at collapse). Reference is made here to Figure 14.11(a) for the beam reactions.

$$M_n = -M_n + \left(w_n \frac{\ell}{2}\right)\left(\frac{\ell}{2} - \frac{\ell}{4}\right) = -M_n + \frac{w_n \ell^2}{8}$$

$$w_n = \frac{16M_n}{\ell^2}$$

The same value could be obtained by considering the diagrams shown in Figure 14.12. You will remember that a fixed-end beam can be replaced with a simply supported beam plus a beam with end

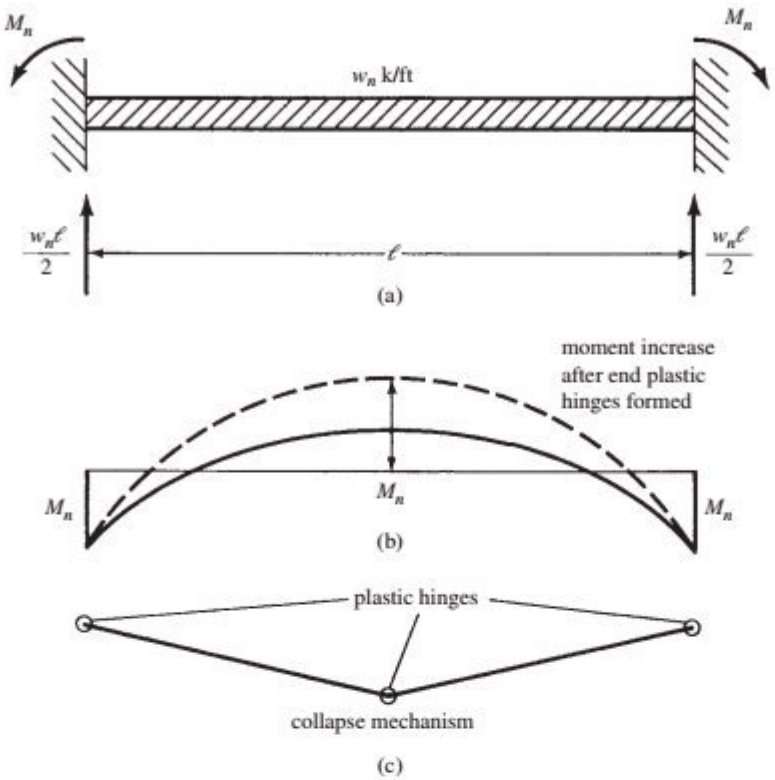

FIGURE 14.11 Formation of a collapse mechanism in a fixed-fixed beam.

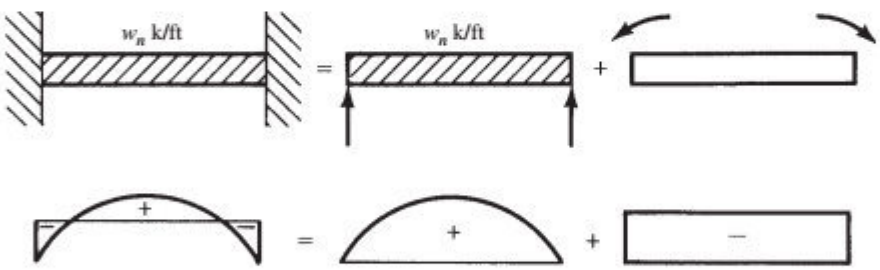

FIGURE 14.12 Moment superposition.

moments. Thus, the final moment diagram for the fixed-end beam equals the moment diagram if the beam had been simply supported plus the end moment diagram.

For the fixed-fixed beam under consideration, the value of M_n can be calculated as follows (see Figure 14.13):

$$2M_n = \frac{w_n\ell^2}{8}$$

$$M_n = \frac{w_n\ell^2}{16}$$

The propped beam of Figure 14.14, which supports a concentrated load, is presented as a second illustration of plastic analysis. The goal is to determine the value of P_n, the theoretical ultimate load the beam can support before collapse. The maximum moment in this beam in the elastic range occurs at the fixed end, as shown in the figure. If the magnitude of the concentrated load is increased, the moments

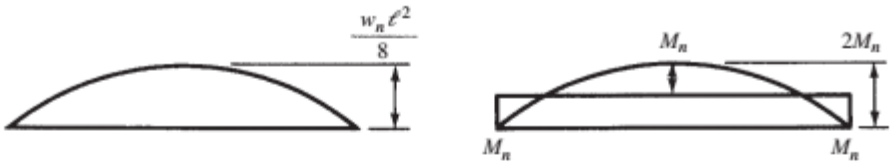

FIGURE 14.13 Determination of required M_n from fixed-fixed beam with hinges forming mechanism.

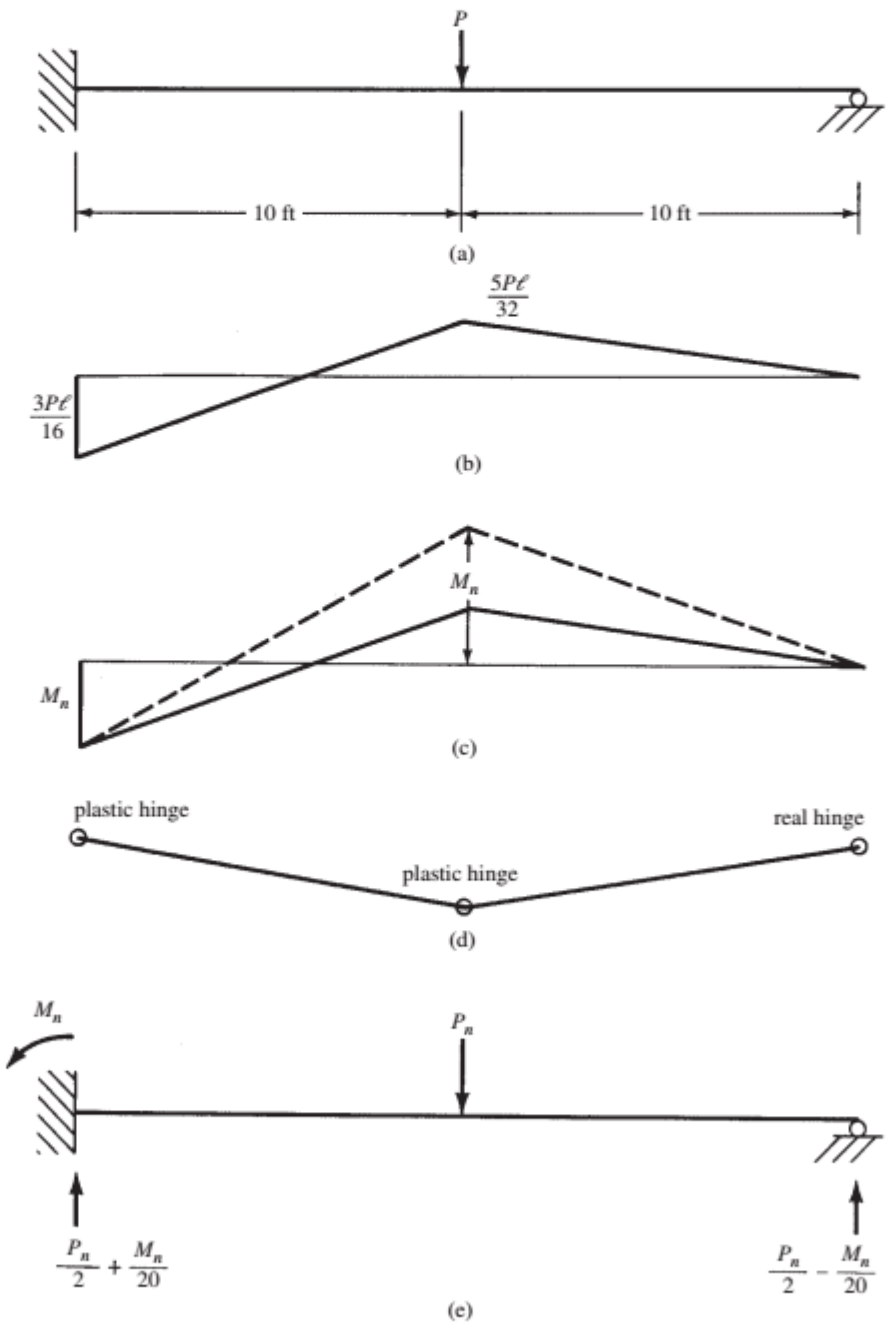

FIGURE 14.14 Hinge formation sequence in a propped cantilever beam.

in the beam will increase proportionately until a plastic moment is eventually developed at some point. This point will be at the fixed end, where the elastic moment diagram has its largest ordinate.

After this plastic hinge is formed, the beam will act as though it is simply supported insofar as load increases are concerned, because it will have a plastic hinge at the left end and a real hinge at the right end. An increase in the magnitude of the load P will not increase the moment at the left end but will increase the moment out in the beam, as it would happen in a simple beam. The increasing simple beam moment is indicated by the dashed line in Figure 14.14(c). Eventually, the moment at the concentrated load will reach M_n and a mechanism will form, consisting of two plastic hinges and one real hinge, as shown in Figure 14.14(d).

The value of the theoretical maximum concentrated load, P_n, that the beam can support can be determined by taking moments to the right or left of the load. Figure 14.14(e) shows the beam reactions for the conditions existing just before collapse. Moments are taken to the right of the load as follows:

$$M_n = \left(\frac{P_n}{2} - \frac{M_n}{20}\right) 10$$

$$P_n = 0.3M_n$$

The subject of plastic analysis can be continued for different types of structures and loadings, as described in several textbooks on structural analysis or steel design.[2] The method has been proved to be satisfactory for ductile structural steels by many tests. Concrete, however, is a relatively brittle material, and the limit design theory has not been fully accepted by the ACI Code. The code does recognize that there is some redistribution of moments and permits partial redistribution based on a rule of thumb that is presented in the next section of this chapter.

14.5 Limit Design under the ACI Code

Tests of reinforced concrete frames have shown that under certain conditions, there is definitely a redistribution of moments before collapse occurs. Recognizing this fact, the ACI Code (6.6.1.2) permits factored moments calculated by elastic theory (*not by an approximate analysis*) to be decreased at locations of maximum negative or positive moment in any span of a continuous structure for any loading arrangement. The amount by which moments can be decreased cannot exceed $1000\epsilon_t\%$ with a maximum of 20%. Redistribution of moments as described herein is not permissible unless ϵ_t is equal to or greater than 0.0075 at the section where moment is reduced (ACI Section 6.6.5). Appendix A, Table A.7 and Appendix B, Table B.7 of this book provide percentages of steel for which ϵ_t will be equal to 0.0075. If ρ is greater than these values, redistribution is not permitted, because ϵ_t will be less than 0.0075. The values given in the table are for rectangular sections with tensile reinforcing.

According to ACI Section 6.6.5.1, negative or positive moments may be decreased for prestressed sections, using the same rules, if bonded reinforcing (as described in Chapter 19 herein) is used.

The ACI Code's percentage of moment redistribution has purposely been limited to a very conservative value to be sure that excessively large concrete cracks do not occur at high steel stresses and to ensure adequate ductility for moment redistribution at the plastic hinges. The ACI Code likely will expand its presently conservative redistribution method after the behavior of plastic hinges is better understood, particularly in regard to shears, deflections, and development of reinforcing. It is assumed here that the sections are satisfactorily reinforced for shears so that the ultimate moments can be reached without shear failure occurring. The adjustments are applied to the moments resulting from each of the different loading conditions. The member in question will then be proportioned on the basis of the resulting moment envelope. Figures 14.15 through 14.18 illustrate the application of the moment redistribution permitted by the code to a three-span continuous beam. It will be noted in these figures that factored loads and elastic analyses are used for all the calculations.

Three different live-load conditions are considered in these figures. To determine the maximum positive moment in span 1, the live load is placed in spans 1 and 3 (Figure 14.15). Similarly, to produce

[2] McCormac, J. C. and Csernak, S. F., 2011, *Structural Steel Design*, 5th ed. (Hoboken, NJ: Pearson Prentice Hall), pp. 240–244.

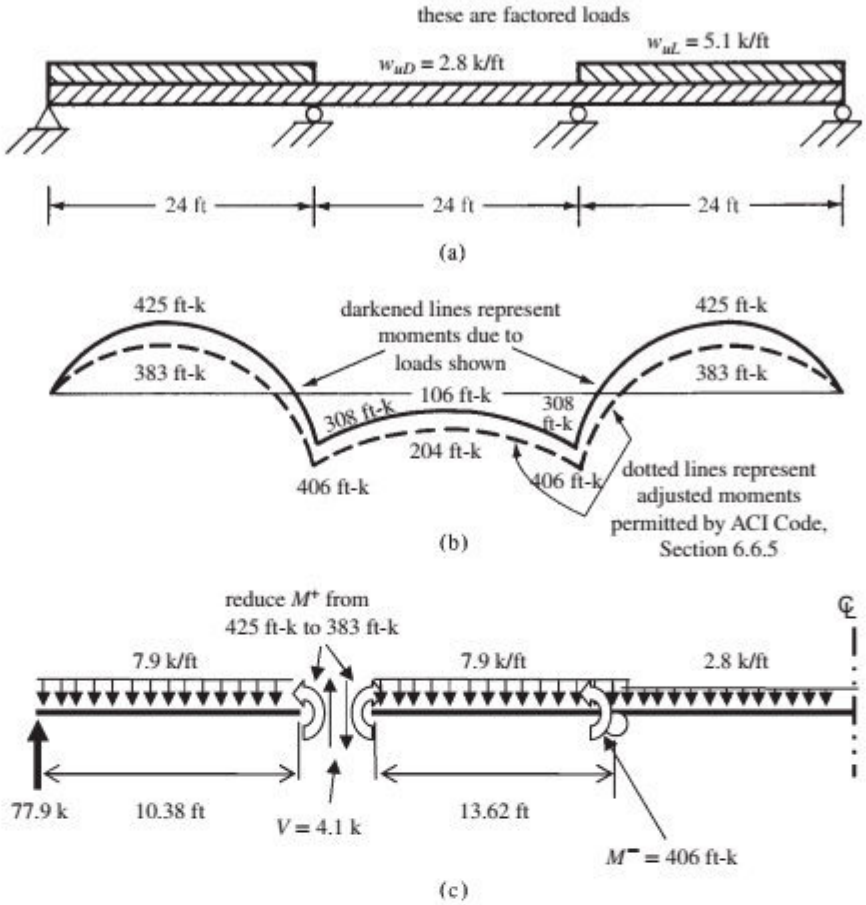

FIGURE 14.15 Maximum positive moment in end spans. (a) Loading patterns for maximum positive moment in both end spans. (b) Moment diagram before and after reducing positive moment in end spans. (c) Shears and moments in left span after M^+ is reduced 10% in end spans.

maximum positive moment in span 2, the live load is placed in that span only (Figure 14.16). Finally, maximum negative moment at the first interior support from the left end is caused by placing the live load in spans 1 and 2 (Figure 14.17).

For this particular beam, it is assumed that the code permits a 10% decrease in the negative or positive moments. This will require that $1000\epsilon_t$ exceed 10 at the sections where the moment is reduced to provide the needed ductility. The result will be smaller design moments at the critical sections. Initially, the loading for maximum positive moment in span 1 is considered as shown in Figure 14.15. If the maximum calculated positive moments of 425 ft-k near midspan of the end spans are each decreased by 10% to 383 ft-k, the negative moments at both interior supports will be increased to 406 ft-k. Even though the negative moment has increased significantly, from 308 ft-k to 406 ft-k, this higher value still will not control the required moment capacity at this location. Hence, the positive design moment is decreased, but the negative moment is not increased.

In the same manner, in Figure 14.16, where the beam is loaded to produce maximum positive moment in span 2, a 10% decrease in positive moment from 261 ft-k to 235 ft-k will increase the negative moment at both interior supports from 339 ft-k to 365 ft-k.

Finally, in Figure 14.17, the live-load placement causes a maximum negative moment at the first interior support of 504 ft-k. If this value is reduced by 10%, the maximum moment there will be −454 ft-k. In this figure, the authors have reduced the negative moment at the other interior support also by 10%.

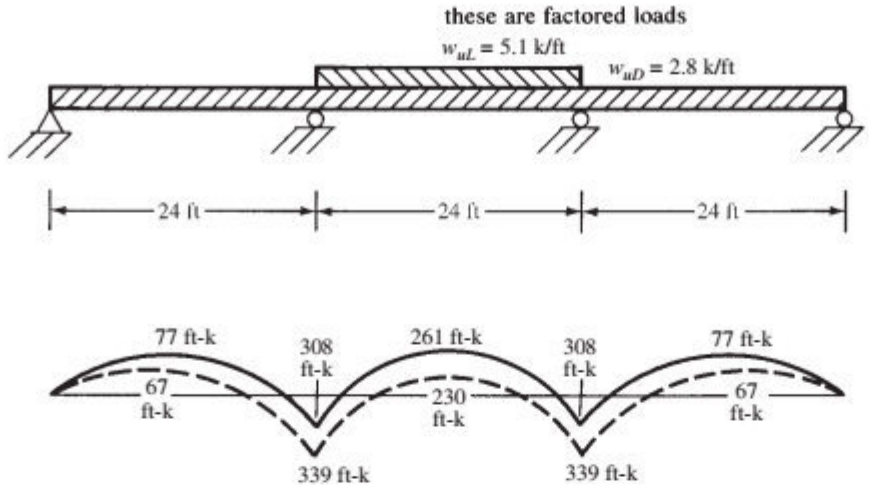

FIGURE 14.16 Maximum positive moment in span 2.

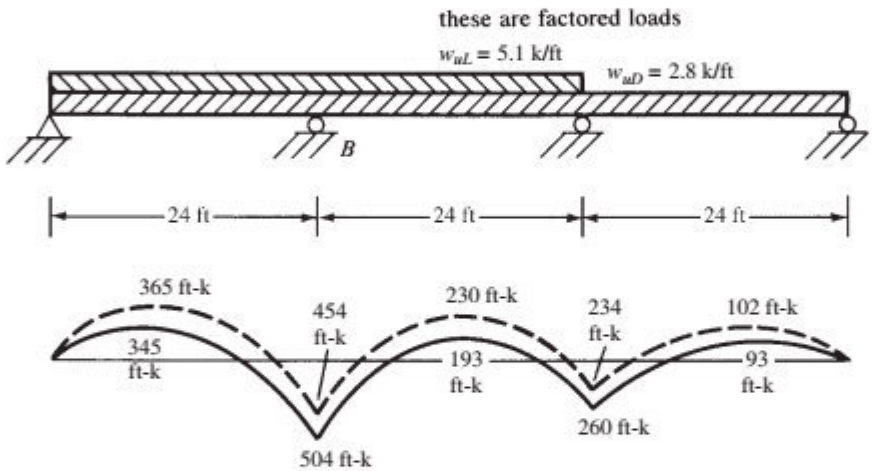

FIGURE 14.17 Maximum negative moment at support *B*.

It will be noticed that the net result of all of the various decreases in the positive or negative moments is a net reduction in both the maximum positive and the maximum negative values. The result of these various redistributions is actually an envelope of the extreme values of the moments at the critical sections. The envelope for the three-span beam considered in this section is presented in Figure 14.18. You can see at a glance the parts of the beams that need positive reinforcement, negative reinforcement, or both.

The reductions in bending moments because of moment redistribution as described here do not mean that the safety factors for continuous members will be less than those for simple spans. Rather,

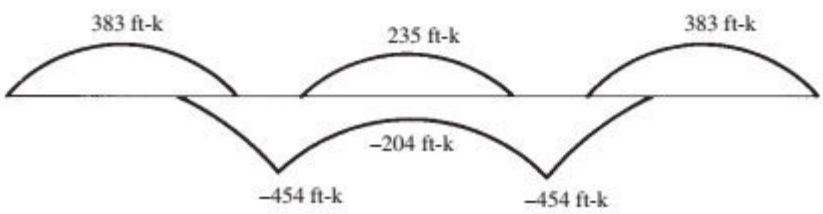

FIGURE 14.18 Moment envelope.

the excess strength that such members have because of this continuity is reduced so that the overall factors of safety are nearer but not less than those of simple spans.

Various studies have shown that cracking and deflection of members selected by the limit design process are no more severe than those for the same members designed without taking advantage of the permissible redistributions.[3,4]

14.6 Preliminary Design of Members

Before an "exact" analysis of a building frame can be made, it is necessary to estimate the sizes of the members. Even if a computer design is used, it is often economically advisable to make some preliminary estimates as to sizes. If an approximate analysis of the structure has been made, it will be possible to make very reasonable member size estimates. The result will be appreciable saving of both computer time and money.

An experienced designer can usually make very satisfactory preliminary size estimates based on previous experience. In the absence of such experience, however, the designer can still make quite reasonable size estimates based on knowledge of structural analysis. For instance, to size columns approximately, a designer can neglect moments and assume an average axial stress, or P_u/A_g, value of about $0.4f'_c$ to $0.6f'_c$. This rough value can be divided into the estimated column load to obtain its estimated area. If moments are large, lower values of average stress ($0.4f'_c$ to $0.5f'_c$) can be used; if moments are small, higher values ($0.55f'_c$ to $0.6f'_c$) can be used.

Preliminary beam sizes can be obtained by considering their approximate moments. A uniformly loaded simple beam will have a maximum bending moment equal to $w_u\ell^2/8$, whereas a uniformly loaded fixed-end beam will have a maximum moment of $w_u\ell^2/12$. For a continuous uniformly loaded beam, the designer might very well estimate a maximum moment somewhere between the values given, perhaps $w_u\ell^2/10$, and use that value to estimate the beam size.

For many structures, it is necessary to conduct at least two different analyses. One analysis is made to consider the effect of gravity loads as described in Section 14.7 of this chapter, while another might be made to consider the effect of lateral loads as discussed in Section 14.8. For the gravity loads only, U usually equals $1.2D + 1.6L$.

Because the gravity loads affect primarily the floor to which they are applied, each floor can be analyzed independently of the others. Such is not the case for lateral loads because lateral loads applied anywhere on the frame affect the lateral displacements throughout the frame and, thus, affect the forces in the frame below. For this situation, the load factor equations involving lateral forces (ACI Equations 5.3.1c through 5.3.1g) must be applied.

Sometimes a third analysis should be made—one that involves the possibility of force reversals on the windward side or even overturning of the structure. If overturning is being considered, the dead and live gravity loads should be reduced to their smallest possible values (i.e., zero live load and $0.9D$, in case the dead loads have been overestimated a little) while the lateral loads are acting. For this case, ACI load factor equations 5.3.1f and 5.3.1g must be considered.

14.7 Approximate Analysis of Continuous Frames for Vertical Loads

Statically indeterminate structures may be analyzed "exactly" or "approximately." Some approximate methods involving the use of simplifying assumptions are presented in this section. Despite the

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[3] Cohn, M. Z., 1964, "Rotational Compatibility in the Limit Design of Reinforced Concrete Continuous Beams," *Proceedings of the International Symposium on the Flexural Mechanics of Reinforced Concrete* (Miami, FL: ASCE-ACI), pp. 359–382.

[4] Mattock, A. H., 1959, "Redistribution of Design Bending Moments in Reinforced Concrete Continuous Beams," *Proceedings of the Institution of Civil Engineers*, 113, pp. 35–46.

increased use of computers for making "exact" analyses, approximate methods are used about as much or more than ever, for several reasons. These include the following:

1. The structure may be so complicated that no one who has the knowledge to make an "exact" analysis is available or no suitable computer software is available.
2. For some structures, either method may be subject to so many errors and imperfections that approximate methods may yield values as accurate as those obtained with an "exact" analysis. A specific example is the analysis of a building frame for wind loads where the walls, partitions, and floors contribute an indeterminate amount to wind resistance. Wind forces calculated in the frame by either method are not accurate.
3. To design the members of a statically indeterminate structure, it is sometimes necessary to make an estimate of their sizes before structural analysis can begin by an exact method. Approximate analysis of the structure will yield forces from which reasonably good initial estimates can be made as to member sizes.
4. Approximate analyses are quite useful in rough-checking exact solutions.

From the discussion of influence lines in Section 14.3, you can see that unless a computer is used (a very practical alternative today), an exact analysis involving several different placements of the live loads would be a long and tedious affair. For this reason, it is common when a computer is not readily available to use some approximate methods of analysis, such as the ACI moment and shear coefficients, the equivalent rigid-frame method, the assumed point-of-inflection-location method, and others discussed in the pages to follow.

ACI Coefficients for Continuous Beams and Slabs

A very common method used for the design of continuous reinforced concrete structures involves the use of the ACI coefficients given in Section 6.5.2 of the ACI code. These coefficients, which are reproduced in Table 14.1, provide estimated maximum shears and moments for buildings of normal proportions.

TABLE 14.1 ACI Coefficients from ACI Tables 6.5.2 and 6.5.4

Positive moment	
End spans	
If discontinuous end is unrestrained	$\frac{1}{11}w_u\ell_n^2$
If discontinuous end is integral with the support	$\frac{1}{14}w_u\ell_n^2$
Interior spans	$\frac{1}{16}w_u\ell_n^2$
Negative moment at exterior face of first interior support	
Two spans	$\frac{1}{9}w_u\ell_n^2$
More than two spans	$\frac{1}{10}w_u\ell_n^2$
Negative moment at other faces of interior supports	$\frac{1}{11}w_u\ell_n^2$
Negative moment at face of all supports for (a) slabs with spans not exceeding 10 ft and (b) beams and girders where ratio of sum of column stiffnesses to beam stiffness exceeds eight at each end of the span	$\frac{1}{12}w_u\ell_n^2$
Negative moment at interior faces of exterior supports for members built integrally with their supports	
Where the support is a spandrel beam or girder	$\frac{1}{24}w_u\ell_n^2$
Where the support is a column	$\frac{1}{16}w_u\ell_n^2$
Shear in end members at face of first interior support	$1.15(w_u\ell_n/2)$
Shear at face of all other supports	$w_u\ell_n/2$

Courtesy of BASF Corporation.

One Shell Plaza, Houston, Texas, which has 52 stories and is 714 ft high.

The values calculated in this manner will usually be somewhat larger than those that would be obtained with an "exact" analysis. As a result, appreciable economy can normally be obtained by taking the time or effort to make such an analysis. In this regard, it should be realized that these coefficients are considered best applied to continuous frames having more than three or four continuous spans.

In developing the coefficients, the negative-moment values were reduced to take into account the usual support widths and also some moment redistribution, as described in Section 14.5 of this chapter.

In addition, the positive-moment values have been increased somewhat to account for the moment redistribution. It will also be noted that the coefficients account for the fact that, in monolithic construction, the supports are not simple and moments are present at end supports, such as where those supports are beams or columns.

In applying the coefficients, w_u is the design load while ℓ_n is the clear span for calculating positive moments and the average of the adjacent clear spans for calculating negative moments. These values can be applied only to members with approximately equal spans (the larger of two adjacent spans not exceeding the smaller by more than 20%) and for cases where the ratio of the uniform service live load to the uniform service dead load is not greater than three. In addition, the values are not applicable to prestressed concrete members. If these limitations are not met, a more precise method of analysis must be used.

For the design of a continuous beam or slab, the moment coefficients provide in effect two sets of moment diagrams for each span of the structure. One diagram is the result of placing the live loads so that they will cause maximum positive moment out in the span, whereas the other is the result of placing the live loads so as to cause maximum negative moments at the supports. To be truthful, however, it is not possible to produce maximum negative moments at both ends of a span simultaneously. It takes one placement of the live loads to produce maximum negative moment at one end of the span and another placement to produce maximum negative moment at the other end. The assumption of both maximums occurring at the same time is on the safe side, however, because the resulting diagram will have greater critical values than are produced by either one of the two separate loading conditions.

The ACI coefficients give maximum points for a moment envelope for each span of a continuous frame. Typical envelopes are shown in Figure 14.19 for a continuous slab, which is assumed to be constructed integrally with its exterior supports, which are spandrel girders.

Example 14.1 presents the design of the slab of Figure 14.20 using the moment coefficients of the ACI Code. The calculations for this problem can be conveniently set up in some type of table, such as the one shown in Figure 14.21. For this particular slab, the authors used an arrangement of reinforcement that included bent bars. It is quite common, however, in slabs—particularly those 5 in. or less in thickness—to use straight bars only in the top and bottom of the slab.

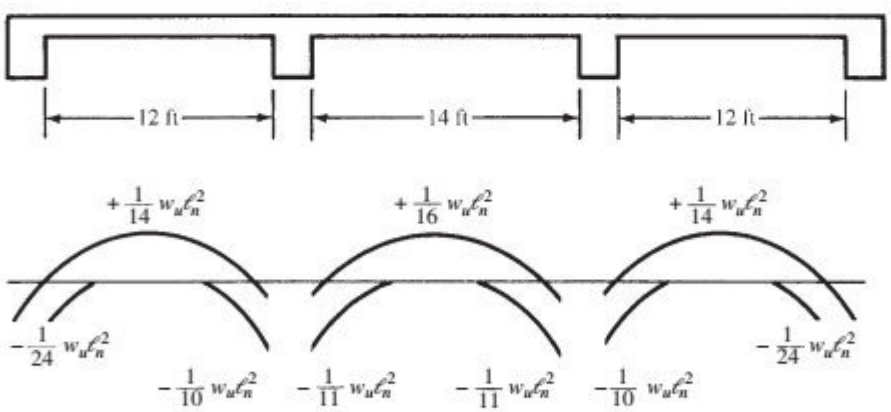


FIGURE 14.19 Moment envelopes for continuous slab constructed integrally with beams.

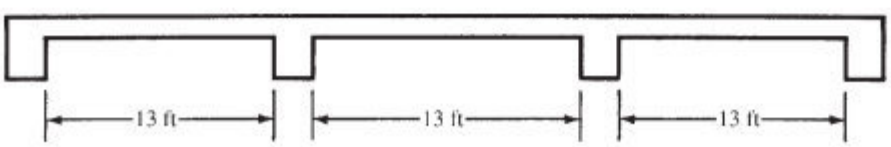
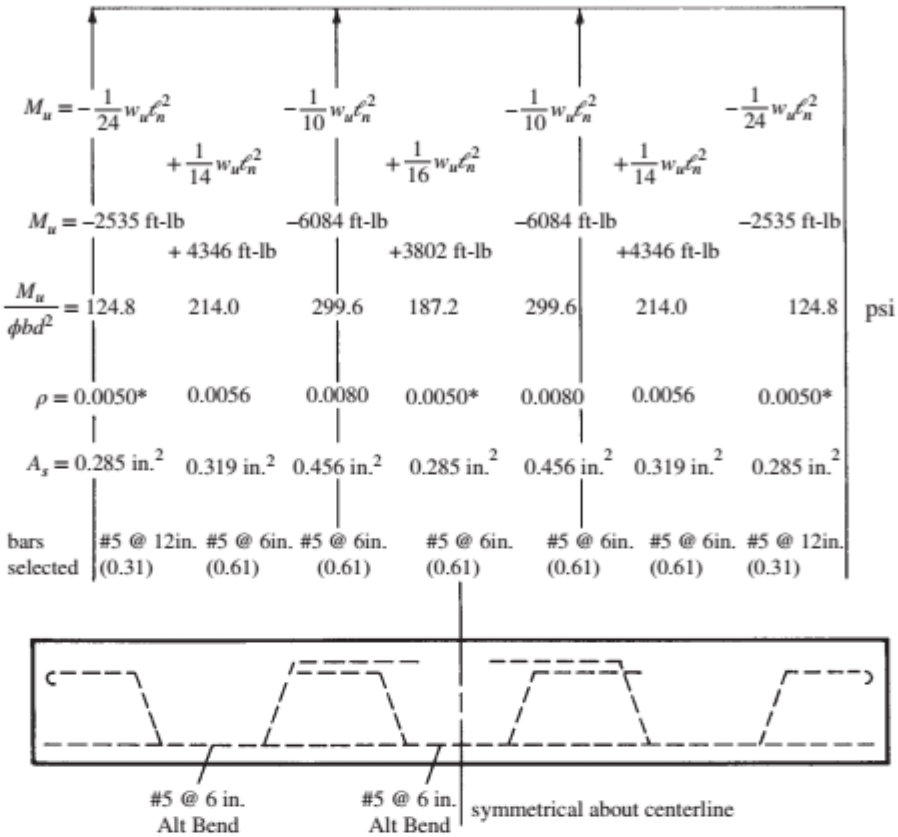


FIGURE 14.20 Clear spans for Example 14.1.



*The minimum ρ values specified in ACI Code 9.6.1 for tensile steel are applicable to both positive- and negative-moment regions (ACI Commentary R9.6.1.1).

FIGURE 14.21 Calculations for Example 14.1.

Example 14.1

Design the continuous slab of Figure 14.20 for moments calculated with the ACI coefficients. The slab is to support a service live load of 165 psf and a superimposed dead load of 5 psf in addition to its own dead weight. $f'_c = 3000$ psi and $f_y = 40{,}000$ psi. The slab is to be constructed integrally with its spandrel girder supports, and the spandrel supports are 12 in. wide.

SOLUTION

Minimum *t* for Deflection by ACI Code, Section 9.3.1.1 (Note that this table uses ℓ, not ℓ_n, as the span.)

$$\text{Deflection multiplier for 40,000-psi steel} = 0.4 + \frac{40{,}000 \text{ psi}}{100{,}000 \text{ psi}} = 0.80$$

$$\left.\begin{aligned} \text{Minimum } t \text{ for end span} &= 0.8\frac{\ell}{24} = \frac{(0.8)\,(12 \text{ in./ft})\,(13 \text{ ft} + 1 \text{ ft})}{24} = 5.6 \text{ in.} \\ \text{Minimum } t \text{ for interior span} &= 0.8\frac{\ell}{28} = \frac{(0.8)\,(12 \text{ in./ft})\,(13 \text{ ft} + 1 \text{ ft})}{28} = 4.80 \text{ in.} \end{aligned}\right\} \text{Select 6-in. slab}$$

Loads and Maximum Moment for a 6-in. Slab ($d = 4\frac{3}{4}$ in.)

$$w_D = \text{slab weight} = \left(\frac{6 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) = 75 \text{ psf}$$

$$w_L = 165 \text{ psf}$$

$$w_u = (1.2)\,(75 \text{ psf} + 5 \text{ psf}) + (1.6)\,(165 \text{ psf}) = 360 \text{ psf}$$

$$\text{Max } M_u = \left(\frac{1}{10}\right) w_u \ell_n^2 = \left(\frac{1}{10}\right)(360 \text{ psf})\,(13 \text{ ft})^2 = 6084 \text{ ft-lb}$$

Computed values of moments, ρ values, A_s requirements, and bar selections at each section are shown in Figure 14.21.

For floor slabs, we are not concerned with the design of web reinforcing. For continuous beams, however, web reinforcing must be carefully designed. Such designs are based on the maximum shears occurring at various sections along the span.

From previous discussions, you will remember that to determine the maximum shear occurring at section 1–1 in the beam of Figure 14.22, the uniform dead load would extend all across the span while the uniform live load would be placed from the section to the most distant support.

If the live load is placed so as to cause maximum shears at various points along the span and the shear is calculated for each point, a maximum shear curve can be drawn. Practically speaking, however, it is unnecessary to go through such a lengthy process for buildings of normal proportions because the values that would be obtained do not vary significantly from the values given by the ACI Code, which are shown in Figure 14.23.

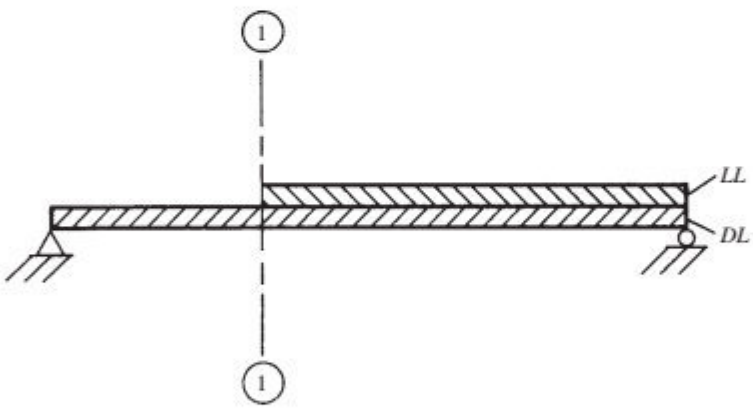


FIGURE 14.22 Position of live load for maximum shear at section 1–1.

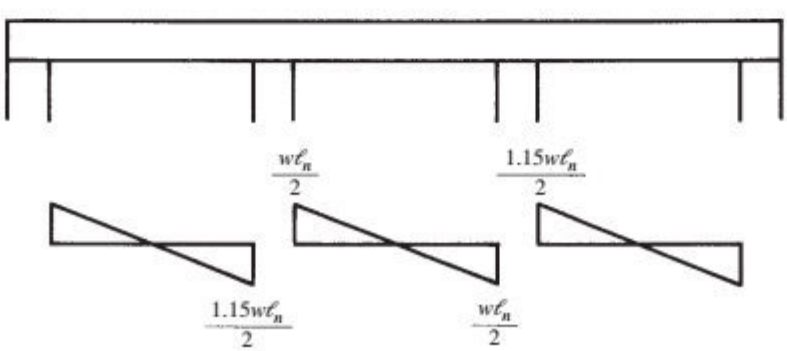


FIGURE 14.23 Shear diagrams resulting from use of shear coefficients in Table 14.1.

Equivalent Rigid-Frame Method

When continuous beams frame into and are supported by girders, the normal assumption made is that the girders provide only vertical support. Thus, they are analyzed purely as continuous beams, as shown in Figure 14.24. The girders do provide some torsional stiffness, and if the calculated torsional moments exceed $\phi\lambda\sqrt{f'_c}(A^2_{cp}/\rho_{cp})$ as specified in ACI Code, Section 22.7.4.1, they must be considered, as will be described in Chapter 15.

Where continuous beams frame into columns, the bending stiffnesses of the columns together with the torsional stiffnesses of the girders are of such magnitude that they must be considered. An approximate method frequently used for analyzing such reinforced concrete members is the *equivalent rigid-frame method*. In this method, which is applicable only to gravity loads, the loads are assumed to be applied only to the floor or roof under consideration and the far ends of the columns are assumed to be fixed, as shown in Figure 14.25. The sizes of the members are estimated, and an analysis is frequently made with moment distribution.

For this type of analysis, it is necessary to estimate the sizes of the members and compute their relative stiffness, or I/ℓ, values. From these values, distribution factors can be computed and the method of moment distribution applied. The moments of inertia of both columns and beams are normally calculated on the basis of gross concrete sections, with no allowance made for reinforcement.

There is a problem involved in determining the moment of inertia to be used for continuous T beams. The moment of inertia of a T beam is much greater where there is positive moment with the flanges in compression than where there is negative moment with the flanges cracked because of tension. Because the moment of inertia varies along the span, it is necessary to use an equivalent value. A practice often used is to assume that the equivalent moment of inertia equals twice the moment of inertia of the web, assuming that the web depth equals the full effective depth of the beam.[5] Some designers use other equivalent values, such as assuming an equivalent T section with flanges of effective widths equal to so many (say, two to six) times the web width. These equivalent sections can be varied over a rather wide range without appreciably affecting the final moments.

Section 6.4.2 of the ACI Code states that for such an approximate analysis, only two live-load combinations need to be considered. These are (1) live load placed on two adjacent spans and (2) live load placed on alternate spans. Example 14.2 illustrates the application of the equivalent rigid-frame method to a continuous T beam.

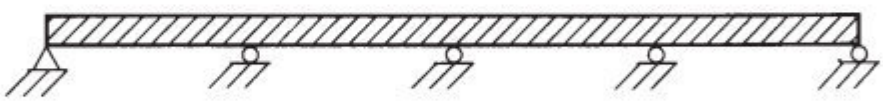


FIGURE 14.24 Continuous with only vertical support.

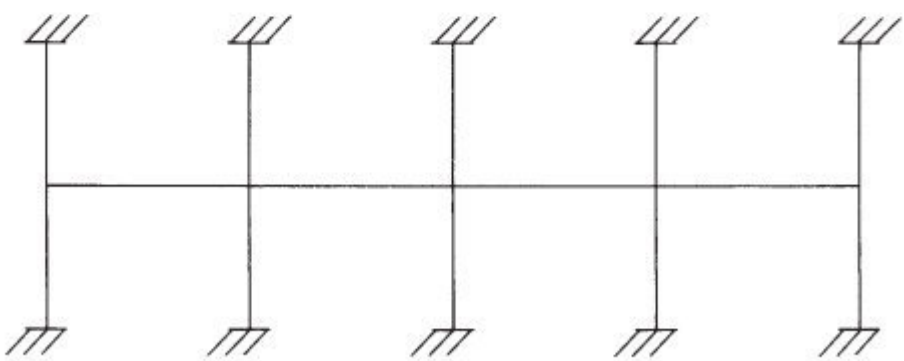


FIGURE 14.25 Continuous with both vertical and rotational supports.

[5] Portland Cement Association, 1959, *Continuity in Concrete Building Frames*, 4th ed. (Chicago, IL: PCA), pp. 17–20.

Computer results appear to indicate that the model shown in Figure 14.25 (as permitted by the ACI Code) may not be trustworthy for unsymmetrical loading. Differential column shortening can completely redistribute the moments obtained from the model (i.e., positive moments can become negative moments). As a result, designers should take into account possible axial deformations in their designs.

Example 14.2

Using the equivalent rigid-frame method, draw the shear and moment diagrams for the continuous T beam of Figure 14.26. The beam is assumed to be framed into 16-in. × 16-in. columns and is to support a service dead load of 2.33 k/ft (including beam weight) and a service live load of 3.19 k/ft. Assume that the live load is applied in the center span only. The girders are assumed to have a depth of 24 in. and a web width of 12 in. Assume that the I of the T beam equals two times the I of its web.

SOLUTION

Computing Fixed-End Moments

$$w_u \text{ in first and third spans} = (1.2)\,(2.33 \text{ klf}) = 2.8 \text{ k/ft}$$

$$M_u = \frac{(2.8 \text{ klf})(24 \text{ ft})^2}{12} = 134.4 \text{ ft-k}$$

$$w_u \text{ in center span} = (1.2)\,(2.33 \text{ klf}) + (1.6)\,(3.19 \text{ klf}) = 7.9 \text{ k/ft}$$

$$M_u = \frac{(7.9 \text{ klf})\,(24 \text{ ft})^2}{12} = 379.2 \text{ ft-k}$$

Computing Stiffness Factors

$$I \text{ of columns} = \left(\frac{1}{12}\right)(16 \text{ in.})\,(16 \text{ in.})^3 = 5461 \text{ in.}^4$$

$$k \text{ of columns} = \frac{I}{\ell} = \frac{5461 \text{ in.}^4}{12 \text{ ft}} = 455 \text{ in.}^4/\text{ft}$$

$$\text{Equivalent } I \text{ of T beam} = (2)\left(\frac{1}{12} b_w h^3\right) = (2)\left(\frac{1}{12}\right)(12 \text{ in.})\,(24 \text{ in.})^3 = 27{,}648 \text{ in.}^4$$

$$k \text{ of T beam} = \frac{27{,}648 \text{ in.}^4}{24 \text{ ft}} = 1152 \text{ in.}^4/\text{ft}$$

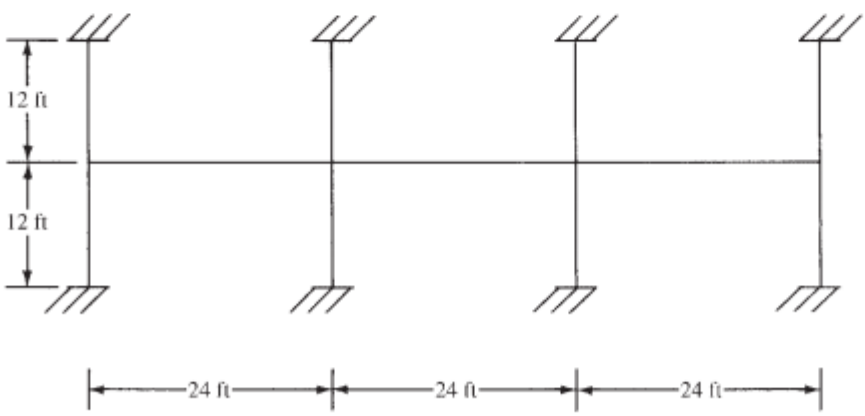


FIGURE 14.26 Frame dimensions for Example 14.2.

Record stiffness factors on the frame and compute distribution factors, as shown in Figure 14.27. Balance fixed-end moments and draw shear and moment diagrams, as shown in Figure 14.28.

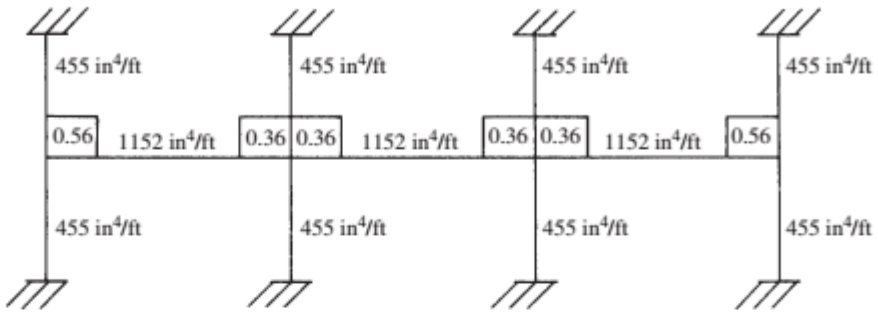


FIGURE 14.27 Stiffness and carry-over factors for Example 14.2.

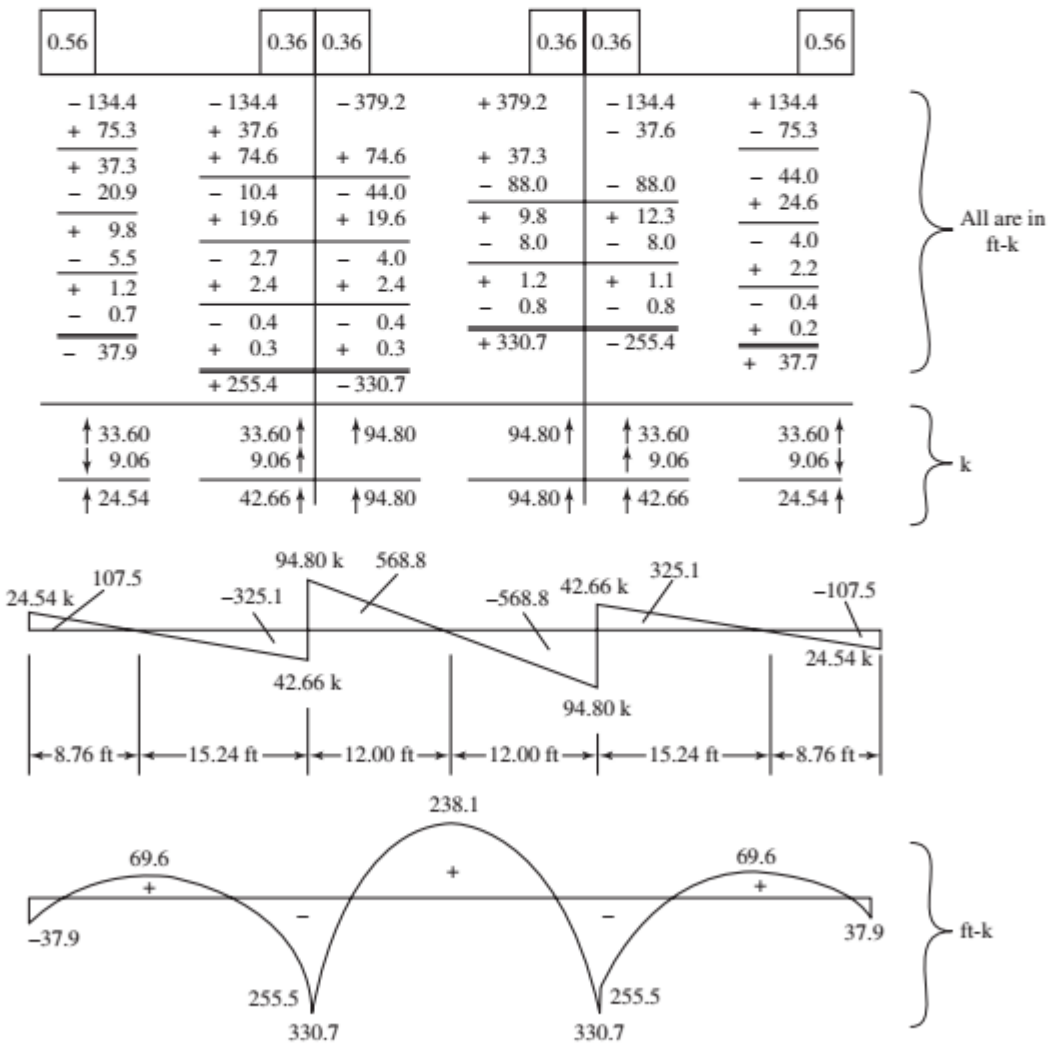


FIGURE 14.28 Results of moment distribution of Example 14.2.

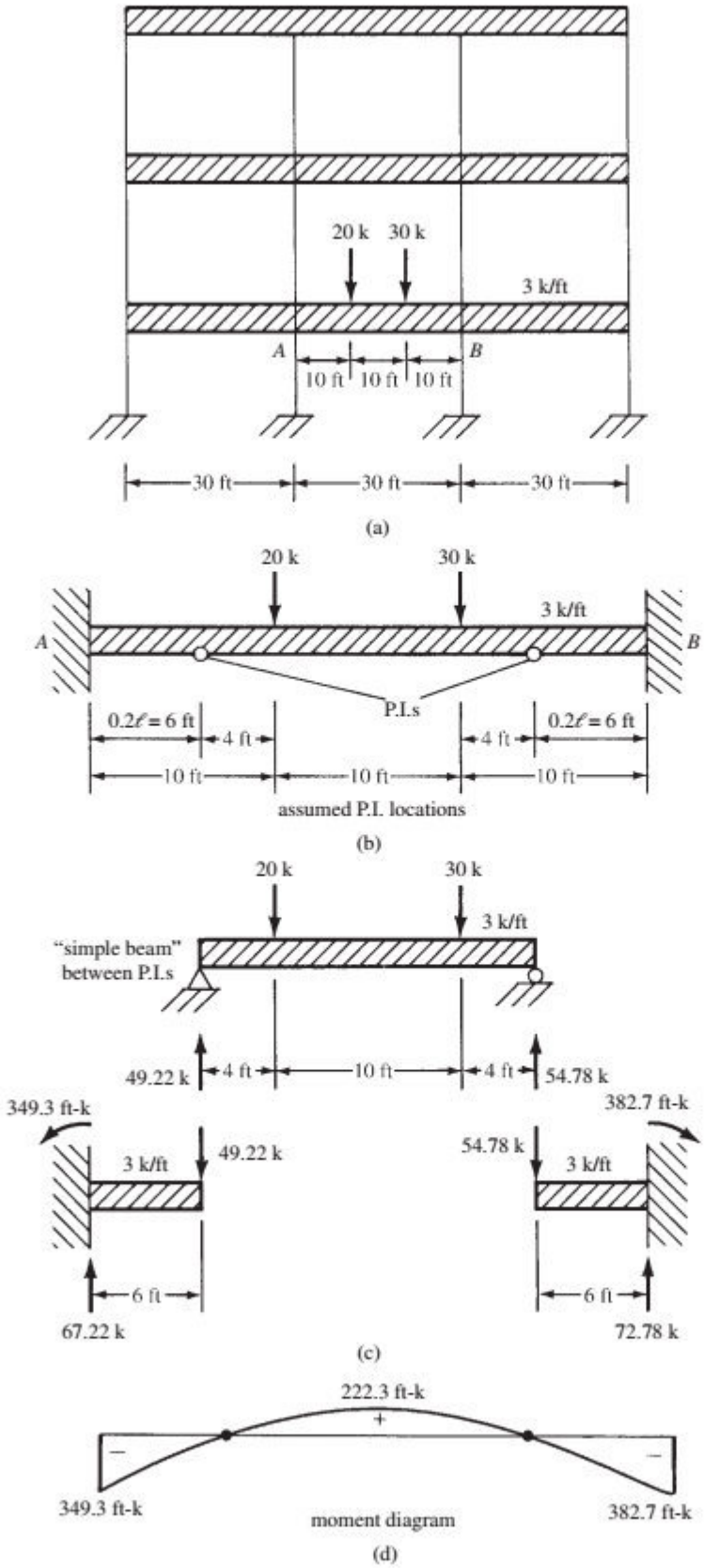


FIGURE 14.29 Assumed points-of-inflection method applied to middle span of Example 14.2.

Assumed Points of Inflection

Another approximate method of analyzing statically indeterminate building frames is to assume the locations of the points of inflection in the members. Such assumptions have the effect of creating simple beams between the points of inflection in each span, and the positive moments in each span can be determined by statics. Negative moments occur in the girders between their ends and the points of inflection. They may be computed by considering the portion of each beam out to the point of inflection (P.I.) to be a cantilever. The shear at the end of each of the girders contributes to the axial forces in the columns. Similarly, the negative moments at the ends of the girders are transferred to the columns.

In Figure 14.29, beam *AB* of the building frame shown is analyzed by assuming points of inflection at the one-fifth points and assuming fixed supports at the beam ends.

14.8 Approximate Analysis of Continuous Frames for Lateral Loads

Building frames are subjected to lateral loads as well as to vertical loads. The necessity for careful attention to these forces increases as buildings become taller. Buildings must not only have sufficient lateral resistance to prevent failure, but also must have sufficient resistance to deflections to prevent injuries to their various parts. Rigid-frame buildings are highly statically indeterminate; their analysis by "exact" methods (unless computers are used) is so lengthy as to make the approximate methods very popular.

The approximate method presented here is called the *portal method*. Because of its simplicity, it has probably been used more than any other approximate method for determining wind forces in building frames. This method, which was presented by Albert Smith in the *Journal of the Western Society of Engineers* in April 1915, is said to be satisfactory for most buildings up to 25 stories in height. Another method very similar to the portal method is the cantilever method. It is thought to give slightly better results for high narrow buildings and can be used for buildings not in excess of 25 to 35 stories.[6]

Courtesy of EFCO Corp.

The 16-story Apoquindo Tower, Santiago, Chile.

[6] "Wind Bracing in Steel Buildings," 1940, *Transactions of the American Society of Civil Engineers*, 105, pp. 1723–1727.

The portal method is merely a variation of the method described in Section 14.7 for analyzing beams in which the location of the points of inflection was assumed. For the portal method, the loads are assumed to be applied at the joints only. If this loading condition is correct, the moments will vary linearly in the members, and points of inflection will be located fairly close to member midpoints.

No consideration is given in the portal method to the elastic properties of the members. These omissions can be very serious in unsymmetrical frames and in very tall buildings. To illustrate the seriousness of the matter, the changes in member sizes are considered in a very tall building. In such a building, there will probably not be a great deal of change in beam sizes from the top floor to the bottom floor. For the same loadings and spans, the changed sizes would be the result of the large wind moments in the lower floors. The change in column sizes from top to bottom, however, would be tremendous. The result is that the relative sizes of columns and beams on the top floors are entirely different from the relative sizes on the lower floors. When this fact is not considered, it causes large errors in the analysis.

In the portal method, the entire wind loads are assumed to be resisted by the building frames, with no stiffening assistance from the floors, walls, and partitions. Changes in the lengths of girders and columns are assumed to be negligible. They are not negligible, however, in tall slender buildings, the height of which is five or more times the least horizontal dimension.

If the height of a building is roughly five or more times its least lateral dimension, it is generally felt that a more precise method of analysis should be used. There are several approximate methods that make use of the elastic properties of structures and give values closely approaching the results of the "exact" methods. These include the factor method,[7] the Witmer method of K percentages, and the Spurr method.[8]

The building frame shown in Figure 14.30 is analyzed by the portal method, as described in the following paragraphs.

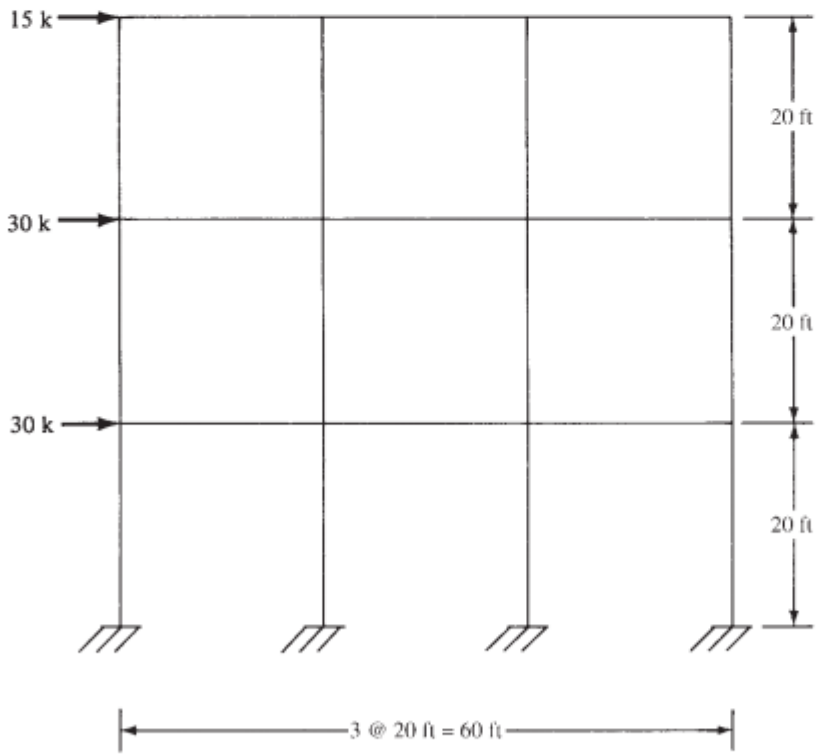


FIGURE 14.30 Frame analyzed by the portal method.

[7] Norris, C. H., Wilbur, J. B., and Utku, S., 1976, *Elementary Structural Analysis*, 3rd ed. (New York: McGraw-Hill), pp. 207–212.
[8] "Wind Bracing in Steel Buildings," pp. 1723–1727.

FIGURE 14.31 One level of frame of Figure 14.30.

At least three assumptions must be made for each individual portal or for each girder. In the portal method, the frame is theoretically divided into independent portals (Figure 14.31), and the following three assumptions are made:

1. The columns are bend in such a manner that there is a point of inflection at middepth.
2. The girders are bend in such a manner that there is a point of inflection at their centerlines.
3. The horizontal shears on each level are arbitrarily distributed between the columns. One commonly used distribution (and the one illustrated here) is to assume that the shear divides among the columns in the ratio of one part to exterior columns and two parts to interior columns.

The reason for the ratio in assumption 3 can be seen in Figure 14.31. Each of the interior columns is serving two bents, whereas the exterior columns are serving only one. Another common distribution is to assume that the shear, V, taken by each column is in proportion to the floor area it supports. The shear distribution by the two procedures would be the same for a building with equal bays, but for one with unequal bays the results would differ, with the floor area method probably giving more realistic results.

Frame Analysis by Portal Method

The frame of Figure 14.30 is analyzed in Figure 14.32 on the basis of the preceding assumptions. The arrows shown on the figure give the direction of the girder shears and the column axial forces. You can visualize the stress condition of the frame if you assume that the wind is tending to push it over from left to right, stretching the left exterior columns and compressing the right exterior columns. Briefly, the calculations were made as follows:

1. *Column shears*—The shears in each column on the various levels were obtained first. The total shear on the top level is 15 k. Because there are two exterior and two interior columns, the following expression may be written:

$$x + 2x + 2x + x = 15 \text{ k}$$
$$x = 2.5 \text{ k}$$
$$2x = 5.0 \text{ k}$$

   The shear in column *CD* is 2.5 k, in *GH* it is 5.0 k, and so on. Similarly, the member shears were determined for the columns on the first and second levels, where the total shears are 75 k and 45 k, respectively.

2. *Column moments*—The columns are assumed to have points of inflection at their middepths; therefore, their moments, top and bottom, equal the column shears times half the column heights.

3. *Girder moments and shears*—At any joint in the frame, the sum of the moments in the girders equals the sum of the moments in the columns. The column moments have been previously determined. Beginning at the upper-left corner of the frame and working across from left to right, adding or subtracting the moments as the case may be, the girder moments were found in this order: *DH*, *HL*, *LP*, *CG*, *GK*, and so on. It follows that with points of inflection at girder centerlines, the girder shears equal the girder moments divided by half-girder lengths.

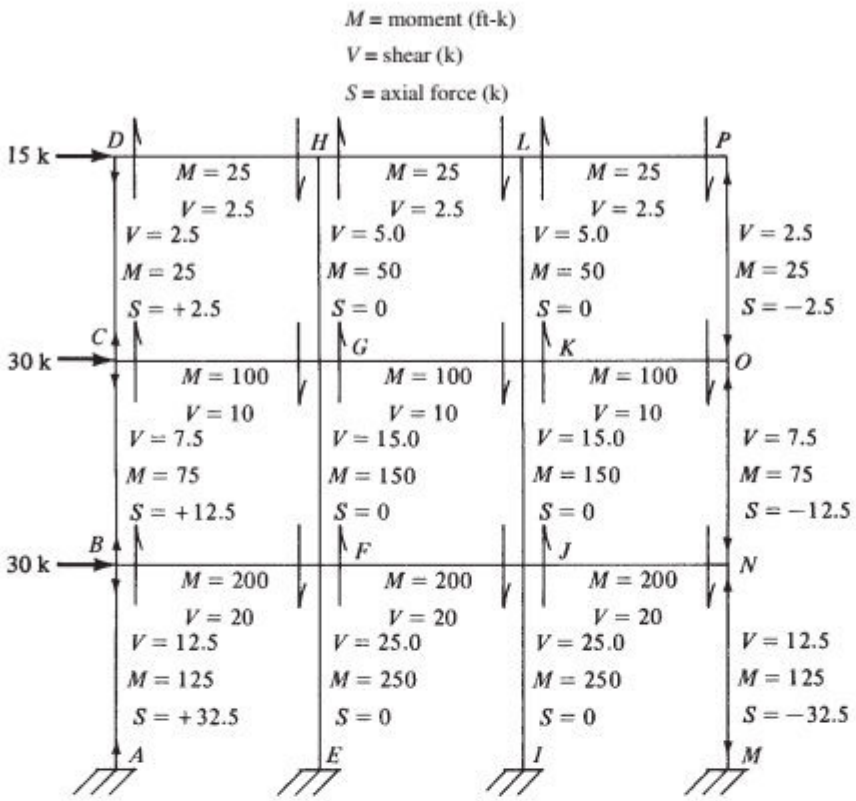


FIGURE 14.32 Results of portal analysis.

4. *Column axial forces*—The axial forces in the columns may be directly obtained from the girder shears. Starting at the upper-left corner, the column axial force in *CD* is equal to the shear in girder *DH*. The axial force in column *GH* is equal to the difference between the two girder shears *DH* and *HL*, which equals zero in this case. (If the width of each of the portals is the same, the shears in the girder on one level will be equal and the interior columns will have no axial force, because only lateral loads are considered.)

14.9 Computer Analysis of Building Frames

All structures are three-dimensional, but theoretical analyses of such structures by hand calculation methods are so lengthy as to be impractical. As a result, such systems are normally assumed to consist of two-dimensional or planar systems, and they are analyzed independently of each other. The methods of analysis presented in this chapter were handled in this manner.

Today's computers have greatly changed the picture, and it is now possible to analyze complete three-dimensional structures. As a result, more realistic analyses are available, and the necessity for high safety factors is reduced. The application of computers is not restricted merely to analysis; they are used in almost every phase of concrete work from analysis to design to detailing to specification writing to material take-offs to cost estimating and so on.

Another major advantage of computer analysis for building frames is that the designer is able to consider many different loading patterns quickly. The results are sometimes rather surprising.

14.10 Lateral Bracing for Buildings

For the usual building, the designer will select relatively small columns. Although such a procedure results in more floor space, it also results in buildings with small lateral stiffnesses or resistance to wind and earthquake loads. Such buildings may have detrimental lateral deflections and vibrations during windstorms unless definite lateral stiffness or bracing is otherwise provided in the structure.

To provide lateral stiffness, it is necessary for the roof and floor slabs to be attached to rigid walls, stairwells, or elevator shafts. Sometimes structural walls, called *shear walls*, are added to a structure to provide the necessary lateral resistance. (The design of shear walls is considered in Chapter 18 of this text.) If it is not possible to provide such walls, stairwells or elevator shafts may be designed as large, box-shaped beams to transmit lateral loads to the supporting foundations. These members will behave as large cantilever beams. In designing such members, the designer should try to keep resistance symmetrical so as to prevent uneven lateral twisting or torsion in the structure when lateral loads are applied.[9]

14.11 Development Length Requirements for Continuous Members

In Chapter 7, a general introduction to the subject of development lengths for simple and cantilever beams was presented. This section examines the ACI development length requirements for continuous members for both positive and negative reinforcement. After studying this information, the reader probably will be pleased to find at the end of this section a discussion of simplified design office practices for determining bar lengths.

Positive-Moment Reinforcement

Section 9.7.3.8 of the ACI code provides several detailed requirements for the lengths of positive-moment reinforcement. These are briefly summarized in the following paragraphs.

1. At least one-third of the positive steel in simple beams and one-fourth of the positive steel in continuous members must extend uninterrupted along the same face of the member at least 6 in. in the support (Section 9.7.3.8.1). The purpose of this requirement is to make sure that the moment resistance of a beam will not be reduced excessively in parts of beams where the moments may be changed due to settlements, lateral loads, and so on.
2. The positive reinforcement required in the preceding paragraph must, *if the member is part of a seismic load resisting system*, be extended into the support a sufficient distance to develop the yield stress in tension of the bars at the face of the support. This requirement is included in Section 18.3.2 of the ACI Code to ensure a ductile response to severe overstress, as might occur with moment reversal during an earthquake. As a result of this requirement, it is necessary to have bottom bars lapped at interior supports and to use additional embedment lengths and hooks at exterior supports.
3. Section 9.7.3.8.3 of the ACI Code says that at simple supports and at points of inflection, the positive-moment tension bars must have their diameters limited to certain maximum sizes. The purpose of the limitation is to keep bond stresses within reason at these points of low moments

[9] Leet, K., 1991, *Reinforced Concrete Design*, 2nd ed. (New York: McGraw-Hill), pp. 453–454.

and large shears. (It is to be remembered that when bar diameters are smaller, the bars have greater surface area in proportion to their cross-sectional areas. Thus, for bonding to concrete, the larger the bars' diameters, the larger must be their development lengths. This fact is reflected in the expressions for ℓ_d.) It has not been shown that long anchorage lengths are fully effective in developing bars in a short distance between a point of inflection and a point of maximum bar stress, a condition that might occur in heavily loaded short beams with large bottom bars. It is specified that ℓ_d as computed by the requirements presented in Chapter 7 may not exceed the following:

$$\ell_d \leq \frac{M_n}{V_u} + \ell_a \qquad \text{(ACI Section 9.7.3.8.3b)}$$

In this expression, M_n is the computed theoretical flexural strength of the member if all reinforcement in that part of the beam is assumed stressed to f_y, and V_u is the factored shear at the section. At a support, ℓ_a is equal to the sum of the embedment length beyond the centerline of the support and the equivalent embedment length of any furnished hooks or mechanical anchorage. Headed and mechanically anchored deformed bars, which are permitted in Section 25.4.4 of the ACI Code, consist of bars or plates or angles or other pieces welded or otherwise attached transversely to the flexural bars in locations where sufficient anchorage length is not available. See Sections 1.15 and 7.15 of this text for details on headed anchorage. At a point of inflection, ℓ_a is equal to the larger of the effective depth of the member or $12d$ (ACI Code, Section 9.7.3.8.3). When the ends of the reinforcement are confined by a compression reaction, such as when there is a column below but not when a beam frames into a girder, an increase of 30% in the value M_n/V_u to $1.3M_n/V_u$ is allowed. The values described here are summarized in Figure 14.33. A brief numerical illustration for a simple end-supported beam is provided in Example 14.3.

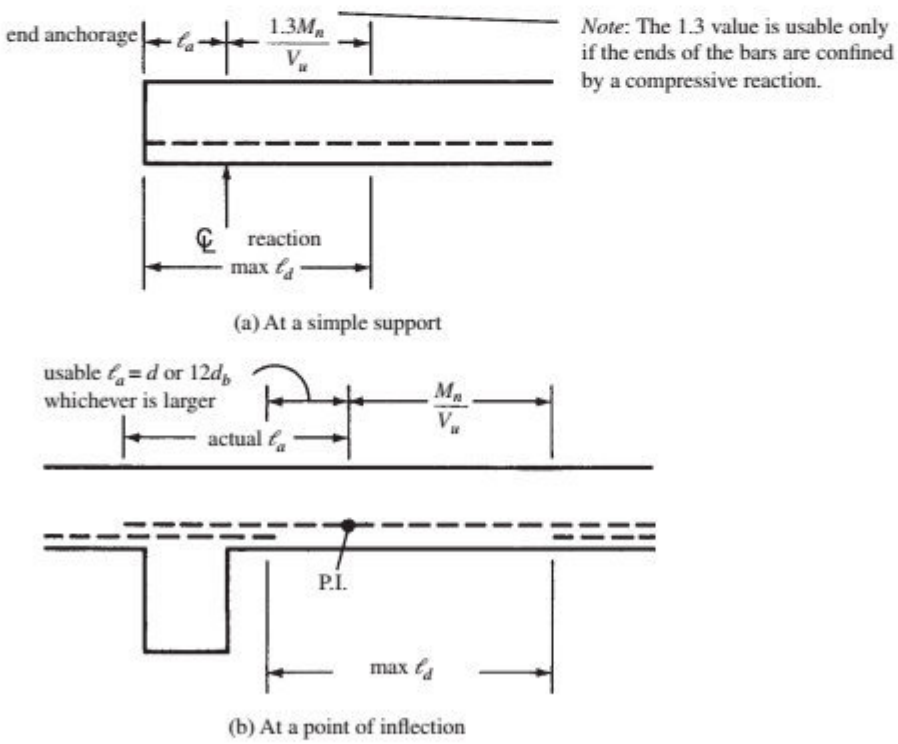


FIGURE 14.33 Development length requirements for positive-moment reinforcement.

Example 14.3

At the simple support shown in Figure 14.34, two uncoated #9 bars have been extended from the maximum moment area and into the support. Are the bar sizes satisfactory if $f_y = 60$ psi, $f'_c = 3$ ksi, $b = 12$ in., $d = 24$ in., and $V_u = 65$ k; if normal sand-gravel concrete is used; and if the reaction is compressive? Assume cover for bars = 2 in. clear.

SOLUTION

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(2.00 \text{ in.}^2)\ (60 \text{ ksi})}{(0.85)\ (3 \text{ ksi})\ (12 \text{ in.})} = 3.92 \text{ in.}$$

$$M_n = A_s f_y \left(d - \frac{a}{2}\right) = (2.00 \text{ in.}^2)\ (60 \text{ ksi})\ \left(24 \text{ in.} - \frac{3.92 \text{ in.}}{2}\right) = 2644.8 \text{ in.-k}$$

$$c = \text{side cover} = 2 \text{ in.} + \frac{1.128 \text{ in.}}{2} = 2.564 \text{ in.} \leftarrow$$

$$c = \text{one-half to c. to c. spacing of bars} = \left(\frac{1}{2}\right)\ (12 \text{ in.} - 2 \times 2 \text{ in.} - 1.128 \text{ in.}) = 3.436 \text{ in.}$$

Assume $K_{tr} = 0$.

$$\frac{\ell_d}{d_b} = \frac{3}{40} \frac{f_y}{\lambda \sqrt{f'_c}} \frac{\psi_t \psi_e \psi_s}{\dfrac{c_b + K_{tr}}{d_b}}$$

$$= \left(\frac{3}{40}\right)\ \left[\frac{60{,}000 \text{ psi}}{(1.0)\ \sqrt{3000} \text{ psi}}\right] \frac{(1.0)\ (1.0)\ (1.0)}{\dfrac{2.564 \text{ in.} + 0}{1.128 \text{ in.}}} = 36.14 \text{ diameters}$$

$$\ell_d = (36.14)\ (1.128 \text{ in.}) = 41 \text{ in.}$$

$$\text{Maximum permissible } \ell_d = \frac{M_n}{V_u} + \ell_a$$

$$= (1.3)\ \left(\frac{2644.8 \text{ in.-k}}{65 \text{ k}}\right) + 7 \text{ in.} = 59.90 \text{ in.} > 41 \text{ in.} \quad \underline{\underline{\text{OK}}}$$

Note: If this condition has not been satisfied, the permissible value of ℓ_d could have been increased by using smaller bars or by increasing the end anchorage, ℓ_a, for example, by the use of hooks.

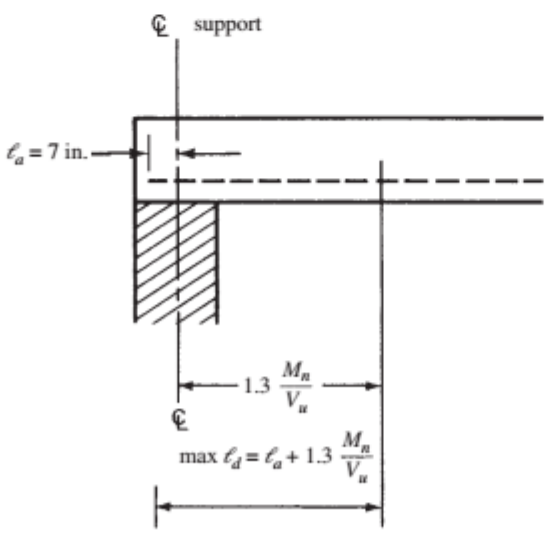


FIGURE 14.34 Support details for Example 14.3.

Negative-Moment Reinforcement

Section 9.7.3.1 of the ACI Code says that the tension reinforcement required by negative moments must be properly anchored into or through the supporting member. The tension in these negative bars shall be developed on each side of the section in question by embedment length, hooks, or mechanical anchorage. Hooks may be used because these are tension bars.

Section 9.7.3.3 of the ACI Code says that the reinforcement must extend beyond the point where it is no longer required for moment by a distance equal to the effective depth d of the member or 12 bar diameters, whichever is greater, except at the supports of simple beams or at the free ends of cantilevers. Section 9.7.3.8.4 of the ACI Code says that at least one-third of the total reinforcement provided for negative moment at the support must have an embedment length beyond the point of inflection no less than the effective depth of the member, 12 bar diameters, or one-sixteenth of the clear span of the member, whichever is greatest. The location of the point of inflection can vary for different load combinations; therefore, be certain to use the most critical case.

Example 14.4, which follows, presents a detailed consideration of the lengths required by the code for a set of straight negative-moment bars at an interior support of a continuous beam.

Example 14.4

At the first interior support of the continuous beam shown in Figure 14.35, six #8 straight uncoated top bars (4.71 in.^{2}) are used to resist a moment M_u of 390 ft-k for which the calculated A_s required is 4.24 in.^{2} If $f_y = 60$ ksi, $f'_c = 3$ ksi, and normal sand-gravel concrete is used, determine the length of the bars as required by the ACI Code. The dimensions of the beam are $b_w = 14$ in. and $d = 24$ in. Assume bar cover = $2d_b$ and clear spacing = $3d_b$.

Note: The six negative-tension bars are actually placed in one layer in the top of the beam, but they are shown for illustration purposes in Figures 14.35 and 14.36 as though they are arranged in three layers of two bars each. In the solution that follows, the "bottom" two bars are cut off first, at the point where the moment plus the required development length is furnished; the "middle" two are cut when the moment plus the needed development length permits; and the "top" two are cut off at the required distance beyond the P.I.

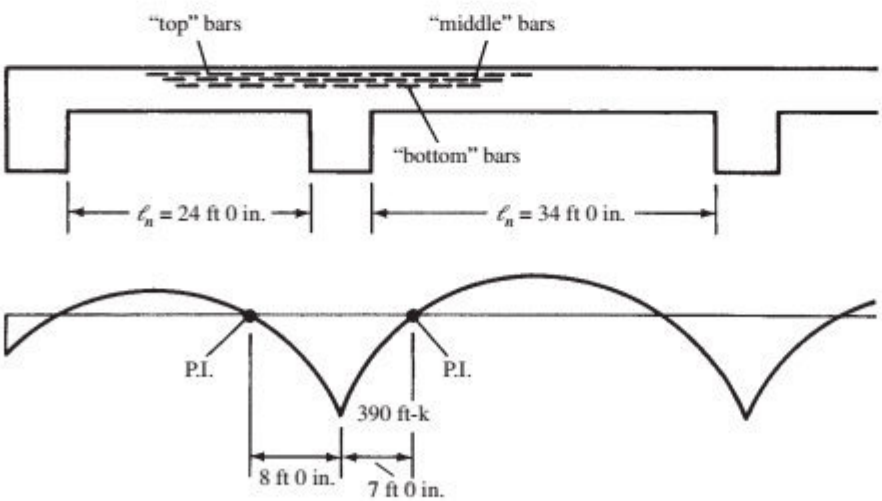


FIGURE 14.35 Continuous beam for Example 14.4.

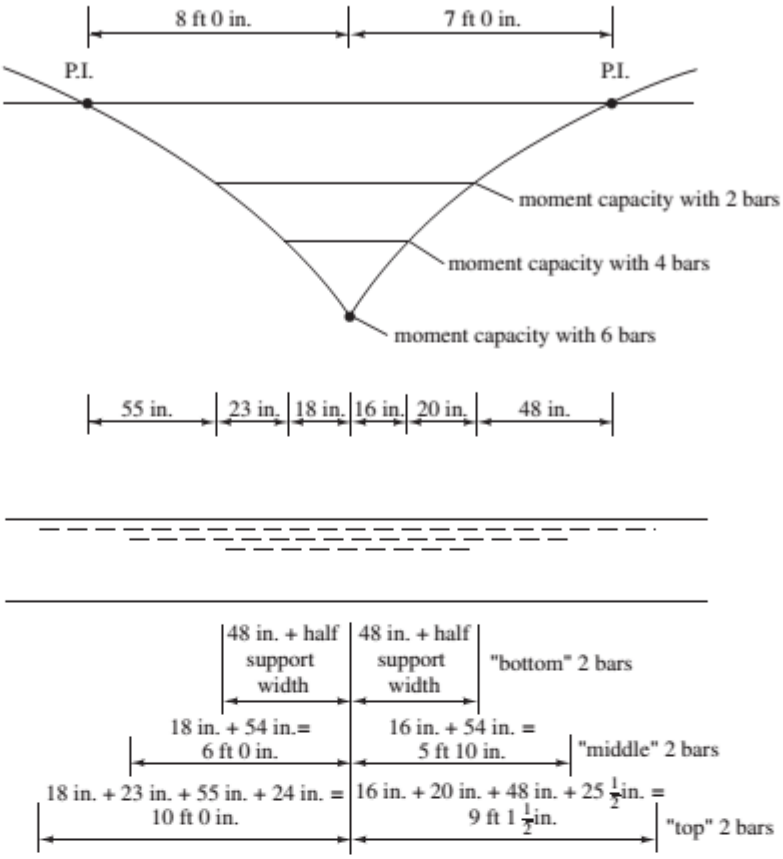

8 ft 0 in.
7 ft 0 in.
P.I.
P.I.
moment capacity with 2 bars
moment capacity with 4 bars
moment capacity with 6 bars
55 in.
23 in.
18 in.
16 in.
20 in.
48 in.
48 in. + half support width
48 in. + half support width
"bottom" 2 bars
18 in. + 54 in.= 6 ft 0 in.
16 in. + 54 in. = 5 ft 10 in.
"middle" 2 bars
18 in. + 23 in. + 55 in. + 24 in. = 10 ft 0 in.
16 in. + 20 in. + 48 in. + 25 1/2 in. = 9 ft 1 1/2 in.
"top" 2 bars

FIGURE 14.36 Bar cutoff details for Example 14.4.

SOLUTION

1. Required Development Length

$$c_b = \text{side cover} = (2)\ (1.0\text{ in.}) + \frac{1.0\text{ in.}}{2} = 2.5\text{ in.}$$

$$c_b = \text{one-half of c. to c. spacing of bars} = \left(\frac{3}{2}\right)\ (1.0\text{ in.}) + \left(\frac{1.0\text{ in.}}{2}\right) = 2.0\text{ in.} \leftarrow \text{controls}$$

Assume $K_{tr} = 0$

$$\frac{\ell_d}{d_b} = \left(\frac{3}{40}\right)\left[\frac{60{,}000\text{ psi}}{(1.0)\sqrt{3000}\text{ psi}}\right]\frac{(1.3)\ (1.0)\ (1.0)}{\frac{2.0+0}{1.0}} = 53.4\text{ diameters}$$

$$\ell_d = (53.41)\ (1.0\text{ in.}) = 53.4\text{ in.} \quad \text{Say } 54\text{ in.} \quad \underline{\text{OK}}$$

2. Section 9.7.3.8.4 of the ACI Code requires one-third of the bars to extend beyond P.I. $\left(\frac{1}{3} \times 6 = 2\text{ bars}\right)$ for a distance equal to the largest of the following values (applies to "top" bars in figure).

(a) d of web = 24 in.

(b) $12d_b = (12)\ (1\text{ in.}) = 12\text{ in.}$

(c) $\left(\frac{1}{16}\right)$ (24 ft × 12 in./ft) for 24 in. span = 18 in.

(d) $\left(\frac{1}{16}\right)$ (34 ft × 12 in./ft) for 34 in. span = $25\frac{1}{2}$ in.

Use 24 in. for short span and $25\frac{1}{2}$ in. for long span

3. Section 9.7.3.3 of the ACI Code requires that the bars shall extend beyond the point where they are no longer required by moment for a distance equal to the greater of

   (a) $d = 24$ in.

   (b) $12d_b = (12)\ (1\text{ in.}) = 12$ in.

4. Section 12.3.3 of the ACI Code says required ℓ_d can be multiplied by $A_{s\ reqd}/A_{s\ furn}$. This applies only to the first bars cut off ("bottom" bars here).

$$\text{Reduced } \ell_d = (53.4\text{ in.}) \left(\frac{4.24\text{ in.}^2}{4.71\text{ in.}^2}\right) = 48\text{ in.}$$

5. In Figure 14.36, these values are applied to the bars in question and the results are given.

The structural designer is seldom involved with a fixed-moment diagram—the loads move and the moment diagram changes. Therefore, Section 9.7.3.3 of the ACI Code says that reinforcing bars should be continued for a distance of 12 bar diameters or the effective depth of the member, whichever is greater (except at the supports of simple spans and the free ends of cantilevers), beyond their theoretical cutoff points.

As previously mentioned, the bars must be embedded a distance ℓ_d from their point of maximum stress.

Next, Section 9.7.3.8.1 of the ACI Code says that at least one-third of the positive steel in simple spans and one-fourth of the positive steel in continuous spans must be continued along the same face of the beam at least 6 in. into the support.

Somewhat similar rules are provided by the ACI Code, Section 9.7.3.8.4 for negative-moment steel. At least one-third of the negative steel provided at a support must be extended beyond its point of inflection a distance equal to one-sixteenth of the clear span or 12 bar diameters or the effective depth of the member, whichever is greatest. Other negative bars must be extended beyond their theoretical point of cutoff by the effective depth, 12 bar diameters and at least ℓ_d from the face of the support.

Trying to go through these various calculations for cutoff or bend points for all of the bars in even a modest-size structure can be a very large job. Therefore, the average designer or perhaps the structural draftsperson will cut off or bend bars by certain general guidelines, which have been developed to meet the code rules described here. In Figure 14.37, a sample set of such rules is given for continuous beams. In the *CRSI Handbook*,[10] such rules are provided for several different types of structural members, such as solid one-way slabs, one-way concrete joists, two-way slabs, and so on.

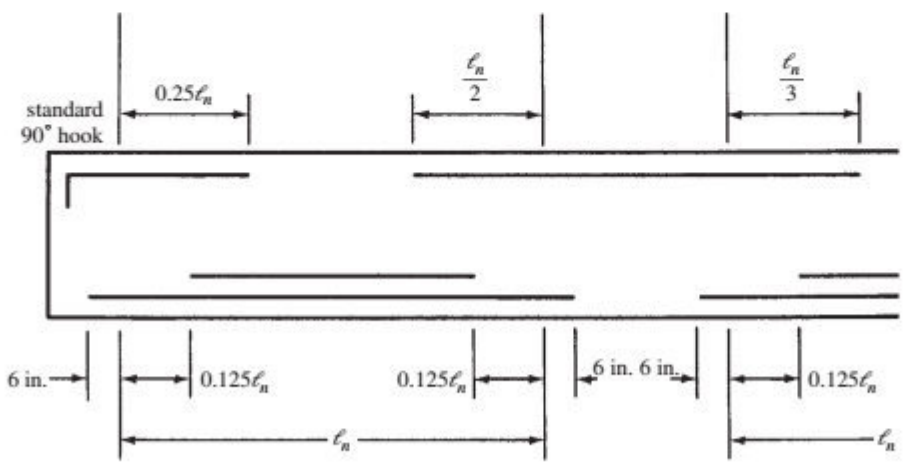


FIGURE 14.37 Recommended bar details for continuous beams.

[10] Concrete Reinforcing Steel Institute, 2002 *CRSI Handbook*, 9th ed., (Chicago), p. 12-1.

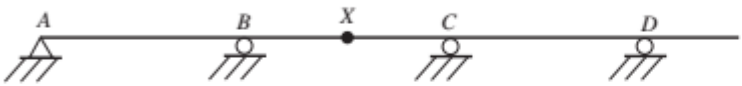
PROBLEMS

Qualitative Influence Lines

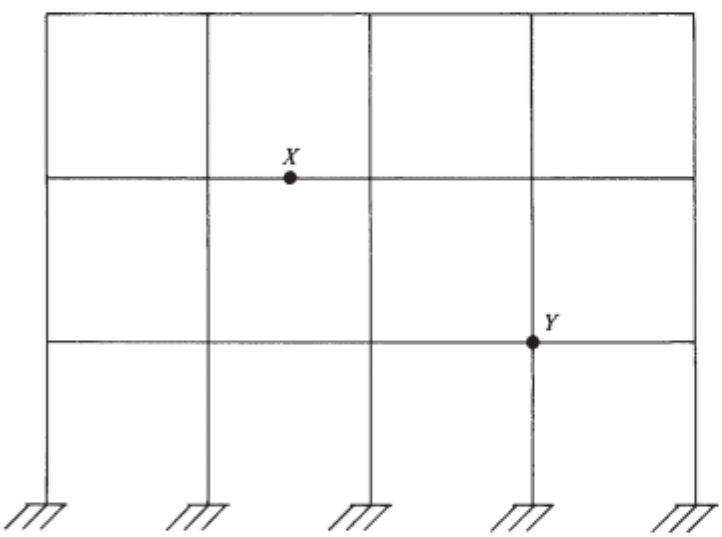
For Problems 14.1 to 14.3, draw qualitative influence lines for the functions indicated in the structures shown.

Problem 14.1 Reactions at A and B, positive moment and positive shear at X.

Problem 14.2 Reactions at B and D, negative moment and negative shear at X.

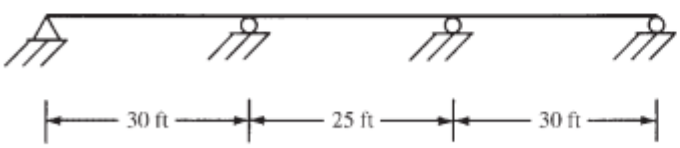


Problem 14.3 Positive moment at X, positive shear at X, negative moment just to the left of Y.



Moment Envelopes

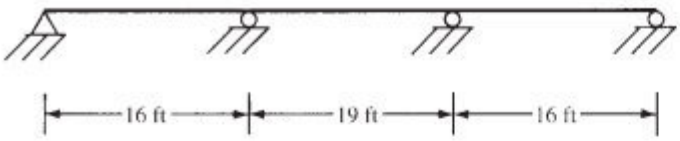
Problem 14.4 For the continuous beam shown and for a service dead load of 2 k/ft and a service live load of 4 k/ft, draw the moment envelope using factored loads assuming a permissible 10% up-or-down redistribution of the maximum negative moment. Do not use ACI coefficients.



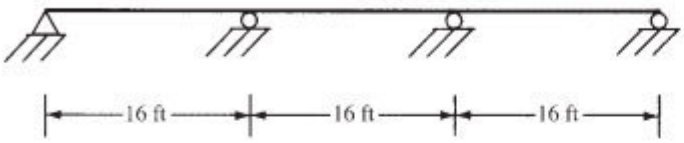
Slab Designs with ACI Coefficients

For Problems 14.5 and 14.6, design the continuous slabs shown using the ACI moment coefficients assuming that a service live load of 200 psf is to be supported in addition to the weight of the slabs. The slabs are to be built integrally with the end supports, which are spandrel beams. Assume $f_y = 60{,}000$ psi, $f'_c = 4000$ psi, and normal-weight concrete. Centerline spans are shown. Assume supports are 12 in. wide. Discontinuous ends are supported on spandrel beams.

Problem 14.5 (*One ans:* 8-in. slab, $d = 6.75$ in., $A_{s-} = 0.267$ in.^{2}/ft at each end; $A_{s-} = 0.420$ in.^{2}/ft at interior supports, $A_{s+} = 0.267$ in.^{2}/ft in end spans, $A_{s+} = 0.307$ in.^{2}/ft in middle span)

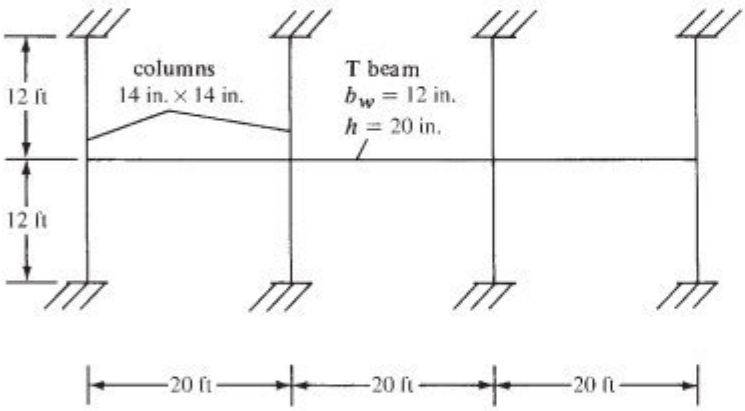


Problem 14.6

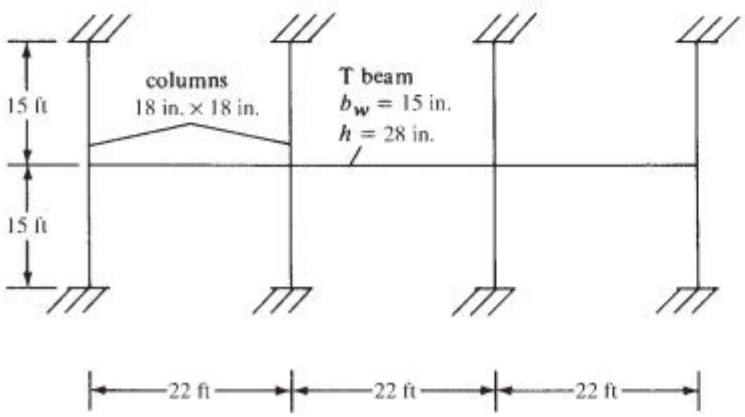


Equivalent Frame Analysis

Problem 14.7 With the equivalent rigid-frame method, draw the shear and moment diagrams for the continuous beam shown using factored loads. Service dead load, including beam weight, is 1.5 k/ft and service live load is 3 k/ft. Place the live load in the center span only. Assume the moments of inertia of the T beams equal two times the moments of inertia of their webs. (*Ans.* max $M^- = 184.8$ ft-k, max $M^+ = 145.2$ ft-k)

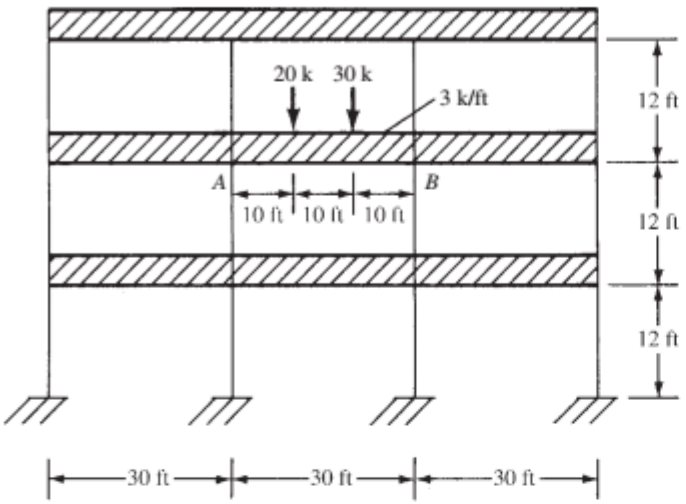


Problem 14.8 With the equivalent rigid-frame method, draw the shear and moment diagrams for the continuous beam shown using factored loads. Service dead load, including beam weight, is 2.4 k/ft and service live load is 3.2 k/ft. Place the live load in spans 1 and 2 only. Assume that the moments of inertia of the T beams equal two times the moments of inertia of their webs.



Analyses by Assuming Points of Inflection Locations

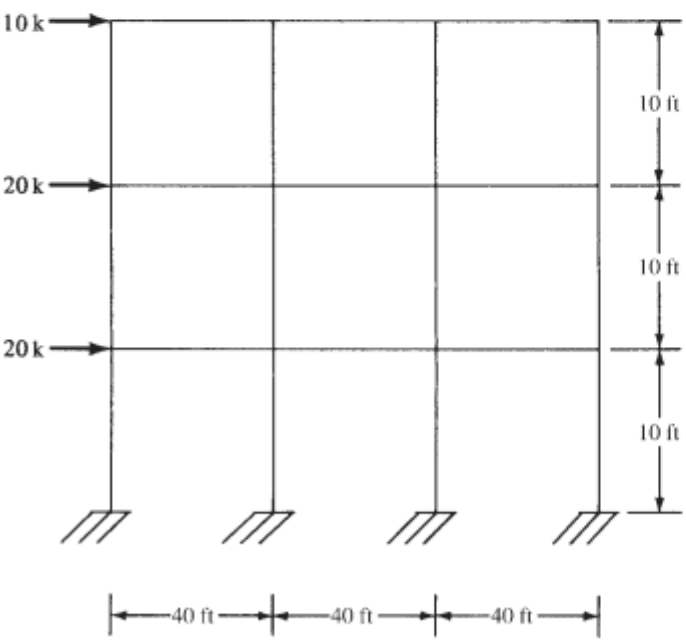
Problem 14.9 Draw the shear and moment diagrams for member *AB* of the frame shown in the accompanying illustration if points of inflection are assumed to be located at 0.15*L* from each end of the span. (*Ans*. max $M = 304$ ft-k @ centerline)



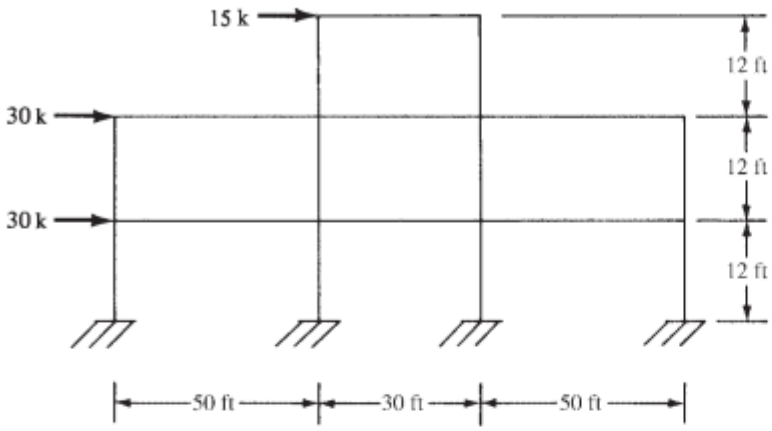
Portal Method of Analysis

For Problems 14.10 to 14.12, compute moments, shears, and axial forces for all the members of the frames shown using the portal method.

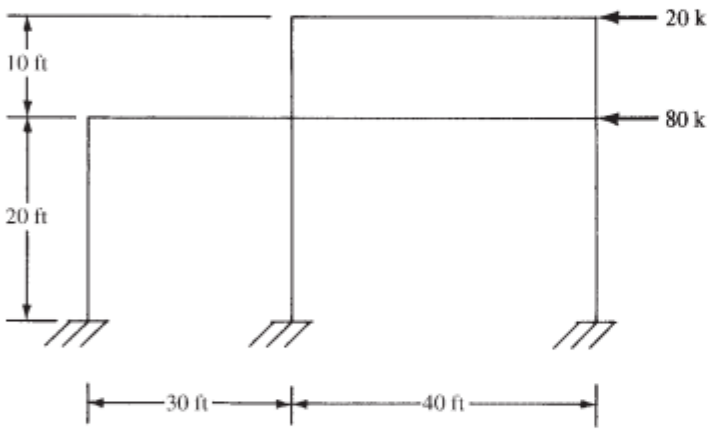
Problem 14.10



Problem 14.11 (*Ans.* $V = 12.5$ k, $M = 75$ ft-k, and $S = 6.6$ k for lower left column)

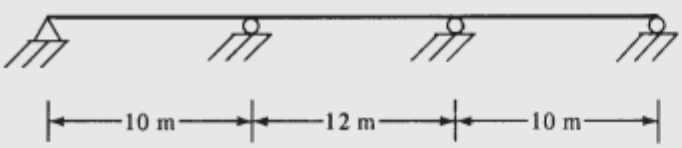


Problem 14.12

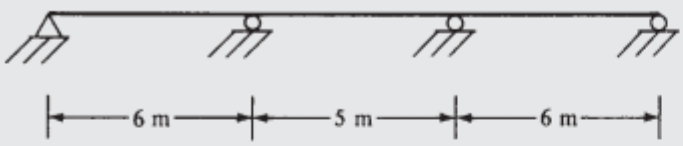


Problems in SI Units

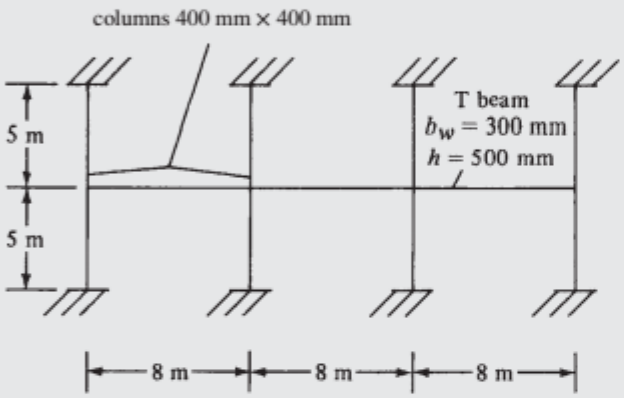
Problem 14.13 For the continuous beam shown and for a service dead load of 26 kN/m and a service live load of 32 kN/m, draw the moment envelope assuming a permissible 10% up-or-down redistribution of the maximum negative moment. (*Ans.* max $M^- = 1110.7$ N•m, max $M^+ = 748.8$ N•m)



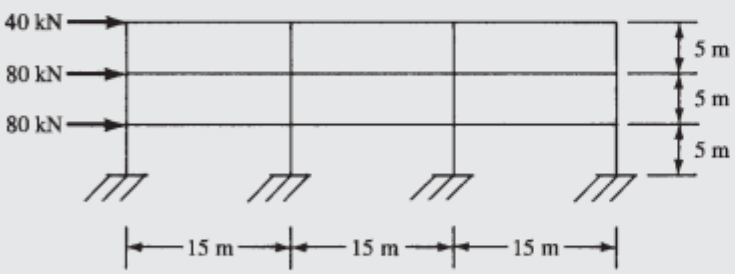
Problem 14.14 Design the continuous slab shown using the ACI moment coefficients assuming the concrete weighs 23.5 kN/m^3 and that a service live load of 9 kN/m is to be supported. The slabs are built integrally with the end supports, which are spandrel beams. $f_y = 420$ MPa and $f'_c = 21$ MPa. Clear spans are given.



Problem 14.15 Using the equivalent rigid-frame method, draw the shear and moment diagrams for the continuous beams shown using factored loads. Service dead load, including beam weight, is 10 kN/m and service live load is 14 kN/m. Place the live load in spans 1 and 3 only. Assume that the I of T beams equals 1.5 times the I of their webs. (*Ans*. max $M^- = 171.6$ N•m, max $M^+ = 127.3$ N•m)



Problem 14.16 Compute moments, shears, and axial forces for all the members of the frame shown using the portal method.



Torsion

CHAPTER 15

15.1 Introduction

The average designer probably does not worry about torsion very much. He or she thinks almost exclusively of axial forces, shears, and bending moments, and yet most reinforced concrete structures are subject to some degree of torsion. Until recent years, the safety factors required by codes for the design of reinforced concrete members for shear, moment, and so forth were so large that the effects of torsion could be safely neglected in all but the most extreme cases. Today, however, overall safety factors are fewer than they used to be and members are smaller, with the result that torsion is a more common problem.

Appreciable torsion does occur in many structures, such as in the main girders of bridges, which are twisted by transverse beams or slabs. It occurs in buildings where the edge of a floor slab and its beams are supported by a spandrel beam running between the exterior columns. This situation is illustrated in Figure 15.1, where the floor beams tend to twist the spandrel beam laterally. Earthquakes can cause dangerous torsional forces in all buildings. This is particularly true in asymmetrical structures, where the centers of mass and rigidity do not coincide. Other cases where torsion may be significant are in curved bridge girders, spiral stairways, and balcony girders, and whenever large loads are applied to any beam "off center." An off-center case where torsional stress can be very large is illustrated in Figure 15.2. It should be realized that if the supporting member is able to rotate, the resulting torsional stresses will be fairly small. If, however, the member is restrained, the torsional stresses can be quite large.

Should a plain concrete member be subjected to pure torsion, it will crack and fail along 45° spiral lines because of the diagonal tension corresponding to the torsional stresses. For a very effective demonstration of this type of failure, you can take a piece of chalk in your hands and twist it until it breaks. Although the diagonal tension stresses produced by twisting are very similar to those caused by

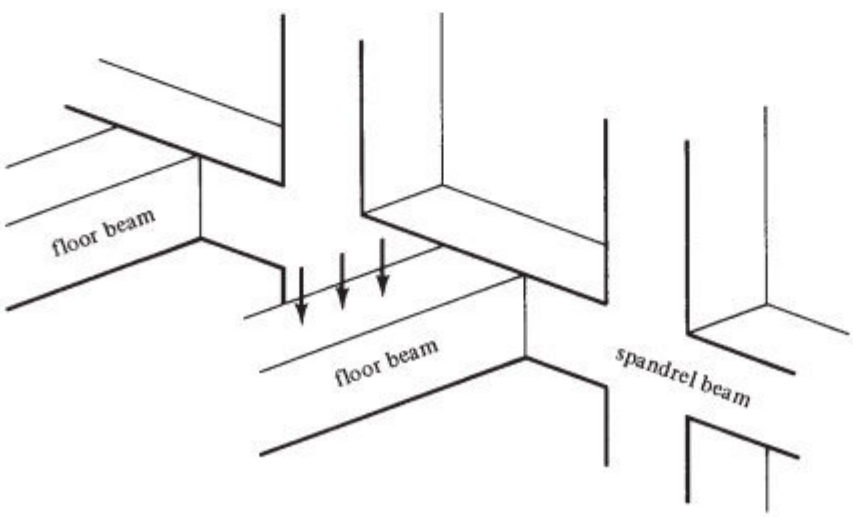


FIGURE 15.1 Torsion in spandrel beams.

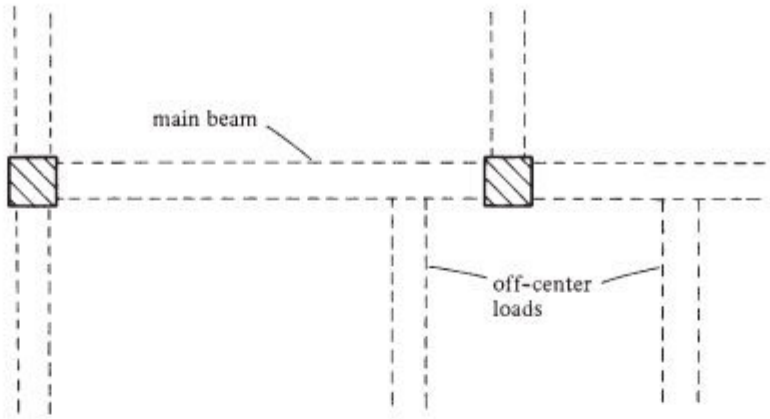


FIGURE 15.2 Off-center loads causing torsion in main beam.

shear, they will occur on all faces of a member. As a result, they add to the stresses caused by shear on one side and subtract from them on the other.[1]

Recently, there have been more reports of structural failures attributed to torsion. As a result, a rather large amount of research has been devoted to the subject, and thus there is a much improved understanding of the behavior of structural members subjected to torsion. On the basis of this rather extensive experimental work, the ACI Code includes very specific requirements for the design of reinforced concrete members subjected to torsion or to torsion combined with shear and bending. It should be realized that maximum shears and torsional forces may occur in areas where bending moments are small. For such cases, the interaction of shear and torsion can be particularly important as it relates to design.

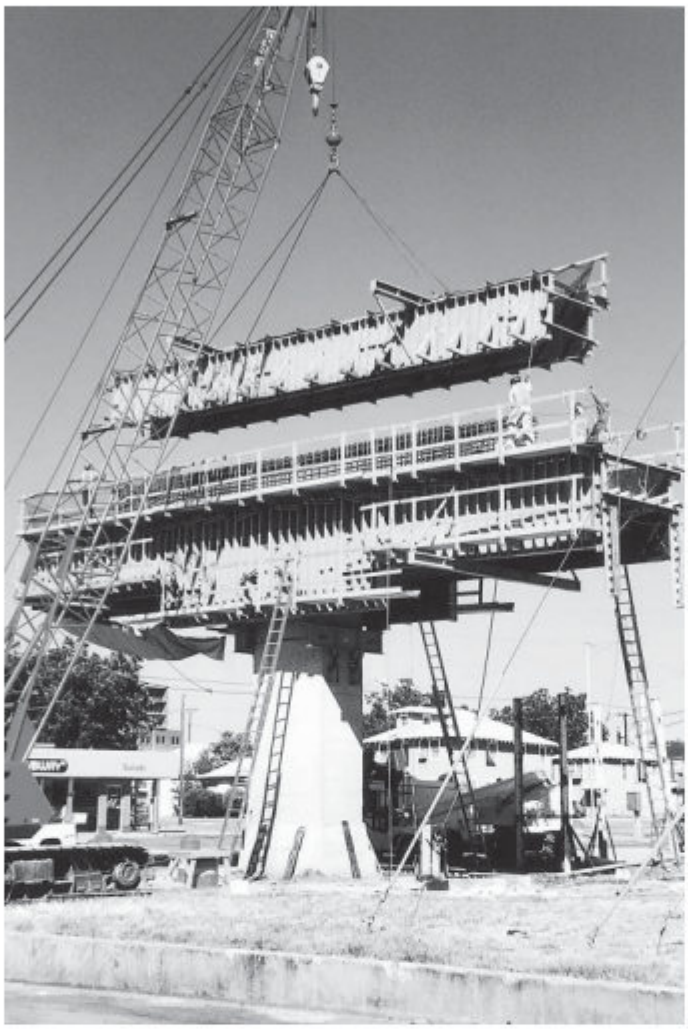
15.2 Torsional Reinforcing

Reinforced concrete members subjected to large torsional forces may fail quite suddenly if they are not specially provided with torsional reinforcing. The addition of torsional reinforcing does not change the magnitude of the torsion that will cause diagonal tension cracks, but it does prevent the members from tearing apart. As a result, they will be able to resist substantial torsional moments without failure. Tests have shown that both longitudinal bars and closed stirrups (or spirals) are necessary to intercept the numerous diagonal tension cracks that occur on all surfaces of members subject to appreciable torsional forces.

The normal ⊔-shaped stirrups are not satisfactory. They must be closed either by welding their ends together to form a continuous loop, as illustrated in Figure 15.3(a), or by bending their ends around a longitudinal bar, as shown in part (b) of the same figure. If one-piece stirrups such as these are used, the entire beam cage may have to be prefabricated and placed as a unit (and that may not be feasible if the longitudinal bars have to be passed between column bars) or the longitudinal bars will have to be threaded one by one through the closed stirrups and perhaps the column bars. It is easy to see that some other arrangement is usually desirable.

In the recent past, it was rather common to use two overlapping ⊔ stirrups arranged as shown in Figure 15.4. Although this arrangement simplifies the placement of longitudinal bars, it has not proved very satisfactory. As described in the ACI Commentary (R9.7.6.3.1), members primarily subjected to torsion lose their concrete side cover by spalling at high torques. Should this happen, the lapped spliced ⊔ stirrups of Figure 15.4 will prove ineffective, and premature torsion failure may occur.

[1] White, R. N., Gergely, P., and Sexsmith, R. G., 1974, *Structural Engineering*, vol. 3 (Hoboken, NJ: John Wiley & Sons), pp. 423–424.



Courtesy of EFCO Corp.

Elevated highway bridge, Wichita Falls, Texas.

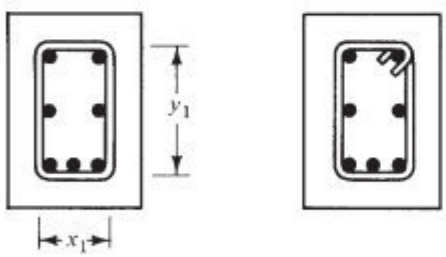


FIGURE 15.3 Closed stirrups (these types frequently impractical).

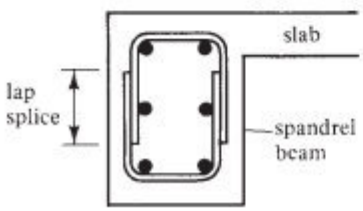


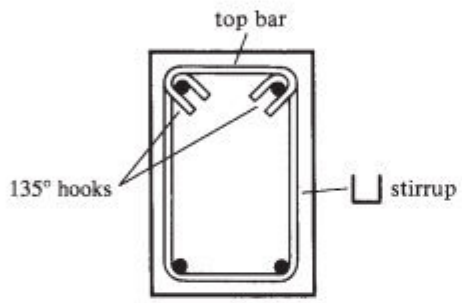
FIGURE 15.4 Overlapping ⊔ stirrups used as torsional reinforcing *but not desirable.*

A much better type of torsion reinforcement consists of ⊔ stirrups, each with a properly anchored top bar, such as the ones shown in Figure 15.5. It has been proved by testing that the use of torsional stirrups with 90° hooks results in spalling of the concrete outside the hooks. The use of 135° hooks for both the ⊔ stirrups and the top bars is very helpful in reducing this spalling.

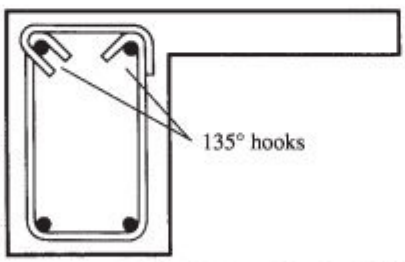
Should lateral confinement of the stirrups be provided as shown in parts (b) and (c) of Figure 15.5, the 90° hooks will be satisfactory for the top bars.

Should beams with wide webs (say, 2 ft or more) be used, it is common to use multiple-leg stirrups. For such situations, the outer stirrup legs will be proportioned for both shear and torsion, whereas the interior legs will be designed to take only vertical shear.

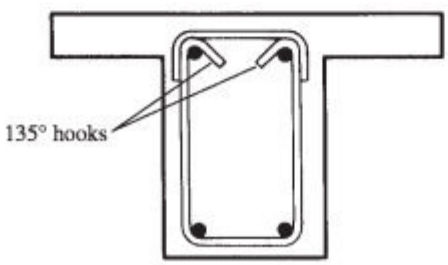
The strength of closed stirrups cannot be developed unless additional longitudinal reinforcement is supplied. Longitudinal bars should be spaced uniformly around the insides of the stirrups, not more than 12 in. apart. There must be at least one bar in each corner of the stirrups to provide anchorage for the stirrup legs (ACI Code, Section 9.7.5.1); otherwise, if the concrete inside the corners were to be crushed, the stirrups would slip and the result would be even larger torsional cracks. These longitudinal bars must have diameters at least equal to 0.042 times the stirrup spacing. Their size may not be less than #3.



(a) No confining concrete either side—thus 135° hooks needed for both ends of top bar.



(b) Lateral confinement provided by slab on right-hand side—thus 90° hook permissible for top bar on that side.



(c) Lateral confinement provided on both sides by concrete slab—thus 90° hooks permissible at both ends of top bar.

FIGURE 15.5 Recommended torsion reinforcement.

15.3 Torsional Moments That Have to Be Considered in Design

The reader is well aware from his or her studies of structural analysis that if one part of a statically indeterminate structure "gives" when a particular force is applied to that part, the amount of force that the part will have to resist will be appreciably reduced. For instance, if three men are walking along with a log on their shoulders (a statically indeterminate situation) and one of them lowers his shoulder a little under the load, there will be a major redistribution of the internal forces in the "structure" and a great deal less load for him to support. On the other hand, if two men are walking along with a log on their shoulders (a statically determinate situation) and one of them lowers his shoulder slightly, there will be little change in force distribution in the structure. These are similar to the situations that occur in statically determinate and indeterminate structures subject to torsional moments. They are referred to respectively as *equilibrium torsion* and *compatibility torsion*.

1. *Equilibrium torsion*—For a statically determinate structure, there is only one path along which a torsional moment can be transmitted to the supports. This type of torsional moment, which is referred to as *equilibrium torsion* or *statically determinate torsion*, cannot be reduced by a redistribution of internal forces or by a rotation of the member. Equilibrium torsion is illustrated in Figure 15.6, which shows an edge beam supporting a concrete canopy. The edge beam must be designed to resist the full calculated torsional moment.

2. *Compatibility torsion*—The torsional moment in a particular part of a statically indeterminate structure may be substantially reduced if that part of the structure cracks under the torsion and "gives," or rotates. The result will be a redistribution of forces in the structure. This type of torsion, which is illustrated in Figure 15.7, is referred to as *statically indeterminate torsion* or *compatibility torsion*, in the sense that the part of the structure in question twists in order to keep the deformations of the structure compatible.

Where a reduction or redistribution of torsion is possible in a statically indeterminate structure, the maximum factored moment, T_u, can be reduced as follows for non-prestressed members according to ACI Code, Section 22.7.3.2. In the expression to follow, A_{cp} is the area enclosed by the outside perimeter of the concrete cross section and p_{cp} is the outside perimeter of that cross section. It is assumed that torsional cracking will occur when the principal tension stress reaches the tensile strength of the concrete in biaxial tension-compression. This cracking value is taken to be $4\lambda\sqrt{f'_c}$, and thus the torque at cracking, T_{cr}, is

$$T_{cr} = \phi 4\lambda\sqrt{f'_c}\left(\frac{A^2_{cp}}{p_{cp}}\right)$$

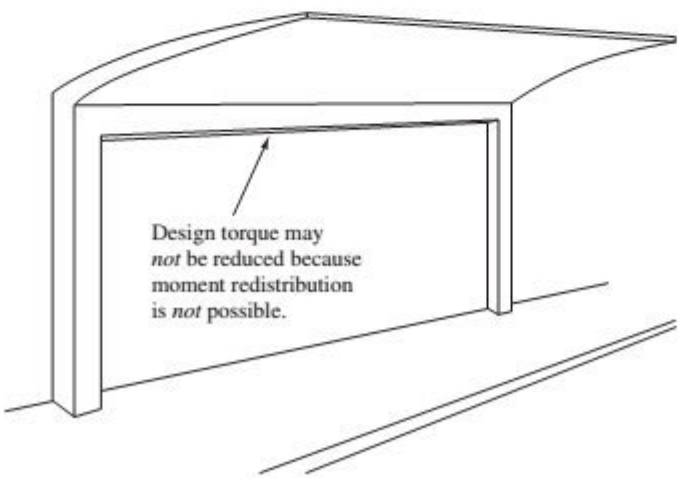


FIGURE 15.6 Equilibrium torsion.

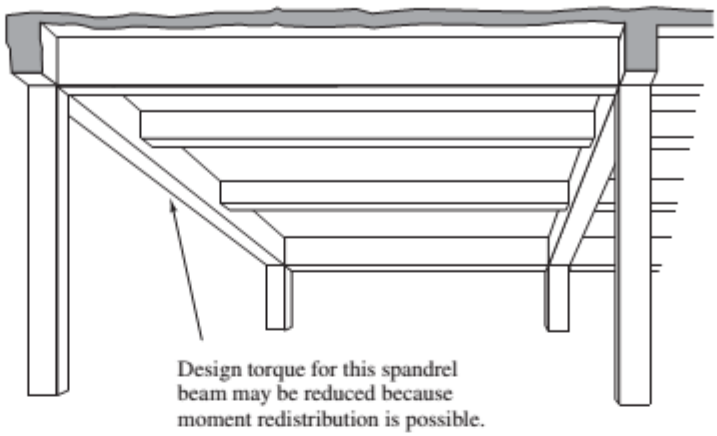


FIGURE 15.7 Compatibility torsion.

When reinforced concrete members are subjected to axial tensile or compressive forces, T_{cr} is to be computed with the expression to follow in which N_u is the factored axial force taken as positive if the force is compressive and negative if it is tensile.

$$T_{cr} = \phi 4\lambda\sqrt{f'_c}\left(\frac{A_{cp}^2}{p_{cp}}\right)\sqrt{1+\frac{N_u}{4\lambda\sqrt{f'_c}}}$$

After cracking occurs, the torsional moments in the spandrel beam shown in Figure 15.7 are reduced as a result of the redistribution of the internal forces. Consequently, the torsional moment used for design in the spandrel beam can be reduced.

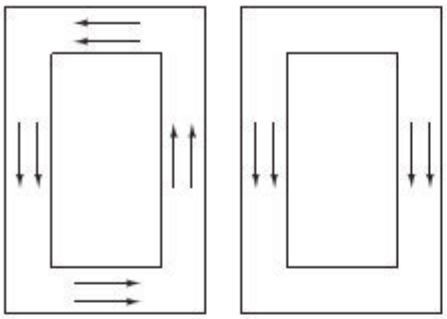
15.4 Torsional Stresses

As previously mentioned, the torsional stresses add to the shear stresses on one side of a member and subtract from them on the other. This situation is illustrated for a hollow beam in Figure 15.8.

Torsional stresses are quite low near the center of a solid beam. Because of this, hollow beams (assuming that the wall thicknesses meet certain ACI requirements) are assumed to have almost exactly the same torsional strengths as solid beams with the same outside dimensions.

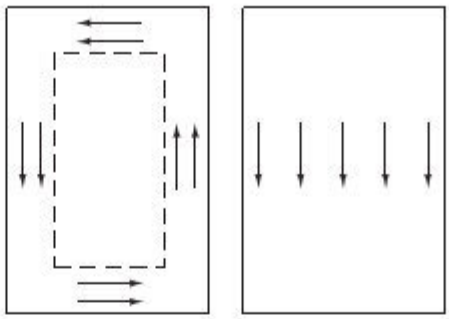
In solid sections, the shear stresses due to the torsion, T_u, are concentrated in an outside "tube" of the member, a shown in Figure 15.9(a), whereas the shear stresses resulting from V_u are spread across the width of the solid section, as shown in part (b) of the figure. As a result, the two types of shear stresses (those from shear and torsion) are combined using a square root expression given in the next section of this chapter.

After cracking, the resistance of concrete to torsion is assumed to be negligible. The torsion cracks tend to spiral around members (hollow or solid) located at approximately 45° angles with the longitudinal edges of those members. Torsion is assumed to be resisted by an imaginary space truss located in the outer "tube" of concrete of the member. Such a truss is shown in Figure 15.10. The longitudinal steel in the corners of the member and the closed transverse stirrups act as tension members in the "truss," whereas the diagonal concrete between the stirrups acts as struts of compression members. The cracked concrete is still capable of taking compression stresses.



(a) Torsional stresses (b) Shear stresses

FIGURE 15.8 Torsion and shear stresses in a hollow beam.



(a) Torsional stresses (b) Shear stresses

FIGURE 15.9 Torsion and shear stresses in a solid beam.

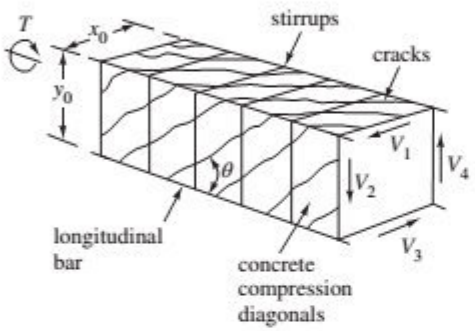


FIGURE 15.10 Imaginary space truss [From ACI Figure R22.7.6.1(a)].

15.5 When Torsional Reinforcement Is Required by the ACI

The design of reinforced concrete members for torsion is based on a thin-walled tube space truss analogy in which the inside or core concrete of the members is neglected. After torsion has caused a member to crack, its resistance to torsion is provided almost entirely by the closed stirrups and the longitudinal reinforcement located near the member surface. Once cracking occurs, the concrete is assumed to have negligible torsional strength left. (This is not the case in shear design, where the concrete is assumed to carry the same amount of shear as it did before cracking.)

If torsional stresses are less than about one-fourth of the cracking torque, T_{cr}, of a member, they will not appreciably reduce either its shear strength or flexural strength. Torsional cracking is assumed to occur when the principal tensile stress reaches $4\lambda\sqrt{f'_c}$. In ACI Code, Section 9.5.4.1, it is stated that torsion effects may be neglected for non-prestressed members if $T_u < \phi T_{th}$, where T_{th} is the threshold torsion as defined in ACI Code, Section 22.7.4.1a. Thus

$$T_u < \phi\lambda\sqrt{f'_c}\left(\frac{A^2_{cp}}{p_{cp}}\right)$$

For statically indeterminate structures where reductions in torsional moments can occur because of redistribution of internal forces, Section 22.7.3.2 of the ACI Code permits the maximum factored torsional moment to be reduced to the following value:

$$\phi 4\lambda\sqrt{f'_c}\left(\frac{A^2_{cp}}{p_{cp}}\right)$$

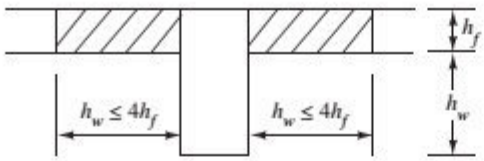


FIGURE 15.11 Portions of monolithic T beam that may be used for torsion calculations.

In other words, the applied torque may be limited to a member's calculated cracking moment. (If the computed torque for a particular member is larger than the preceding value, the preceding value may be used in design.) Should the torsional moments be reduced as described here, it will be necessary to redistribute these moments to adjoining members. The ACI Commentary R22.7.3 does say that when the layout of structures is such as to impose significant torsional rotations within a short length of a member (as where a large torque is located near a stiff column), a more exact analysis should be used.

For isolated members with or without flanges, A_{cp} equals the area of the entire cross sections (including the area of any voids in hollow members), and p_{cp} represents the perimeters of the entire cross sections. Should a beam be cast monolithically with a slab, the values of A_{cp} and p_{cp} may be assumed to include part of the adjacent slabs of the resulting T- or L-shaped sections. The widths of the slabs that may be included as parts of the beams are described in ACI Code, Section 9.2.4.4 and illustrated in Figure 15.11. Those widths or extensions may not exceed the projections of the beams above or below the slab or four times the slab thickness, whichever is smaller.

When appreciable torsion is present, it may be more economical to select a larger beam than would normally be selected so that torsional reinforcing does not have to be used. Such a beam may very well be more economical than a smaller one with the closed stirrups and additional longitudinal steel required for torsion design. On other occasions, such a practice may not be economical, and sometimes architectural considerations may dictate the use of smaller sections.

15.6 Torsional Moment Strength

The sizes of members subject to shear and torsion are limited by the ACI Code so that unsightly cracking is reduced and crushing of the surface concrete caused by inclined compression stresses is prevented. This objective is accomplished with the equations that follow, in which the left-hand portions represent the shear stresses from shear and torsion. The sum of these two stresses in a particular member may not exceed the stress that will cause shear cracking ($8\sqrt{f'_c}$ as per ACI R11.5.3.1). In these expressions, $V_c = 2\lambda\sqrt{f'_c}b_w d$ (ACI Equation 22.5.5.1). For solid sections

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right)^2} \le \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f'_c}\right) \qquad \text{(ACI Equation 22.7.7.1a)}$$

For hollow sections

$$\left(\frac{V_u}{b_w d}\right) + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right) \le \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f'_c}\right) \qquad \text{(ACI Equation 22.7.7.1b)}$$

Should the wall thickness of a hollow section be less than A_{oh}/p_h, the second term in ACI Equation 22.7.7.1b is to be taken not as $T_u p_h/1.7A_{oh}^2$ but as $T_u/1.7A_{oh}t$, where t is the thickness of the wall where stresses are being checked (ACI Code, Section 22.7.7.2).

Another requirement given in ACI Section 9.7.6.3.4 for hollow sections is that the distance from the centerline of the transverse torsional reinforcing to the inside face of the wall must not be

less than $0.5A_{oh}/p_h$. In this expression, p_h is the perimeter of the centerline of the outermost closed torsional reinforcing and A_{oh} is the cross-sectional area of the member that is enclosed within this centerline. The letters *oh* stand for *outside hoop* (of stirrups).

15.7 Design of Torsional Reinforcing

The torsional strength of reinforced concrete beams can be greatly increased by adding torsional reinforcing consisting of closed stirrups and longitudinal bars. If the factored torsional moment for a particular member is larger than the value given in ACI Code, Section 22.7.4.1 $[\phi\lambda\sqrt{f'_c}(A^2_{cp}/P_h)]$, the code provides an expression to compute the absolute minimum area of transverse closed stirrups that may be used (ACI Code, Section 9.6.4.2).

$$(A_v + 2A_t) = 0.75\sqrt{f'_c}\,\frac{b_w s}{f_{yt}} \geq \frac{50 b_w s}{f_{yt}}$$

In this expression, A_v is the area of reinforcing required for shear in a distance s (which represents the stirrup spacing). You will remember from shear design that the area A_v obtained is for both legs of a two-legged stirrup (or for all legs of a four-legged stirrup, etc.). The value A_t, which represents the area of the stirrups needed for torsion, is for only one leg of the stirrup. Therefore, the value $A_v + 2A_t$ is the total area of both legs of the stirrup (for two-legged stirrups) needed for shear plus torsion. It is considered desirable to use equal volumes of steel in the stirrups and the added longitudinal steel so that they will participate equally in resisting torsional moments. This theory was followed in preparing the ACI equations used for selecting torsional reinforcing. The ACI Code requires that the nominal torsional moment strength, T_n, be computed with the equations that follow:

$$T_n = \frac{2A_o A_t f_{yt}}{s}\cot\theta \qquad \text{(ACI Equation 22.7.6.1a)}$$

$$T_n = \frac{2A_o A_\ell f_y}{p_h}\tan\theta \qquad \text{(ACI Equation 22.7.6.1b)}$$

T_n is the lesser of the two equations. By equating T_u to ϕT_n the required minimum values of A_t and A_ℓ can be determined.

Equation 22.7.6.1a is usually written in the following form:

$$\frac{A_t}{s} = \frac{T_n}{2A_o f_{yt}\cot\theta}$$

Setting ACI Equations 22.7.6.1a and 22.7.6.1b equal and solving for A_ℓ results in the following:

$$A_\ell = \frac{A_t}{s}p_h\left(\frac{f_{yt}}{f_y}\right)\cot^2\theta \qquad \text{(Equation 15-1)}$$

This equation appeared in ACI Code 318-11, but not in ACI Code 318-14. For this reason, it is given the designation of Equation 15-1.

The term A_o represents the gross area enclosed by the shear flow path around the perimeter of the tube. This area is defined in terms of A_{oh}, which is the area enclosed by the outermost closed hoops. Figure 15.12 illustrates this definition of A_{oh} for several beam cross sections.

The value of A_o may be determined by analysis, or it may be taken as $0.85A_{oh}$ [ACI Code, Section 22.7.6.1(a)]. The term θ represents the angle of the concrete "compression diagonals" in the analogous space truss in Figure 15.10 of the text. It may not be smaller than 30° or larger than 60°, and it may be

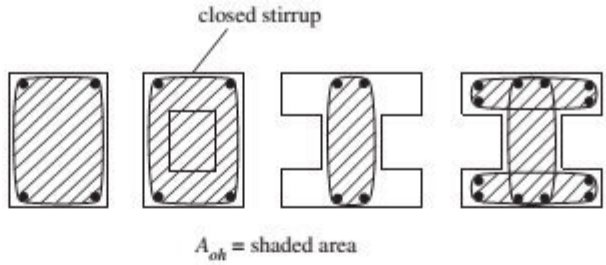


FIGURE 15.12 Values of A_{oh}.

taken equal to 45°, according to ACI 318 Code, Section 22.7.6.1.2a. Suggested θ values for prestressed concrete are given in this same ACI section.

As given in the ACI Commentary R9.5.4.3, the required stirrup areas for shear and torsion are added together as follows for a two-legged stirrup:

$$\text{Total } \left(\frac{A_{v+t}}{s}\right) = \frac{A_v}{s} + \frac{2A_t}{s}$$

The spacing of transverse torsional reinforcing may not be larger than $p_h/8$ or 12 in., where p_h is the perimeter of the centerline of the outermost closed transverse reinforement (ACI Code, Section 22.7.6.1). Remember also that the maximum spacings of stirrups for shear ($d/2$ and $d/4$) are given in ACI Code, Section 9.7.6.2.

It has been found that reinforced concrete specimens with less than about 1% torsional reinforcing by volume that are loaded in pure torsion fail as soon as torsional cracking occurs. The percentage is smaller for members subject to both torsion and shear. The equation to follow, which provides a minimum total area of longitudinal torsional reinforcing, is based on using about 0.5% torsional reinforcing by volume. In this expression, A_{cp} is the area enclosed by the outside concrete cross section. The value A_t/s may not be taken as less than $25b_w/f_{yv}$, according to ACI Code, Section 9.6.4.3b.

$$A_{\ell\ \min} = \frac{5\sqrt{f'_c}A_{cp}}{f_y} - \left(\frac{A_t}{s}\right)p_h\frac{f_{yt}}{f_y} \qquad \text{(ACI Section 9.6.4.3a)}$$

ACI Code, Section 9.7.5.4 states that the longitudinal torsional reinforcement must be developed at both ends.

Maximum torsion generally acts at the ends of beams, and as a result the longitudinal torsion bars should be anchored for their yield strength at the face of the supports. To do this, it may be necessary to use hooks or horizontal ⊔-shaped bars lap spliced with the longitudinal torsional reinforcing. *A rather common practice is to extend the bottom reinforcing in spandrel beams subjected to torsion 6 in. into the supports. Usually this is insufficient.*

15.8 Additional ACI Requirements

Before we present numerical examples for torsion design, it is necessary to list several other ACI requirements. These are as follows:

1. Sections located at a distance less than d from the face of support may be designed for the torque at a distance d. Should, however, a concentrated torque be present within this distance, the critical design section will be at the face of the support (ACI Code, Section 9.4.4.3).
2. The design yield strength of torsional reinforcing for non-prestressed members may not be greater than 60,000 psi. The purpose of this maximum value is to limit the width of diagonal cracks (ACI Code, Section 20.2.2.4).

3. The longitudinal tension created by torsion moments is partly offset in the flexural compression zones of members. In these zones, the computed area of longitudinal torsional reinforcing may be reduced by an amount equal to $M_u/0.9df_y$, according to ACI Section 9.5.4.5. In this expression, M_u is the factored moment acting at the section in combination with T_u. The reinforcement provided, however, may not be less than the minimum values required in ACI Code, Section 9.6.4.
4. The longitudinal reinforcing must be distributed around the inside perimeter of the closed stirrups and must be spaced no farther apart than 12 in. At least one bar must be placed in each corner of the stirrups to provide anchorage for the stirrup legs. These bars have to be #3 or larger in size, and they must have diameters no less than 0.042 times the stirrup spacings (ACI Code, Section 9.7.5.2).
5. Torsional reinforcing must be provided for a distance no less than $b_t + d$ beyond the point where it is theoretically no longer required. The term b_t represents the width of that part of the member cross section which contains the closed torsional stirrups (ACI Code, Section 9.7.6.3.2).

15.9 Example Problems Using U.S. Customary Units

In this section, the design of torsional reinforcing for a beam is presented using U.S. customary units; an example using SI units is presented in the next section.

Example 15.1

Design the torsional reinforcing needed for the beam shown in Figure 15.13 if $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, $T_u = 30$ ft-k, and $V_u = 60$ k. Assume 1.5-in. clear cover, #4 stirrups, and a required A_s for M_u of 3.52 in.^{2}. Select #8 bars for flexural reinforcement. Normal-weight concrete is specified.

SOLUTION

1. Is Torsional Reinforcing Needed?

A_{cp} = area enclosed by outside perimeter of concrete cross section = (16) (26) = 416 in.^{2}
(the letter *c* stands for concrete, and the letter *p* stands for perimeter of cross section)

$$p_{cp} = \text{outside perimeter of the cross section}$$
$$= (2)\ (16 \text{ in.} + 26 \text{ in.}) = 84 \text{ in.}$$

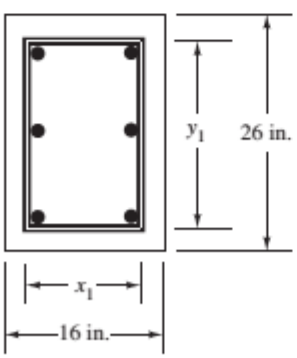


FIGURE 15.13 Beam cross section for Example 15.1.

Torsion, T_u, can be neglected if less than

$$\phi\lambda\sqrt{f'_c}\frac{A_{cp}^2}{p_{cp}} = (0.75)(1.0)\left(\sqrt{4000}\text{ psi}\right)\left[\frac{(416\text{ in.}^2)^2}{84\text{ in.}}\right]$$

$$= 97{,}723\text{ in.-lb} = 97.72\text{ in.-k} < 30\text{ ft-k} \times 12\text{ in./ft} = 360\text{ in.-k}$$

$\therefore$ Torsion must be considered

2. Compute Sectional Properties

A_{oh} = area enclosed by centerline of the outermost closed stirrups

Noting 1.5-in. clear cover and #4 stirrups

$$\left.\begin{aligned} x_1 &= 16\text{ in.} - (2)(1.5\text{ in.} + 0.25\text{ in.}) = 12.5\text{ in.} \\ y_1 &= 26\text{ in.} - (2)(1.5\text{ in.} + 0.25\text{ in.}) = 22.5\text{ in.} \end{aligned}\right\}\text{ See Figure 15.13}$$

$$A_{oh} = (12.5\text{ in.})(22.5\text{ in.}) = 281.25\text{ in.}^2$$

A_o = gross area enclosed by shear flow path

$= 0.85A_{oh}$ (from ACI Code, Section 22.7.6.1.1)

$= (0.85)(281.25\text{ in.}^2) = 239.06\text{ in.}^2$

d = effective depth of beam

$$= 26\text{ in.} - 1.50\text{ in.} - 0.50\text{ in.} - \frac{1.00\text{ in.}}{2} = 23.50\text{ in.}$$

p_h = perimeter of centerline of outermost closed torsional reinforcing

$= (2)(x_1 + y_1)$

$= (2)(12.5\text{ in.} + 22.5\text{ in.}) = 70\text{ in.}$

3. Is the Concrete Section Sufficiently Large to Support the Torsion?

V_c = nominal shear strength of concrete section

$= 2\lambda\sqrt{f'_c}b_w d = (2)(1.0)\left(\sqrt{4000}\text{ psi}\right)(16\text{ in.})(23.50\text{ in.}) = 47{,}561\text{ lb}$

Applying ACI Equation 22.7.7.1a

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right)^2} \le \phi\left(\frac{V_c}{b_w d} + 8\sqrt{f'_c}\right)$$

$$\sqrt{\left(\frac{60{,}000\text{ lb}}{16\text{ in.} \times 23.50\text{ in.}}\right)^2 + \left[\frac{360{,}000\text{ in.-lb} \times 70\text{ in.}}{1.7 \times (281.25\text{ in.}^2)^2}\right]^2} \le 0.75\left(\frac{47{,}561\text{ lb}}{16\text{ in.} \times 23.50\text{ in.}} + 8\sqrt{4000}\text{ psi}\right)$$

$$246\text{ psi} < 474\text{ psi}$$

$\therefore$ Section is sufficiently large

4. Determine the Transverse Torsional Reinforcing Required

$$T_n = \frac{T_u}{\phi} = \frac{30\text{ ft-k}}{0.75} = 40\text{ ft-k} = 480{,}000\text{ in-lb}$$

Assuming $\theta = 45°$ as per ACI Code, Section 22.7.6.1.2

$$\frac{A_t}{s} = \frac{T_n}{2A_o f_{yt}\cot\theta} = \frac{480{,}000\text{ in.-lb}}{(2)(239.06\text{ in.}^2)(60{,}000\text{ psi})(\cot 45°)}$$

$$= 0.0167\text{ in.}^2/\text{in. for one leg of stirrups}$$

5. Calculate the Area of Shear Reinforcing Required

$$V_u = 60{,}000 \text{ lb} > \frac{1}{2}\phi V_c = \left(\frac{1}{2}\right)(0.75)(47{,}561 \text{ lb}) = 17{,}835 \text{ lb}$$

$\therefore$ Shear reinforcement is required

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{60{,}000 \text{ lb} - (0.75)(1.0)(47{,}561 \text{ lb})}{0.75} = 32{,}439 \text{ lb}$$

Applying ACI Equation 22.5.10.5.3

$$\frac{A_v}{s} = \frac{V_s}{f_{yt}d} = \frac{32{,}439 \text{ lb}}{(60{,}000 \text{ psi})(23.50 \text{ in.})}$$
$$= 0.0230 \text{ in.}^2/\text{in. for 2 legs of stirrup}$$

6. Select Stirrups

Total web reinforcing required for two legs $= \frac{A_{v+t}}{s}$

$$= \frac{2A_t}{s} + \frac{A_v}{s} = (2)(0.0167 \text{ in.}^2/\text{in.}) + 0.0230 \text{ in.}^2/\text{in.}$$
$$= 0.0564 \text{ in.}^2/\text{in. for two legs}$$

Using #4 stirrups

$$s = \frac{(2)(0.20 \text{ in.}^2)}{0.0564 \text{ in.}^2/\text{in.}} = 7.09 \text{ in. o.c.} \qquad \text{Say 7 in.}$$

Maximum allowable spacing of stirrups

$$\text{from ACI Code, Section 9.7.6.3.3} = \frac{p_h}{8} = \frac{70 \text{ in.}}{8} = \text{Say 8.75 in. or 12 in.}$$

Minimum area of stirrups A_v by ACI Code, Section 9.6.4.2

$$A_v + 2A_t = 0.75\sqrt{f'_c}\frac{b_w s}{f_{yt}} < \frac{50 b_w s}{f_{yt}} = \frac{(50)(16 \text{ in.})(7 \text{ in.})}{60{,}000 \text{ psi}} = 0.0933 \text{ in.}$$

$$= 0.75\sqrt{4000}\frac{(16 \text{ in.})(7 \text{ in.})}{60{,}000 \text{ psi}} = 0.089 \text{ in.}^2$$

$$< (2)(0.20 \text{ in.}^2) = 0.40 \text{ in.}^2 \quad \text{OK}$$

Use #4 stirrups @7 in.

7. Selection of Longitudinal Torsional Reinforcing

Additional longitudinal reinforcement required for torsion

$$A_\ell = \frac{A_t}{s}p_h\left(\frac{f_{yt}}{f_y}\right)\cot^2\theta \qquad \text{(Equation 15-1)}$$

$$= (0.0167 \text{ in.}^2/\text{in.})(70 \text{ in.})\left(\frac{60{,}000 \text{ psi}}{60{,}000 \text{ psi}}\right)(1.00)^2 = 1.17 \text{ in.}^2$$

$$\text{min. } A_\ell = \frac{5\sqrt{f'_c}A_{cp}}{f_y} - \left(\frac{A_t}{s}\right)p_h\frac{f_{yt}}{f_y} \qquad \text{(ACI Equation 9.6.4.3a)}$$

$$\left[\text{with min } \frac{A_t}{s} = \frac{25 b_w}{f_{yv}} = \frac{(25)(16 \text{ in.})}{60{,}000 \text{ psi}} = 0.00667 \text{ in.}^2/\text{in.}\right]$$

$$= \frac{(5\sqrt{4000} \text{ psi})(416 \text{ in.}^2)}{60{,}000 \text{ psi}} - (0.0167 \text{ in.}^2/\text{in.})(70 \text{ in.})\left(\frac{60{,}000 \text{ psi}}{60{,}000 \text{ psi}}\right)$$

$$= 1.02 \text{ in.}^2 < 1.17 \text{ in.}^2 \quad \text{OK}$$

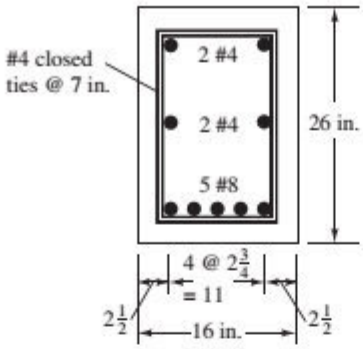


FIGURE 15.14 Reinforcing selected for beam of Example 15.1.

This additional longitudinal steel is spread out into three layers to the four inside corners of the stirrups and vertically in between, with a spacing not in excess of 12 in.

Assume one-third in top = 1.17 in.^{2}/3 = 0.39 in.^{2}, 0.39 in.^{2} + 3.52 in.^{2} = 3.91 in.^{2} in bottom, and remainder 0.39 in.^{2} in between.

Use two #4 bars (0.40 in.^{2}) in top corner, two #4 bars in the middle, and five #8 bars (3.93 in.^{2}) at the bottom, as shown in Figure 15.14.

15.10 SI Equations and Example Problem

The SI equations necessary for the design of reinforced concrete members that are different from those needed for similar designs using U.S. customary units are listed here, and their use is illustrated in Example 15.2.

Torsional moments can be neglected if

$$T_u = \phi(0.083)\lambda\sqrt{f'_c}\,\frac{A_{cp}^2}{p_{cp}} \qquad \text{(ACI Section 9.5.4.1)}$$

$$V_c = 0.17\lambda\sqrt{f'_c}\,b_w d \qquad \text{(ACI Equation 22.5.5.1)}$$

Maximum torsional moment strength for solid sections

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right)^2} \le \phi\left(\frac{V_c}{b_w d} + 0.66\sqrt{f'_c}\right) \qquad \text{(ACI Equation 22.7.7.1a)}$$

Maximum torsional moment strength for hollow sections

$$\frac{V_u}{b_w d} + \frac{T_u p_h}{1.7A_{oh}^2} \le \phi\left(\frac{V_c}{b_w d} + 0.66\sqrt{f'_c}\right) \qquad \text{(ACI Equation 22.7.7.1b)}$$

The expression used to calculate the required area of torsional reinforcing needed for one leg of stirrups

$$T_n = \frac{2A_o A_t f_{yt}}{s}\cot\theta \qquad \text{(ACI Equation 22.7.6.1a)}$$

from which

$$\frac{A_t}{s} = \frac{T_n}{2A_o f_y \cot\theta}$$

Minimum area of transverse reinforcement required

$$A_v + 2A_t = 0.062\sqrt{f'_c}\frac{b_w s}{f_{yt}} \geq \frac{0.35 b_w s}{f_{yt}} \qquad \text{(ACI Section 9.6.4.2)}$$

Additional longitudinal reinforcement required for torsion

$$A_\ell = \frac{A_t}{s} p_h \left(\frac{f_{yt}}{f_y}\right)\cot^2\theta \qquad \text{(Equation 15-1)}$$

Minimum total area of additional longitudinal reinforcement required

$$A_{\ell\ \min} = \frac{0.42\sqrt{f'_c}A_{cp}}{f_y} - \frac{A_t}{s} p_h \frac{f_{yt}}{f_y} \qquad \text{(ACI Equation 9.6.4.3)}$$

In this last expression, the value of (A_t/s) may not be taken less than $(b_w/6f_{yv})$. Other SI requirements for the design of torsional reinforcing are as follows:

1. Max spacing permitted for transverse torsional reinforcing $= \frac{1}{8}p_h$ or 300 mm or $d/2$ or $d/4$ as required for shear design (ACI Code, Section 9.7.6.3.3).
2. The diameter of stirrups may not be less than 0.042 times their spacing, and stirrups smaller than #10 may not be used (ACI Code, Section 9.7.5.2).
3. Maximum yield stresses f_y or $f_{yt} = 420$ MPa (ACI Code, Section 20.2.2.4).

Example 15.2 illustrates the design of the torsional reinforcing for a member using SI units.

Example 15.2

Design the torsional reinforcing for the beam shown in Figure 15.15, for which $f'_c = 28$ MPa, $f_y = 420$ MPa, $V_u = 190$ kN, $T_u = 30$ kN-m, and A_s required for M_u is 2050 mm^2. Assume #13 stirrups and a clear cover equal to 40 mm.

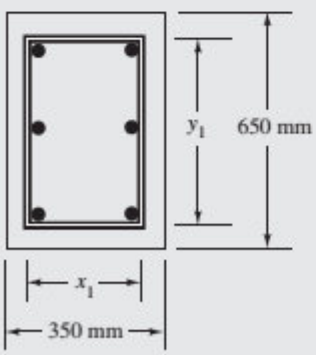


FIGURE 15.15 Beam cross section for Example 15.2.

SOLUTION

1. Is Torsional Reinforcing Necessary?

$$A_{cp} = (350\text{ mm})(650\text{ mm}) = 227\,500\text{ mm}^2$$

$$p_{cp} = (2)(350\text{ mm} + 650\text{ mm}) = 2000\text{ mm}$$

$$\phi(0.083\lambda)\sqrt{f'_c}\,\frac{A_{cp}^2}{p_{cp}} = (0.75)(0.083)(1)\sqrt{28}\text{ MPa}\,\frac{(227\,500\text{ mm}^2)^2}{2000\text{ mm}} \qquad \text{(ACI 9.5.4.1)}$$

$$= 8.524 \times 10^6\text{ N-mm} = 8.52\text{ kN-m} < 30\text{ kN-m}$$

$\therefore$ Torsional reinforcing is required

2. Compute Sectional Properties

With 40-mm clear cover and #13 stirrups (diameter = 12.7 mm)

$$x_1 = 350\text{ mm} - (2)\left(40\text{ mm} + \frac{12.7\text{ mm}}{2}\right) = 257.3\text{ mm}$$

$$y_1 = 650\text{ mm} - (2)\left(40\text{ mm} + \frac{12.7\text{ mm}}{2}\right) = 557.3\text{ mm}$$

$$A_{oh} = (257.3\text{ mm})(557.3\text{ mm}) = 143\,393\text{ mm}^2$$

$$A_o = 0.85A_{oh} = (0.85)(143\,393\text{ mm}^2) = 121\,884\text{ mm}^2$$

Assuming bottom reinforcement consists of #25 bars (diameter = 25.4 mm)

$$d = 650\text{ mm} - 40\text{ mm} - 12.7\text{ mm} - \frac{25.4\text{ mm}}{2} = 584.6\text{ mm}$$

$$p_h = 2(x_1 + y_1) = (2)(257.3\text{ mm} + 557.3\text{ mm}) = 1629\text{ mm}$$

3. Is the Concrete Section Sufficiently Large to Support T_u?

$$V_c = 0.17\lambda\sqrt{f'_c}\,b_w d = (0.17)(1.0)\sqrt{28}\text{ MPa}\,(350\text{ mm})(584.6\text{ mm}) = 180\,449\text{ N}$$

$$= 180.45\text{ kN} \qquad \text{(ACI Equation 22.5.5.1)}$$

$$\sqrt{\left(\frac{V_u}{b_w d}\right)^2 + \left(\frac{T_u p_h}{1.7A_{oh}^2}\right)^2} \le \phi\left(\frac{V_c}{b_w d} + 0.66\sqrt{f'_c}\right) \qquad \text{(ACI Equation 22.7.7.1a)}$$

$$\sqrt{\left(\frac{190\text{ kN} \times 10^3}{350\text{ mm} \times 584.6\text{ mm}}\right)^2 + \left(\frac{30\text{ kN-m} \times 10^6 \times 1629\text{ mm}}{1.7(143\,393\text{ mm}^2)^2}\right)^2}$$

$$< 0.75\left(\frac{180.45\text{ kN} \times 10^3}{350\text{ mm} \times 584.6\text{ mm}} + 0.66\sqrt{28}\text{ MPa}\right)$$

$$1.678\text{ N/mm}^2 < 3.307\text{ N/mm}^2$$

$\therefore$ Section is sufficiently large

4. Determine the Transverse Torsional Reinforcing Required

$$T_n = \frac{T_u}{\phi} = \frac{30\text{ kN-m}}{0.75} = 40\text{ kN-m}$$

Assuming $\theta = 45°$ as per ACI Code, Section 11.5.3.6(a)

$$\frac{A_t}{s} = \frac{T_n}{2A_o f_{yt} \cot\theta} = \frac{40 \text{ kN-m} \times 10^6}{(2)(121\,884 \text{ mm}^2)(420 \text{ MPa})(1.0)}$$

$$= 0.391 \text{ mm}^2/\text{mm for one leg of stirrup} \qquad \text{(ACI Equation 22.7.6.1a)}$$

5. Calculate the Area of Shear Reinforcing Required

$$V_u = 190 \text{ kN} > \frac{1}{2}V_c = \left(\frac{1}{2}\right)(180.45 \text{ kN}) = 90.22 \text{ kN}$$

$\therefore$ Shear reinforcement is required

$$V_s = \frac{V_u - \phi V_c}{\phi} = \frac{190 \text{ kN} - (0.75)(180.45 \text{ kN})}{0.75} = 72.88 \text{ kN}$$

$$\frac{A_v}{s} = \frac{V_s}{f_y d} = \frac{72.88 \text{ kN} \times 10^3}{(420 \text{ MPa})(584.6 \text{ mm})} = 0.297 \text{ mm}^2/\text{mm for two legs of stirrup}$$

6. Select Stirrups

$$\frac{2A_t}{s} + \frac{A_v}{s} = (2)(0.391 \text{ mm}^2) + 0.297 \text{ mm}^2 = 1.079 \text{ mm}^2/\text{mm for two legs of stirrup}$$

Using #13 stirrup ($A_s = 129 \text{ mm}^2$)

$$s = \frac{(2)(129 \text{ mm}^2)}{1.079 \text{ mm}^2/\text{mm}} = 239 \text{ mm}$$

Maximum allowable spacing of stirrup

$$= \frac{p_h}{8} = \frac{1629 \text{ mm}}{8} = 204 \text{ mm} \qquad \text{Use 200 mm}$$

Minimum area of stirrup

$$A_v + 2A_t = 0.062\sqrt{28} \text{ MPa} \frac{(350 \text{ mm})(200 \text{ mm})}{420 \text{ MPa}} = 55.11 \text{ mm}^2$$

$$\leq 0.35\frac{b_w s}{f_y} = 0.35\frac{(350 \text{ mm})(200 \text{ mm})}{420 \text{ MPa}} \qquad \text{(ACI Equation 7.6.4.2)}$$

$$= 55.56 \text{ mm}^2 < (2)(129 \text{ mm}^2) = 258 \text{ mm}^2 \quad \text{OK}$$

7. Selection of Longitudinal Torsional Reinforcing

$$A_\ell = \frac{A_t}{s} p_h \frac{f_{yt}}{f_y} \cot^2\theta = (0.391 \text{ mm}^2/\text{mm})(1629 \text{ mm}) \left(\frac{420 \text{ MPa}}{420 \text{ MPa}}\right)(1.0)^2$$

$$= 637 \text{ mm}^2 \qquad \text{(Equation 15-1)}$$

$$\min A_\ell = \frac{0.42\sqrt{f'_c} A_{cp}}{f_y} - \frac{A_t}{s} p_h \frac{f_{yt}}{f_y} \qquad \text{(ACI Equation 9.6.4.3)}$$

$$= \frac{(0.42\sqrt{28} \text{ MPa})(227\,500 \text{ mm}^2)}{(420 \text{ MPa})} - (0.391 \text{ mm}^2/\text{mm})(1629 \text{ mm}) \left(\frac{420 \text{ MPa}}{420 \text{ MPa}}\right)$$

$$= 557 \text{ mm}^2 \quad \text{OK}$$

Additional longitudinal steel is spread out to the four inside corners of the stirrups and vertically in between. Assume one-third in top = 557 mm^2/3 = 186 mm^2, 186 mm^2 + 2050 mm^2 = 2236 mm^2 in bottom, and remainder 186 mm^2 in between.

Use two #13 bars (258 mm^2) in top corners and at middepth and five #25 bars (2550 mm^2) in bottom.

15.11 Computer Example

Example 15.3

Repeat Example 15.1 using the Excel spreadsheet provided for Chapter 15.

SOLUTION

Open the Excel spreadsheet for Chapter 15 and the worksheet entitled Torsion. Observe that this sheet has U.S. customary units. Enter values for all cells highlighted in yellow (only in the Excel spreadsheets, not the printed example). Note that the results are the same as those obtained in Example 15.1.

Shear and Torsion Design — Rectangular Beams

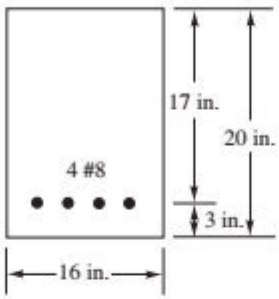
U.S. Customary Units

	V_u =	60	k				
	T_u =	30	ft-k				
	N_u =	0	k				
	f'_c =	4000	psi				
	λ =	1					
	b =	16	in.				
	h =	26	in.				
	cover =	1.5	in.				
	stirrup dia. =	0.5	in.				
	A_v =	0.40	in.^{2}				
	f_{yt} =	60,000	psi				
	f_y =	60,000	psi				
	ϕ =	0.75					
	A_{cp} =	416.00	in.^{2}				
	p_{cp}=	84.00	in.				
	Neglect torsion if T_u <	97,723	in.-lb =	8.144	ft-k	< T_u consider torsion	
	x_1 =	12.50	in.				
	y_1 =	22.50	in.				
	A_{oh} =	281.25	in.^{2}				
	A_o =	239.06	in.^{2}				
	d =	23.5	in.				
	p_h =	70.00	in.				
	V_c =	47,561	lb				
	Eq. 22.7.7.1a left	246					
	Eq. 22.7.7.1a right	474	section is large enough				
	A_t/s =	0.0167	in.^{2}/in. for 2 legs of stirrup				
	$V_u > \phi V_c/2$?	yes, shear reinforcement needed					
	V_s =	32,439	lb				
	A_v/s =	0.0230	in.^{2}/in. for 2 legs of stirrup				
	A_{v+t}/s =	0.0565	in.^{2}/in. for 2 legs of stirrup				
	s =	7.08	in.				
	$s \le p_h/8$ =	8.75	in.				
	$s \le 12$ =	12.00	in.				
	s =	7.08	in.				
	Use s =	7.00	in.				
ACI 9.6.4.2	$0.75S_{qrt}(f'_c)bs/f_{yt}$ =	0.089	<A_v – OK				
	A_ℓ =	1.17	in.^{2}	Select additional longitudinal bars			
	Min A_ℓ =	1.02	OK				

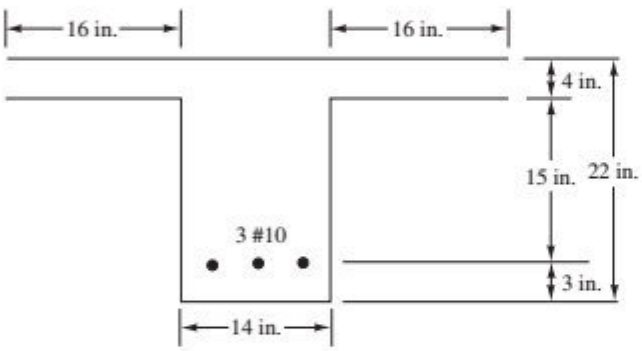
PROBLEMS

For Problems 15.1 to 15.3, determine the equilibrium torsional capacity of the sections if no torsional reinforcing is used. $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and normal-weight concrete.

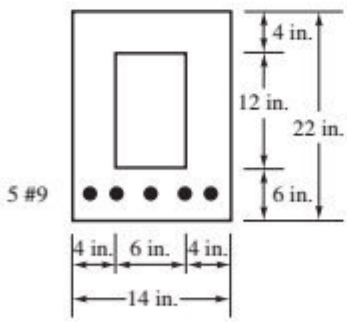
Problem 15.1 (*Ans*. 5.62 ft-k)



Problem 15.2



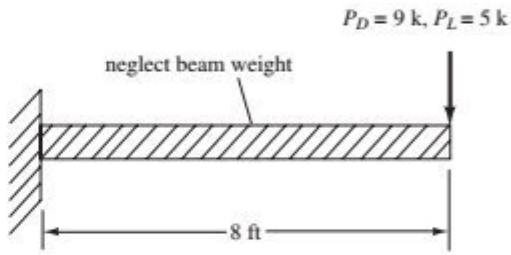
Problem 15.3 (*Ans*. 5.21 ft-k)



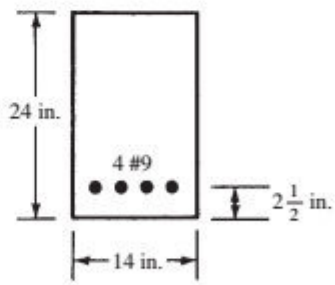
Problem 15.4 Repeat Problem 15.1 if the width is changed from 16 in. to 12 in. and the depth from 20 in. to 28 in.

Problem 15.5 Repeat Problem 15.3 if $f'_c = 3000$ psi and sand-lightweight concrete. (*Ans*. 3.83 ft-k)

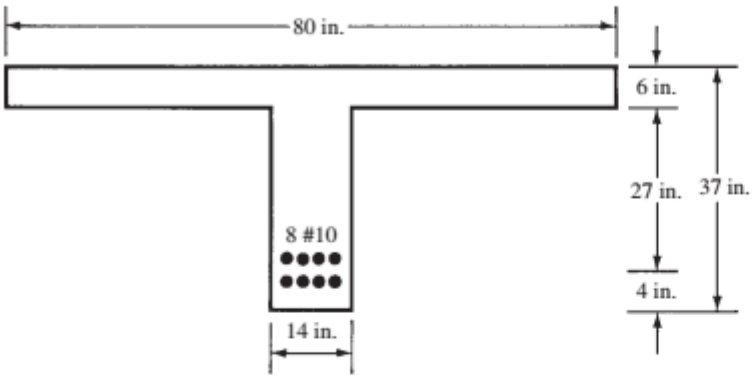
Problem 15.6 What minimum total theoretical depth is needed for the beam shown if no torsional reinforcing is to be used? The cross section is not shown, but it is rectangular with $b = 20$ in. and the depth to be determined. The concentrated load is located at the end of the cantilever 8 in. to one side of the beam centerline, $f_y = 60{,}000$ psi, $f'_c = 4000$ psi, and all-lightweight concrete.



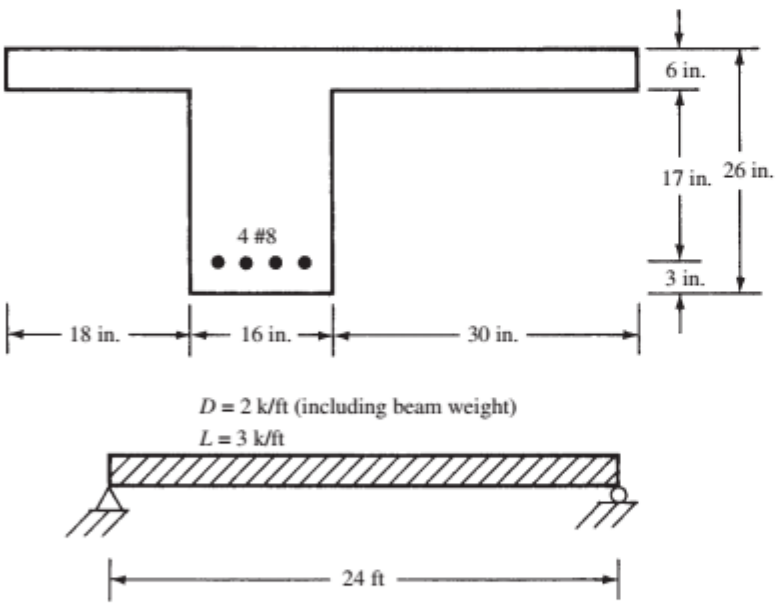
Problem 15.7 If the reinforced concrete spandrel beam shown has $f'_c = 4000$ psi, normal-weight concrete, and $f_y = 60{,}000$ psi, determine the theoretical spacing required for #3 stirrups at a distance d from the face of the support where $V_u = 32$ k and $T_u = 10$ ft-k. Clear cover = $1\frac{1}{2}$ in. (*Ans*. 5.87 ft-k < 10 ft-k ∴ torsion reinf. reqd @ 7.81 in. on center)



Problem 15.8 Design the torsional reinforcement for the beam shown at a section a distance d from the face of the support for a torsional moment of 24 ft-k. $V_u = 90$ k, $f'_c = 3000$ psi, and $f_y = 60{,}000$ psi. Clear cover = 1.50 in. Use #4 stirrups.

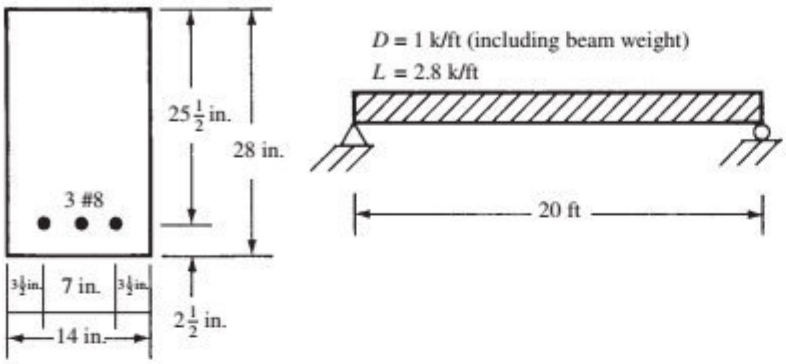


Problem 15.9 Determine the theoretical spacing of #4 closed □ stirrups at a distance d from the face of the support for the edge beam shown if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. T_u equals 36 ft-k at the face of the support and is assumed to vary along the beam in proportion to the shear. Clear cover = $1\frac{1}{2}$ in. and normal-weight concrete. (*Ans.* 6.82 in.)

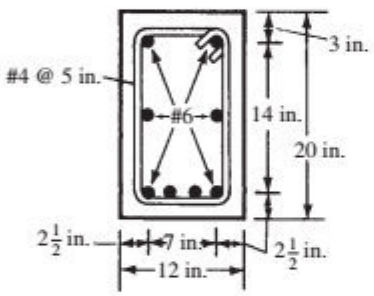


Problem 15.10 A 12-in. × 22-in. spandrel beam ($d = 19.5$ in.) with a 20-ft simple span has a 4-in. slab 16 in. wide on one side acting as a flange. It must carry a maximum V_u of 60 k and a maximum T_u of 20 ft-k at the face of the support. Assuming these values are zero at the beam centerline, select #4 stirrups if $f'_c = 3000$ psi, normal-weight concrete, $f_y = 60{,}000$ psi, and clear cover = 1.5 in.

Problem 15.11 Determine the theoretical spacing of #4 closed □ stirrups at a distance d from the face of the support for the beam shown if the load is acting 3 in. off center of the beam. Assume that the torsion equals the uniform load times the 3 in., $f'_c = 3000$ psi, all-lightweight aggregate concrete, and $f_y = 60{,}000$ psi. Assume that the torsion value varies from a maximum at the support to zero at the beam centerline as does the shear. Assume 1.5-in. clear cover. (*Ans.* 8.51 in.)



Problem 15.12 Is the beam shown satisfactory to resist a T_u of 15 ft-k and a V_u of 60 k if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi? The bars shown are used in addition to those provided for bending moment.

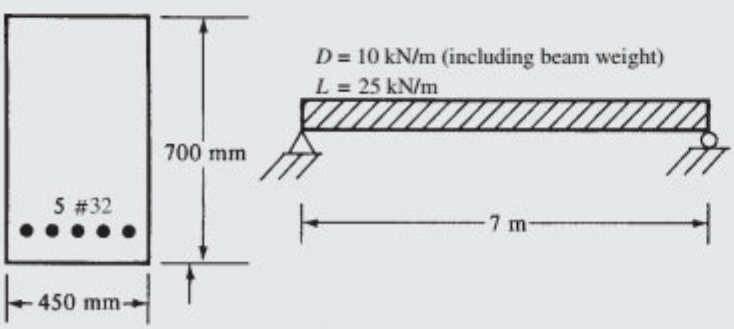


Problem 15.13 Using Chapter 15 Excel spreadsheet determine the required spacing at a distance d from the support for Problem 15.7. Use the same materials, but change concrete to all-lightweight aggregate. (*Ans.* 7.81 in.)

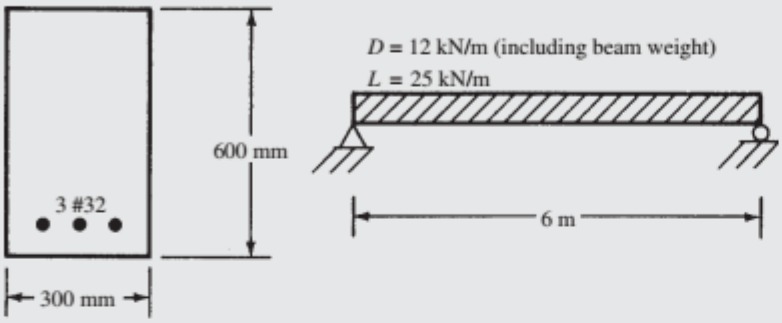
Problem 15.14 Repeat Problem 15.11 using #3 closed stirrups, $f'_c = 4000$ psi, semi-lightweight concrete, and Chapter 15 spreadsheet.

Problems in SI Units

Problem 15.15 Determine the required spacing of #13 closed □ stirrups at a distance d from the face of the support for the beam shown assuming that the torsion decreases uniformly from the beam end to the beam centerline. The member is subjected to a 34-kN-m service dead load torsion and a 40-kN-m service line load torsion at the face of the support, $f_y = 420$ MPa, $f'_c = 24$ MPa, and clear cover = 40 mm. (*Ans.* 175 mm)



Problem 15.16 Determine stirrup spacing at a distance d from support for the beam shown if the load is acting 100 mm off center of the beam. Assume the torsion at the support equals the uniform load times 100 mm, $f'_c = 28$ MPa, and $f_y = 420$ MPa. Use #13 stirrups and assume that the torsion and shear vary from a maximum at the support to zero at the beam centerline. Clear cover = 40 mm.



Two-Way Slabs, Direct Design Method

CHAPTER 16

16.1 Introduction

In general, slabs are classified as being one way or two way. Slabs that primarily deflect in one direction are referred to as *one-way slabs*; see Figure 16.1(a). Simple-span, one-way slabs have been discussed previously in Section 4.7 of this text, while the design of continuous one-way slabs was considered in Section 14.7. When slabs are supported by columns arranged generally in rows so that the slabs can deflect in two directions, they are usually referred to as *two-way slabs*.

Two-way slabs can be strengthened by the addition of beams between the columns, by thickening the slabs around the columns (*drop panels*), and by flaring the columns under the slabs (*column capitals*). These situations are shown in Figure 16.1 and discussed in the following paragraphs.

Flat plates, as shown in Figure 16.1(b), are solid concrete slabs of uniform depths that transfer loads directly to the supporting columns without the aid of beams or capitals or drop panels. Flat plates can be constructed quickly because of their simple formwork and reinforcing bar arrangements. They need the smallest overall story heights to provide specified headroom requirements, and they give the most flexibility in the arrangement of columns and partitions. They also provide little obstruction to light and have high fire resistance because there are few sharp corners where spalling of the concrete might occur. Flat plates are probably the most commonly used slab system today for multistory reinforced concrete hotels, motels, apartment houses, hospitals, and dormitories.

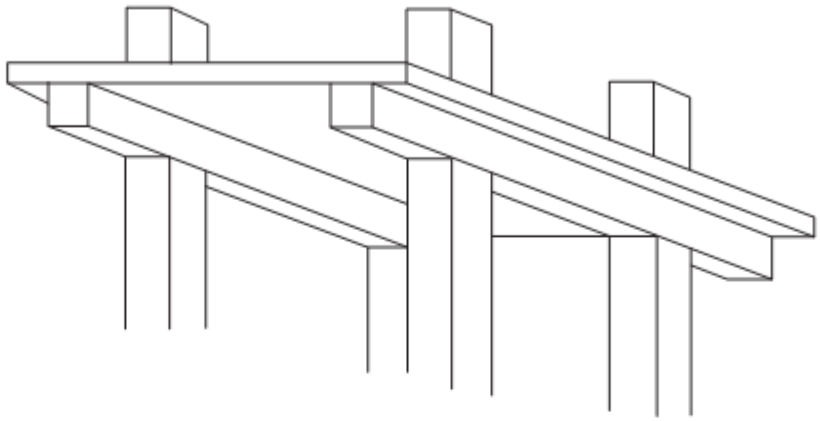
Flat plates present a possible problem in transferring the shear at the perimeter of the columns. In other words, there is a danger that the columns may punch through the slabs. As a result, it is frequently necessary to increase column sizes or slab thicknesses or to use *shearheads*. Shearheads consist of steel I or channel shapes placed in the slab over the columns, as discussed in Section 16.5 of this chapter. Although such procedures may seem expensive, the simple formwork required for flat plates will usually result in such economical construction that the extra costs required for shearheads are more than canceled. For heavy industrial loads or long spans, however, some other type of floor system may be required.

Flat slabs, shown in Figure 16.1(c), include two-way reinforced concrete slabs with capitals, drop panels, or both. These slabs are very satisfactory for heavy loads and long spans. Although the formwork is more expensive than for flat plates, flat slabs will require less concrete and reinforcement than would be required for flat plates with the same loads and spans. They are particularly economical for warehouses, parking and industrial buildings, and similar structures, where exposed drop panels or capitals are acceptable.

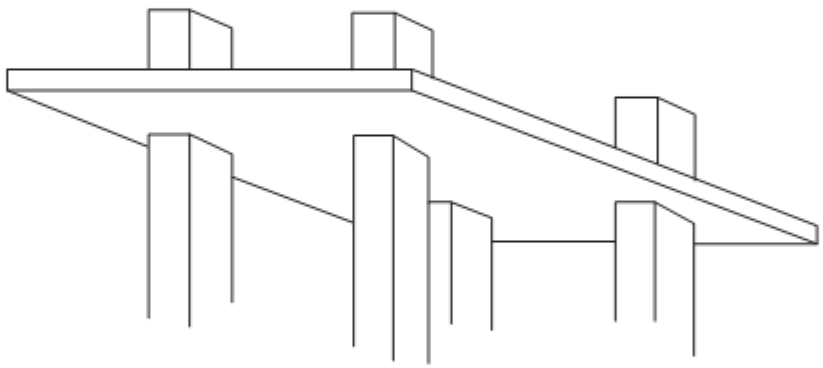
In Figure 16.1(d), a *two-way slab with beams* is shown. This type of floor system is used where its cost is less than the costs of flat plates or flat slabs. In other words, when the loads or spans or both become quite large, the slab thickness and column sizes required for flat plates or flat slabs are of such magnitude that it is more economical to use two-way slabs with beams, despite the higher formwork costs.

Another type of floor system is the *waffle slab*. The floor is constructed by arranging square fiberglass or metal pans with tapered sides with spaces. When the concrete is placed over and between the pans and the forms are removed, the waffle shape is obtained. The intervals or gaps between the pans

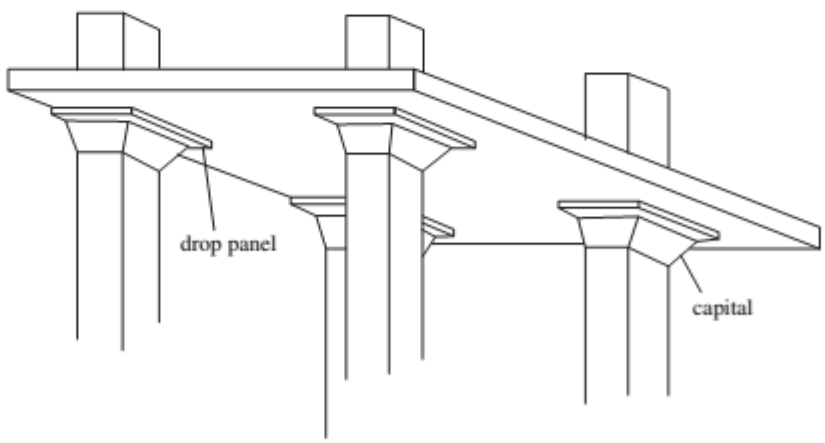
form the beam webs. These webs are rather deep and provide large moment arms for the reinforcing bars. With waffle slabs, the weight of the concrete is greatly reduced without significantly changing the moment resistance of the floor system. As in flat plates, shear can be a problem near columns. Consequently, waffle floors are usually made solid in those areas to increase shear resistance.



(a) One-way slab with beams



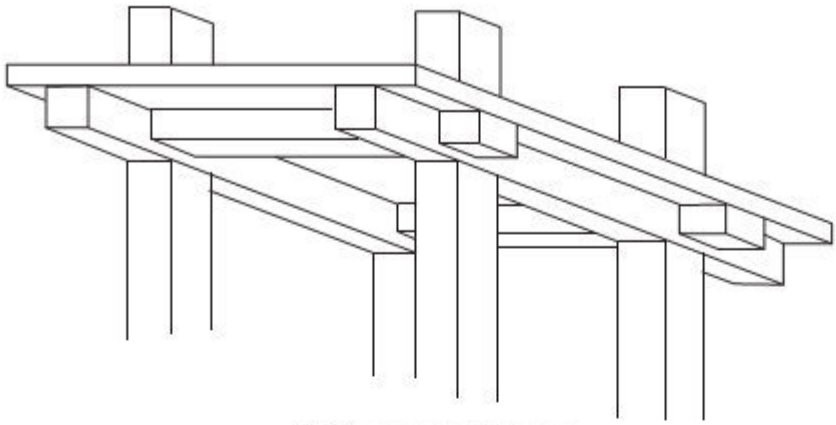
(b) Flat plate



(c) Flat slab

(*continues*)

FIGURE 16.1 Slabs.



(d) Two-way slab with beams

FIGURE 16.1 (*continued*)

View Pictures/UIG via Getty Images, Inc.

Flat plate without edge beams. (Buckley Gray Yeoman, Fashion Street, London, 2010)

16.2 Analysis of Two-Way Slabs

Two-way slabs bend under load into dish-shaped surfaces, so there is bending in both principal directions. As a result, they must be reinforced in both directions by layers of bars that are perpendicular to each other. A theoretical elastic analysis for such slabs is a very complex problem because of their highly indeterminate nature. Numerical techniques such as finite difference and finite elements are required, but such methods require sophisticated software to be practical in design. The methods described in this chapter can be done by hand or with simple spreadsheets and are sufficiently accurate for most design problems.

Actually, the fact that a great deal of stress redistribution can occur in such slabs at high loads makes it unnecessary to make designs based on theoretical analyses. Therefore, the design of two-way slabs is generally based on empirical moment coefficients, which, although they might not accurately predict stress variations, result in slabs with satisfactory overall safety factors. In other words, if too much reinforcement is placed in one part of a slab and too little somewhere else, the resulting slab behavior will probably still be satisfactory. *The total amount of reinforcement in a slab seems more important than its exact placement*.

You should clearly understand that, although this chapter and the next are devoted to two-way slab design based on approximate methods of analysis, there is no intent to prevent the designer from using more exact methods. Designers may design slabs on the basis of numerical solutions, yield-line analysis, or other theoretical methods, provided that it can be clearly demonstrated that they have met all the necessary safety and service ability criteria required by the ACI Code.

Although it has been the practice of designers for many years to use approximate analyses for design and to use average moments rather than maximum ones, two-way slabs so designed have proved to be very satisfactory under service loads. Furthermore, they have been proved to have appreciable overload capacity.

16.3 Design of Two-Way Slabs by the ACI Code

Section 8.2.1 of the ACI Code specifies two methods for designing two-way slabs for gravity loads. These are the direct design method and the equivalent frame method.

Direct Design Method

Section 8.10 of the ACI Code provides a procedure with which a set of moment coefficients can be determined. The method, in effect, involves a single-cycle moment distribution analysis of the structure based on (a) the estimated flexural stiffnesses of the slabs, beams (if any), and columns and (b) the torsional stiffnesses of the slabs and beams (if any) transverse to the direction in which flexural moments are being determined. Some types of moment coefficients have been used satisfactorily for many years for slab design. They do not, however, give very satisfactory results for slabs with unsymmetrical dimensions and loading patterns.

Equivalent Frame Method

In this method, a portion of a structure is taken out by itself, as shown in Figure 16.2, and analyzed much as a portion of a building frame was handled in Example 14.2. The same stiffness values used for the direct design method are used for the equivalent frame method. This latter method, which is very satisfactory for symmetrical frames as well as for those with unusual dimensions or loadings, is discussed in Chapter 17 of this text.

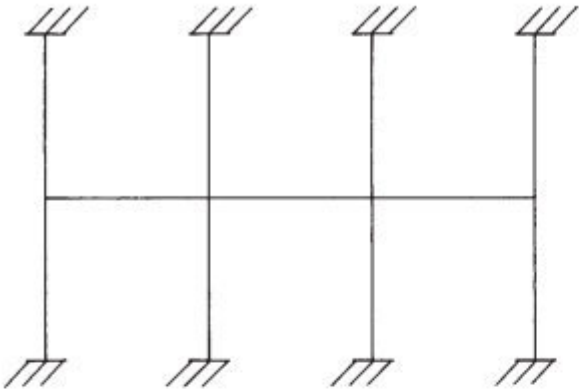


FIGURE 16.2 Equivalent frame method.

Design for Lateral Loads

The ACI Code permits considerable freedom for the designer to model two-way slab systems for lateral loads. The method must satisfy equilibrium and geometric compatibility and be in reasonable agreement with test data. The effects of cracking and such parameters as the slab aspect ratio and ratio of column-to-slab span dimensions should be considered[1] (ACI Code, Sections 6.6.3.2.1 and 24.2).

16.4 Column and Middle Strips

After the design moments have been determined by either the direct design method or the equivalent frame method, they are distributed across each panel. The panels are divided into column and middle strips, as shown in Figure 16.3, and positive and negative moments are estimated in each strip. The *column strip* is a slab with a width on each side of the column centerline equal to one-fourth

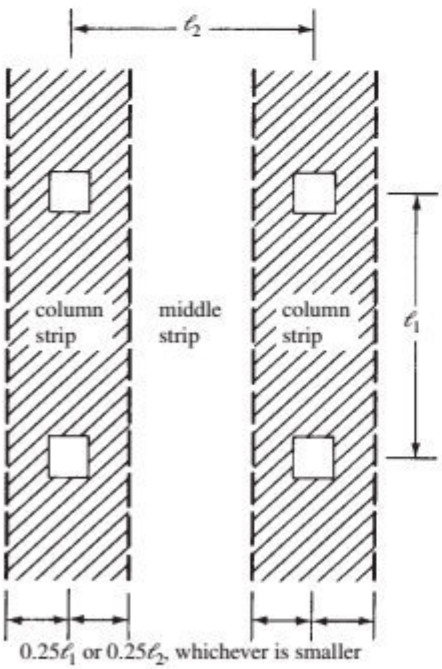


FIGURE 16.3 Column and middle strips.

[1] Vanderbilt, M. D. and Corley, W. G., 1983. "Frame Analysis of Concrete Buildings," *Concrete International: Design and Construction*, Vol. 5, No. 12, pp. 33–43.

Courtesy of Professor Larry Kahn, Georgia Institute of Technology.

Flat Plate Construction—Pharr Road Condominiums, Atlanta, Georgia.

the smaller of the panel dimensions ℓ_1 or ℓ_2. It includes beams if they are present. The middle strip is the part of the slab between the two column strips.

The part of the moments assigned to the column and middle strips may be assumed to be uniformly spread over the strips. As will be described later in this chapter, the percentage of the moment assigned to a column strip depends on the effective stiffness of that strip and on its *aspect ratio*, ℓ_2/ℓ_1 (where ℓ_1 is the length of span, center to center, of supports in the direction in which moments are being determined and ℓ_2 is the span length, center to center, of supports in the direction transverse to ℓ_1). Note that Figure 16.3 shows column and middle strips in only one direction. A similar analysis must be performed in the perpendicular direction. The resulting analysis will result in moments in both directions.

16.5 Shear Resistance of Slabs

For two-way slabs supported by beams or walls, shears are calculated at a distance d from the faces of the walls or beams. The value of ϕV_c is, as for beams, $\phi 2\lambda\sqrt{f'_c}b_w d$. Shear is not usually a problem for these types of slabs.

For flat slabs and flat plates supported directly by columns, shear may be the critical factor in design. In almost all tests of such structures, failures have been the result of shear or perhaps shear and torsion. These conditions are particularly serious around exterior columns.

Two kinds of shear must be considered in the design of flat slabs and flat plates. These are the same two that were considered in column footings—one-way and two-way shears (i.e., beam shear and punching shear). For beam shear analysis, the slab is considered to act as a wide beam running between the supports. The critical sections are taken at a distance d from the face of the column or capital. For punching shear, the critical section is taken at a distance $d/2$ from the face of the column, capital, or drop panel and the shear strength, as usually used in footings, is $\phi 4\lambda\sqrt{f'_c}b_w d$.

If shear stresses are too large around interior columns, it is possible to increase the shearing strength of the slabs by as much as 75% by using shearheads. A shearhead, as defined in Section 22.6.1.1 of the ACI Code, consists of four steel I or channel shapes fabricated into cross arms and placed in the slabs, as shown in Figure 16.4. The code states that shearhead designs of this type do not apply at exterior columns. Thus, special designs are required, and the code does not provide specific requirements. Shearheads increase the effective b_o for two-way shear, and they also increase the negative moment resistance of the slab, as described in Section 22.6.9.7 of the ACI Code. The negative moment reinforcing bars in the slab are usually run over the top of the steel shapes, whereas the positive reinforcing bars are normally stopped short of the shapes.

The main advantage of shearheads is that they push the critical sections for shear farther out from the columns, thus giving a larger perimeter to resist the shear, as illustrated in Figure 16.5. In this figure, ℓ_v is the length of the shearhead arm from the centroid of the concentrated load or reaction, and c_1 is the dimension of the rectangular or equivalent rectangular column or capital or bracket, measured in the direction in which moments are being calculated. Section 22.6.9.8 of the ACI Code states that the critical section for shear shall cross the shearhead arm at a distance equal to $\frac{3}{4}[\ell_v - (c_1/2)]$ from the column face, as shown in Figure 16.5(b). Although this critical section is to be located so that its perimeter is a minimum, it does not have to be located closer to the column face or edges of capitals or drop panels than $d/2$ at any point. When shearhead reinforcement is provided with reinforcing bars or steel I or channel shapes, the maximum shear strength can be increased to $7\sqrt{f'_c}b_o d$ at a distance $d/2$ from the column. According to Section 22.6.9.10 of the ACI Code, this is permissible only if the maximum computed shear does not exceed $4\sqrt{f'_c}b_o d$ along the dashed critical section for shear shown in Figure 16.5(b).

In Section 16.12, the subject of shear stresses is continued with a consideration of the transfer of moments and shears between slabs and columns. The maximum load that a two-way slab can support is often controlled by this transfer strength.

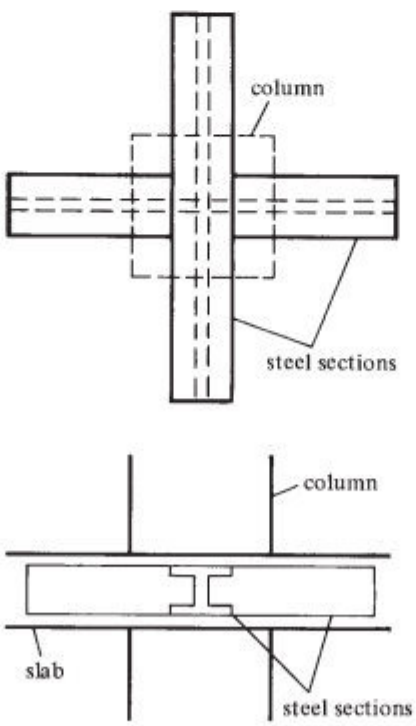


FIGURE 16.4 Shearhead reinforcement for slabs at columns.

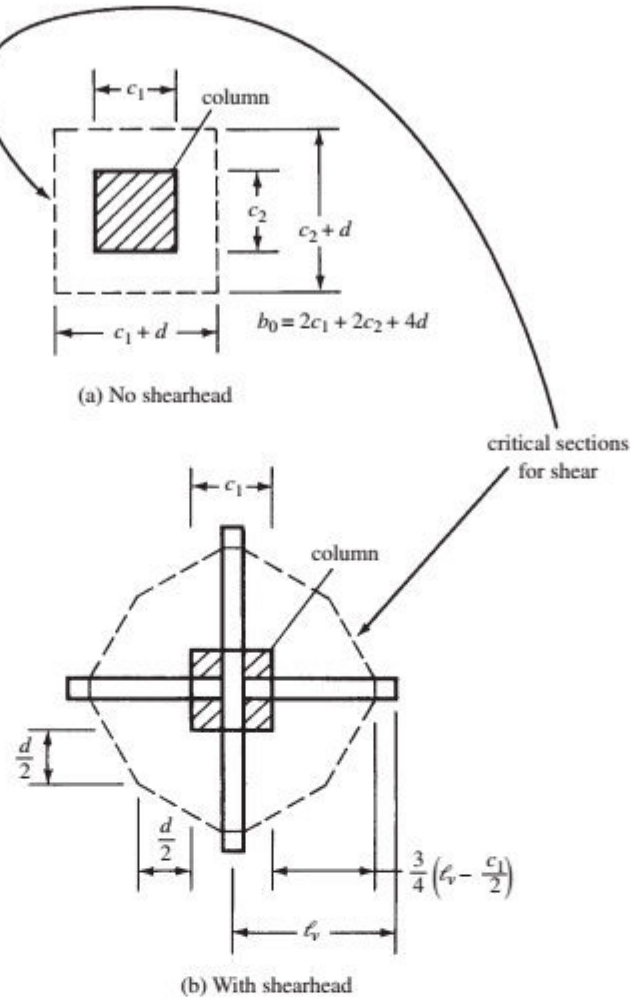


FIGURE 16.5 Critical sections for shear.

16.6 Depth Limitations and Stiffness Requirements

It is obviously very important to keep the various panels of a two-way slab relatively level (i.e., with reasonably small deflections). Thin reinforced two-way slabs have quite a bit of moment resistance, but deflections are often large. As a consequence, their depths are very carefully controlled by the ACI Code so as to limit these deflections. This is accomplished by requiring the designer to either (a) compute deflections and make sure they are within certain limitations or (b) use certain minimum thicknesses as specified in Section 8.3.1 of the ACI Code. Deflection computations for two-way slabs are rather complicated, so the average designer usually uses the minimum ACI thickness values, presented in the next few paragraphs of this chapter.

Slabs without Interior Beams

For a slab without interior beams spanning between its supports and with a ratio of its long span to short span not greater than 2.0, the minimum thickness can be taken from Table 16.1 of this chapter [ACI Table 8.3.1.1]. The values selected from the table, however, must not be less than the following values (ACI Code, Section 8.3.1.1):

1. Slabs without drop panels 5 in.
2. Thickness of those slabs with drop panels outside the panels 4 in.

TABLE 16.1 Minimum Thickness of Slabs without Interior Beams

	Without Drop Panels†			With Drop Panels†		
	Exterior Panels		Interior Panels	Exterior Panels		Interior Panels
Yield strength, f_y, psi*	Without edge beams	With edge beams‡		Without edge beams	With edge beams‡	
40,000	$\frac{\ell_n}{33}$ [§]	$\frac{\ell_n}{36}$	$\frac{\ell_n}{36}$	$\frac{\ell_n}{36}$	$\frac{\ell_n}{40}$	$\frac{\ell_n}{40}$
60,000	$\frac{\ell_n}{30}$	$\frac{\ell_n}{33}$	$\frac{\ell_n}{33}$	$\frac{\ell_n}{33}$	$\frac{\ell_n}{36}$	$\frac{\ell_n}{36}$
75,000	$\frac{\ell_n}{28}$	$\frac{\ell_n}{31}$	$\frac{\ell_n}{31}$	$\frac{\ell_n}{31}$	$\frac{\ell_n}{34}$	$\frac{\ell_n}{34}$

*For values of reinforcement yield strength between the values given in the table, minimum thickness shall be determined by linear interpolation.

†Drop panel is defined in ACI Code, Section 8.10.2.7.

‡Slabs with beams between columns along exterior edges. The value of α_f for the edge beam shall not be less than 0.8.

§For two-way construction, ℓ_n is the length of the clear span in the long direction, measured face to face of the supports in slabs without beams and face to face of beams or other supports in other cases.

Source: Permission of American Concrete Institute.

In Table 16.1, some of the values are given for slabs with drop panels. To be classified as a drop panel, according to Section 8.10.2.7 of the ACI Code, a panel must (a) extend horizontally in each direction from the centerline of the support not less than one-sixth the distance, center to center, of supports in that direction and (b) project vertically below the slab a distance not less than one-fourth the thickness of the slab away from the drop panel. In this table, ℓ_n is the length of the clear span in the long direction of two-way construction, measured face to face of the supports in slabs without beams and face to face of beams or other supports in other cases.

Very often slabs are built without interior beams between the columns but with edge beams running around the perimeter of the building. These beams are very helpful in stiffening the slabs and reducing the deflections in the exterior slab panels. The stiffness of slabs with edge beams is expressed as a function of α_f, which follows.

Throughout this chapter, the letter α_f is used to represent the ratio of the flexural stiffness ($E_{cb}I_b$) of a beam section to the flexural stiffness of the slab ($E_{cs}I_s$) whose width equals the distance between the centerlines of the panels on each side of the beam. If no beams are used, as is the case for the flat plate, α will equal 0. For slabs with beams between columns along exterior edges, α for the edge beams may not be < 0.8, as specified in a footnote to Table 16.1.

$$\alpha_f = \frac{E_{cb}I_b}{E_{cs}I_s} \qquad \text{(ACI Equation 8.10.2.7b)}$$

where

E_{cb} = the modulus of elasticity of the beam concrete

E_{cs} = the modulus of elasticity of the column concrete

I_b = the gross moment of inertia about the centroidal axis of a section made up of the beam and the slab on each side of the beam extending a distance equal to the projection of the beam above or below the slab (whichever is greater) but not exceeding four times the slab thickness (ACI Code, Section 8.4.1.8)

I_s = the moment of inertia of the gross section of the slab taken about the centroidal axis and equal to $h^3/12$ times the slab width, where the width is the same as for α

Example 16.1

Using the ACI Code, determine the minimum permissible total thicknesses required for the slabs in panels ③ and ② for the floor system shown in Figure 16.6. Edge beams are used around the building perimeter, and they are 12 in. wide and extend vertically for 8 in. below the slab, as shown in Figure 16.7. They also extend 8 in. out into the slab as required by Section 8.4.1.8 of the ACI Code. No drop panels are used, and the concrete in the slab is the same as that used in the edge beams. $f_y = 60{,}000$ psi.

SOLUTION

For Interior Panel ③

$$\alpha_f = 0 \text{ (because the interior panels have no perimeter beams)}$$

$$\ell_n = 20 \text{ ft} - \frac{16 \text{ in.}}{12 \text{ in./ft}} = 18.67 \text{ ft (clear distance between columns)}$$

$$\text{Min } h = \frac{\ell_n}{33} \text{ (from Table 16.1)}$$

$$= \frac{18.67 \text{ ft}}{33} = 0.566 \text{ ft} = 6.79 \text{ in.}$$

Cannot be less than 5 in., according to Section 8.3.1.1 of the ACI Code. Try 7 in.

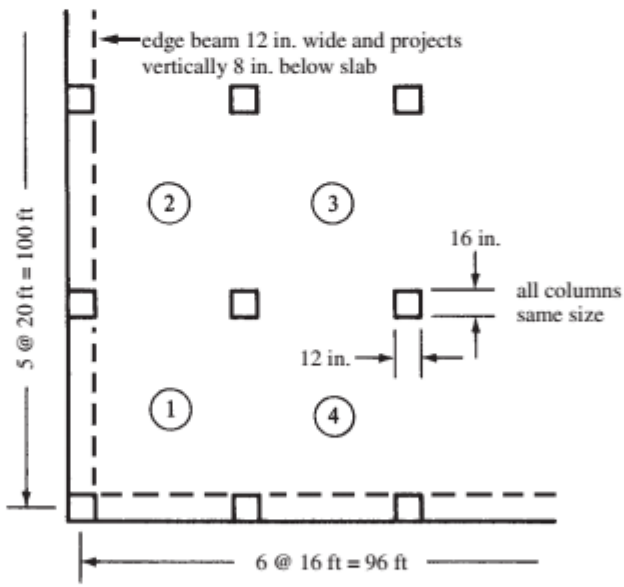
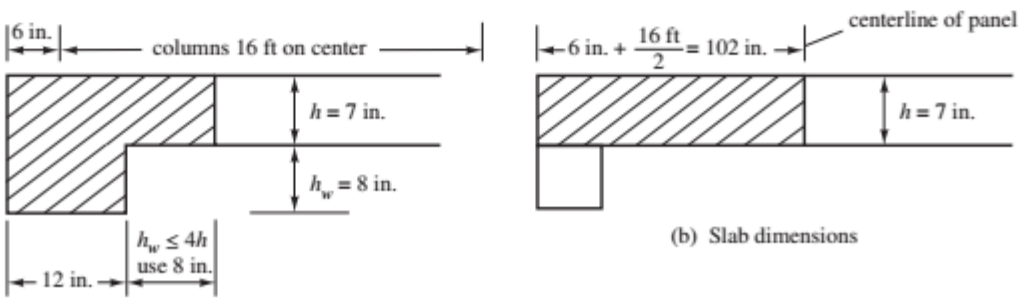


FIGURE 16.6 Flat-plate floor slab for Example 16.1.



(a) Edge beam dimensions

(b) Slab dimensions

FIGURE 16.7 Sections for edge beam and slab.

For Exterior Panel ②

Assume $h = 7$ in. and compute α_f with reference made to Figure 16.7(a). Centroid of cross-hatched beam section located by statics 6.55 in. from top.

$$I_b = \left(\frac{1}{3}\right)(20\text{ in.})(6.55\text{ in.})^3 + \left(\frac{1}{3}\right)(12\text{ in.})(8.45\text{ in.})^3 + \left(\frac{1}{3}\right)(8\text{ in.})(0.45\text{ in.})^3$$

$$= 4287\text{ in.}^4$$

$$I_s = \left(\frac{1}{12}\right)(102\text{ in.})(7\text{ in.})^3 = 2915.5\text{ in.}^4 \quad \text{See Figure 16.7(b)}$$

$$\alpha = \frac{EI_b}{EI_s} = \frac{(E)(4287\text{ in.}^4)}{(E)(2915.5\text{ in.}^4)} = 1.47 > 0.8$$

∴ This is an edge beam as defined in the footnote of Table 16.1.

$$\text{Min } h = \frac{\ell_n}{33} = \frac{20\text{ ft} - \dfrac{16\text{ in.}}{12\text{ in./ft}}}{33} = 0.566\text{ ft} = 6.79\text{ in.} \qquad \underline{\text{Try 7 in.}}$$

Slabs with Interior Beams

To determine the minimum thickness of slabs with beams spanning between their supports on all sides, Section 8.3.1.2 of the ACI Code must be followed. Involved in the expressions presented there are span lengths, panel shapes, flexural stiffness of beams if they are used, steel yield stresses, and so on. In these equations, the following terms are used:

ℓ_n = the clear span in the long direction, measured face to face, of (a) columns for slabs without beams and (b) beams for slab with beams
β = the ratio of the long to the short clear span
α_{fm} = the average value of the ratios of beam-to-slab stiffness on all sides of a panel

The minimum thickness of slabs or other two-way construction may be obtained by substituting into the equations to follow, which are given in Section 8.3.1.2 of the ACI Code. In the equations, the quantity β is used to take into account the effect of the shape of the panel on its deflection, whereas the effect of beams (if any) is represented by α_{fm}. If there are no beams present (as is the case for flat slabs), α_{fm} will equal 0.

1. For $\alpha_{fm} \le 0.2$, the minimum thicknesses are obtained as they were for slabs without interior beams spanning between their supports (ACI Table 8.3.1.2a).
2. For $0.2 \le \alpha_{fm} \le 2.0$, the thickness may not be less than 5 in. or

$$h = \frac{\ell_n\left(0.8 + \dfrac{f_y}{200{,}000}\right)}{36 + 5\beta(\alpha_{fm} - 0.2)} \qquad \text{(ACI Equation 8.3.1.2b)}$$

3. For $\alpha_{fm} > 2.0$, the thickness may not be less than 3.5 in. or

$$h = \frac{\ell_n\left(0.8 + \dfrac{f_y}{200{,}000}\right)}{36 + 9\beta} \qquad \text{(ACI Equation 8.3.1.2d)}$$

where ℓ_n and f_y are in inches and psi, respectively.

For panels with discontinuous edges, Section 8.3.1.2.1 of the ACI Code requires that edge beams be used, which have a minimum stiffness ratio α_f equal to 0.8, or else that the minimum slab thicknesses, as determined by ACI Table 8.3.1.2, must be increased by 10%.

The designer may use slabs of lesser thicknesses than those required by the ACI Code, as described in the preceding paragraphs, if deflections are computed and found to be equal to or less than the limiting values given in Table 24.2.2 of the ACI 318 Code (Table 6.1 in this text).

Should the various rules for minimum thickness be followed but the resulting slab be insufficient to provide the shear capacity required for the particular column size, column capitals will probably be required. Beams running between the columns may be used for some slabs where partitions or heavy equipment loads are placed near column lines. A very common case of this type occurs where exterior beams are used when the exterior walls are supported directly by the slab. Another situation where beams may be used occurs where there is concern about the magnitude of slab vibrations. Example 16.2 illustrates the application of the minimum slab thickness rules for a two-way slab with beams.

Example 16.2

The two-way slab shown in Figure 16.8 has been assumed to have a thickness of 7 in. Section A–A in the figure shows the beam cross section. Check the ACI equations to determine if the slab thickness is satisfactory for an interior panel. $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, and normal-weight concrete.

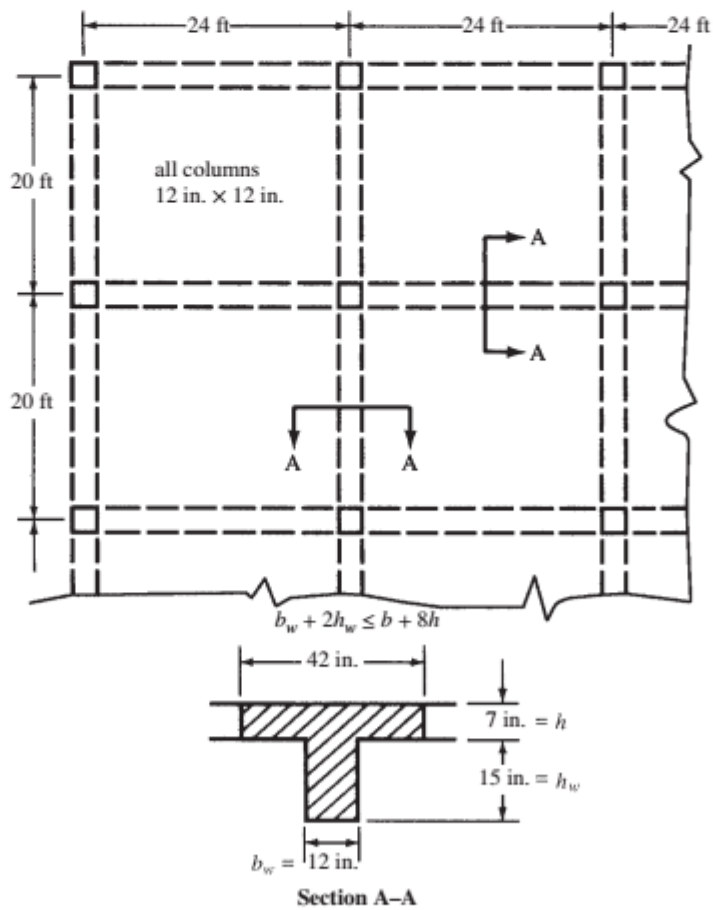


FIGURE 16.8 A two-way slab.

SOLUTION

(Using the Same Concrete for Beams and Slabs)

Computing α_1 for Long (Horizontal) Span for Interior Beams

I_s = gross moment of inertia of slab 20 ft wide

$$= \left(\frac{1}{12}\right)(12 \text{ in./ft} \times 20 \text{ in.})(7 \text{ in.})^3 = 6860 \text{ in.}^4$$

I_b = gross I of T-beam cross section shown in Figure 16.8 about centroidal axis = 18,060 in.^{4}

$$\alpha_1 = \frac{EI_b}{EI_s} = \frac{(E)(18{,}060 \text{ in.}^4)}{(E)(6860 \text{ in.}^4)} = 2.63$$

Computing α_2 for Long Interior Beams

$$I_s \text{ for 24-ft-wide slab} = \left(\frac{1}{12}\right)(12 \text{ in./ft} \times 24 \text{ in.})(7 \text{ in.})^3 = 8232 \text{ in.}^4$$

$$I_b = 18{,}060 \text{ in.}^4$$

$$\alpha_2 = \frac{(E)(18{,}060 \text{ in.}^4)}{(E)(8232 \text{ in.}^4)} = 2.19$$

$$\alpha_{fm} = \frac{\alpha_1 + \alpha_2}{2} = \frac{2.63 + 2.19}{2} = 2.41$$

Determining Slab Thickness per ACI Code, Section 8.3.1.2

$\alpha_{fm} = 2.41 > 2$ ∴ Use ACI Equation 8.3.1.2d

$$h = \frac{\ell_n\left(0.8 + \dfrac{f_y}{200{,}000 \text{ psi}}\right)}{36 + 9\beta}$$

$\ell_{n\,\text{long}} = 24 \text{ ft} - 1 \text{ ft} = 23 \text{ ft}$

$\ell_{n\,\text{short}} = 20 \text{ ft} - 1 \text{ ft} = 19 \text{ ft}$

$$\beta = \frac{23 \text{ ft}}{19 \text{ ft}} = 1.21$$

$$h = \frac{(23 \text{ ft})\left(0.8 + \dfrac{60{,}000 \text{ psi}}{200{,}000 \text{ psi}}\right)}{36 + (9)(1.21)} = 0.540 \text{ ft} = 6.47 \text{ in.}$$

Use 7-in. slab

Note that the interior panel will generally not control the required slab thickness. Usually it will be an edge or corner panel. The interior panel was chosen here to illustrate the calculations and to avoid excess complexity. Had a corner panel been selected, each edge of the panel would have had a different α_f.

16.7 Limitations of Direct Design Method

For the moment coefficients determined by the direct design method to be applicable, Section 8.10.2 of the ACI Code says that the following limitations must be met, unless a theoretical analysis shows that the strength furnished after the appropriate capacity reduction or ϕ factors applied is sufficient to support the anticipated loads and provided that all serviceability conditions, such as deflection limitations, are met:

1. There must be at least three continuous spans in each direction.
2. The panels must be rectangular, with the length of the longer side of any panel not being more than two times the length of its shorter side lengths being measured c to c of supports.
3. Span lengths of successive spans in each direction may not differ in length by more than one-third of the longer span.
4. Columns may not be offset by more than 10% of the span length in the direction of the offset from either axis between center lines of successive columns.
5. The unfactored live load must not be more than two times the unfactored dead load. All loads must be the result of gravity and must be uniformly distributed over an entire panel.
6. If a panel is supported on all sides by beams, the relative stiffness of those beams in the two perpendicular directions, as measured by the following expression, shall not be less than 0.2 or greater than 5.0.

$$\frac{\alpha_{f1}\ell_2^2}{\alpha_{f2}\ell_1^2}$$

The terms ℓ_1 and ℓ_2 were shown in Figure 16.3.

16.8 Distribution of Moments in Slabs

The total moment, M_o, that is resisted by a slab equals the sum of the maximum positive and negative moments in the span. It is the same as the total moment that occurs in a simply supported beam. For a uniform load per unit area, q_u, it is as follows:

$$M_o = \frac{(q_u\ell_2)(\ell_1)^2}{8}$$

In this expression, ℓ_1 is the span length, center to center, of supports in the direction in which moments are being taken and ℓ_2 is the length of the span transverse to ℓ_1, measured center to center of the supports.

The moment that actually occurs in such a slab has been shown by experience and tests to be somewhat less than the value determined by the above M_o expression. For this reason, ℓ_1 is replaced with ℓ_n, the clear span measured face to face of the supports in the direction in which moments are taken. Section 8.10.3.2.1 of the ACI Code states that ℓ_n may not be taken to be less than 65% of the span ℓ_1 measured center to center of supports. If ℓ_1 is replaced with ℓ_n, the expression for M_o, which is called the *static moment*, becomes

$$M_o = \frac{(q_u\ell_2)(\ell_n)^2}{8} \qquad \text{(ACI Equation 8.10.3.2)}$$

When the static moment is being calculated in the long direction, it is convenient to write it as $M_{o\ell}$, and in the short direction as M_{os}.

It is next necessary to know what proportions of these total moments are positive and what proportions are negative. If a slab was completely fixed at the end of each panel, the division would be as it is in a fixed-end beam, two-thirds negative and one-third positive, as shown in Figure 16.9.

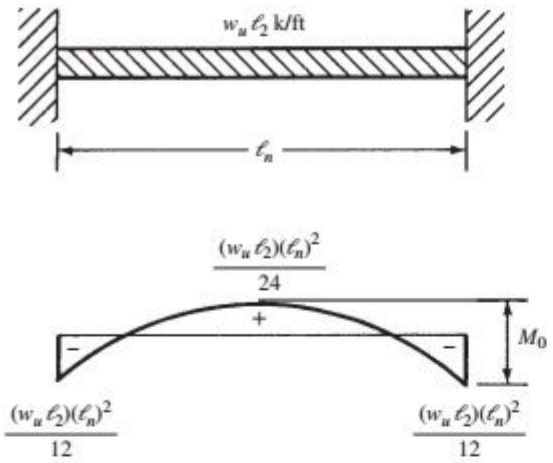


FIGURE 16.9 Distribution of positive and negative moments.

This division is reasonably accurate for interior panels where the slab is continuous for several spans in each direction with equal span lengths and loads. In effect, the rotation of the interior columns is assumed to be small, and values of $0.65M_o$ for negative moment and $0.35M_o$ for positive moment are specified by Section 8.10.4.1 of the ACI Code. For cases where the span lengths and loadings are different, the proportion of positive and negative moments may vary appreciably, and the use of a more detailed method of analysis is desirable. The equivalent frame method (Chapter 17) will provide rather good approximations for such situations.

The relative stiffnesses of the columns and slabs of exterior panels are of far greater significance in their effect on the moments than is the case for interior panels. The magnitudes of the moments are very sensitive to the amount of torsional restraint supplied at the discontinuous edges. This restraint is provided both by the flexural stiffness of the slab and by the flexural stiffness of the exterior column.

Should the stiffness of an exterior column be quite small, the end negative moment will be very close to zero. If the stiffness of the exterior column is very large, the positive and negative moments will still not be the same as those in an interior panel unless an edge beam with a very large torsional stiffness is provided that will substantially prevent rotation of the discontinuous edge of the slab.

If a 2-ft-wide beam were to be framed into a 2-ft-wide column of infinite flexural stiffness in the plane of the beam, the joint would behave as would a perfectly fixed end, and the negative beam moment would equal the fixed-end moment.

If a two-way slab 24 ft wide were to be framed into this same 2-ft-wide column of infinite stiffness, the situation of no rotation would occur along the part of the slab at the column. For the remaining 11 ft widths of slab on each side of the column, there would be rotation varying from zero at the side face of the column to maximum 11 ft on each side of the column. As a result of this rotation, the negative moment at the face of the column would be less than the fixed-end moment. Thus, the stiffness of the exterior column is reduced by the rotation of the attached transverse slab.

To take into account the fact that the rotation of the edge of the slab is different at different distances from the column, the exterior columns and slab edge beam are replaced with an equivalent column that has the same estimated flexibility as the column plus the edge beam. It can be seen that this is quite an involved process; therefore, instead of requiring a complicated analysis, Section 8.10.4.2 of the ACI Code provides a set of percentages for dividing the total factored static moment into its positive and negative parts in an end span. These divisions, which are shown in Table 16.2, include values for unrestrained edges (where the slab is simply supported on a masonry or concrete wall) and for restrained edges (where the slab is constructed integrally with a very stiff reinforced concrete wall so that little rotation occurs at the slab-to-wall connection).

In Figure 16.10, the distribution of the total factored moment for the interior and exterior spans of a flat-plate structure is shown. The plate is assumed to be constructed without beams between interior supports and without edge beams.

TABLE 16.2 Distribution of Total Span Moment in an End Span (ACI Code, Section 8.10.4.2)

	(1)	(2)	(3)	(4)	(5)
			Slab without Beams between Interior Supports		
	Exterior Edge Unrestrained	Slab with Beams between All Supports	Without Edge Beam	With Edge Beam	Exterior Edge Fully Restrained
Interior negative factored moment	0.75	0.70	0.70	0.70	0.65
Positive factored moment	0.63	0.57	0.52	0.50	0.35
Exterior negative factored moment	0	0.16	0.26	0.30	0.65

Source: Permission of American Concrete Institute.

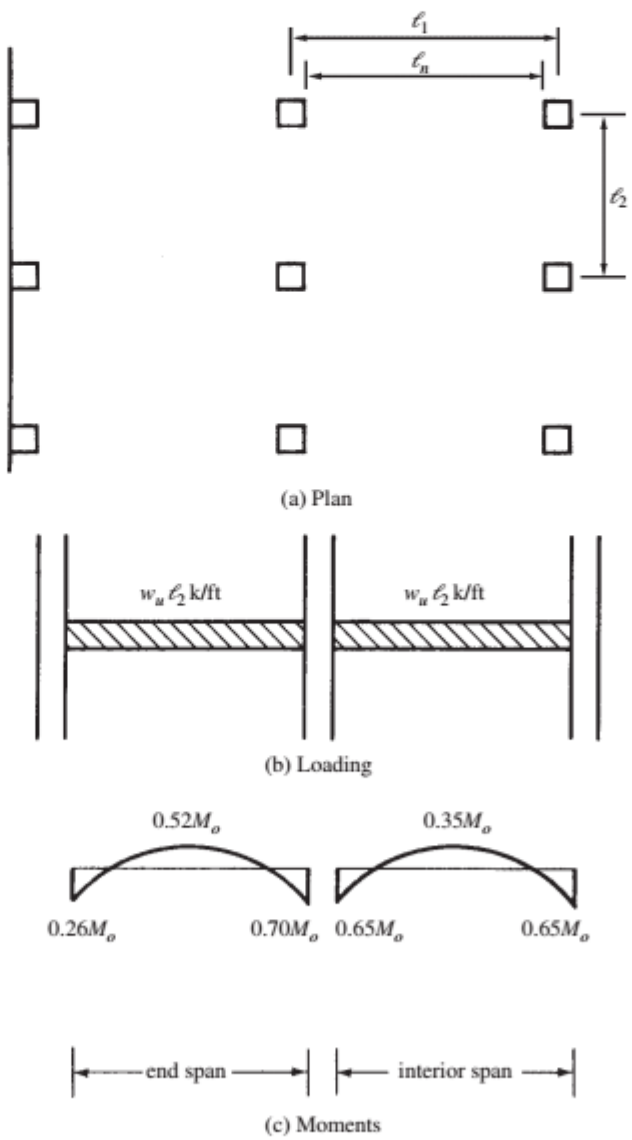


FIGURE 16.10 Sample moments for a flat plate with no edge beams.

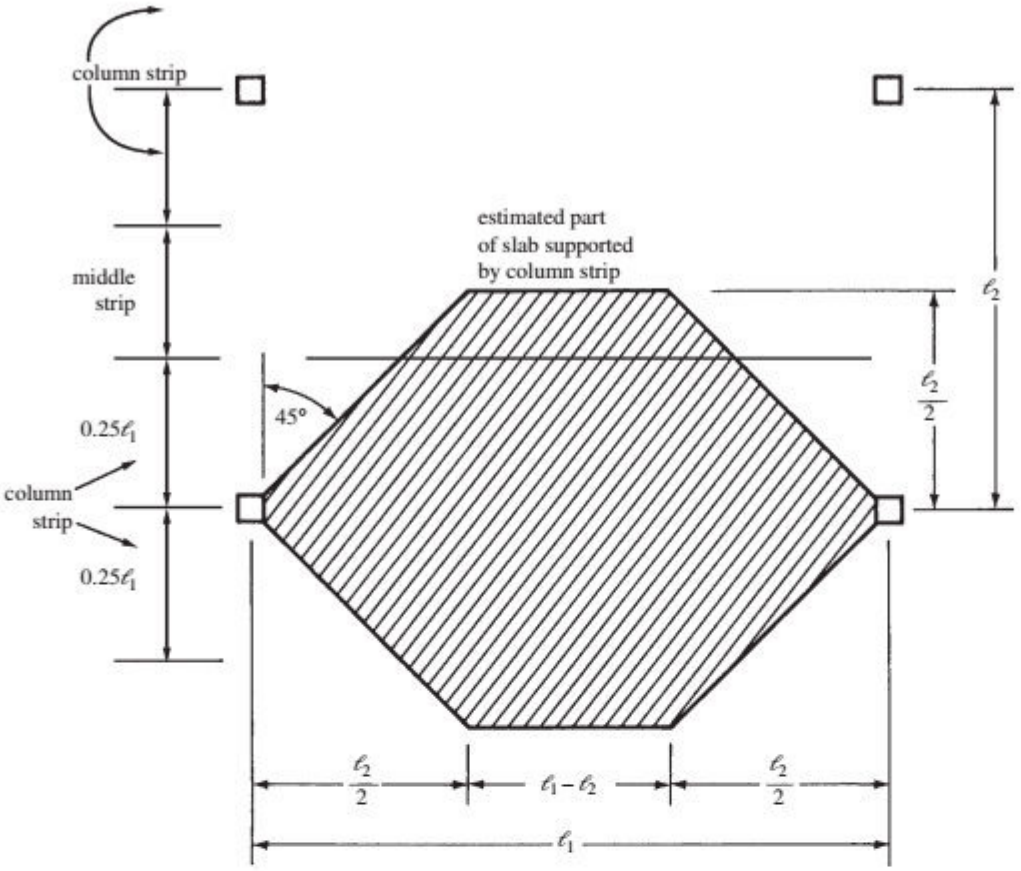


FIGURE 16.11 Tributary areas for a flat plate.

The next problem is to estimate what proportion of these moments is taken by the column strips and what proportion is taken by the middle strips. For this discussion, a flat-plate structure is assumed, and the moment resisted by the column strip is estimated by considering the tributary areas shown in Figure 16.11.

To simplify the mathematics, the load to be supported is assumed to fall within the dashed lines shown in either part (a) or part (b) of Figure 16.12. The corresponding load is placed on the simple span, and its centerline moment is determined as an estimate of the portion of the static moment taken by the column strip.[2]

In Figure 16.12(a), the load is spread uniformly over a length near the midspan of the beam, thus causing the moment to be overestimated a little, whereas in Figure 16.12(b), the load is spread uniformly from end to end, causing the moment to be underestimated. Based on these approximations, the estimated moments in the column strips for square panels will vary from $0.5M_o$ to $0.75M_o$, where M_o equals the absolute sum of the positive and average negative factored moments in each direction, that is, it equals $q_u \ell_2 \ell_n^2/8$. As ℓ_1 becomes larger than ℓ_2, the column strip takes a larger proportion of the moment. For such cases, about 60% to 70% of M_o will be resisted by the column strip.

If you sketch in the approximate deflected shape of a panel, you will see that a larger portion of the positive moment is carried by the middle strip than by the column strip, and vice versa for the negative moments. As a result, about 60% of the positive M_o and about 70% of the negative M_o are expected to be resisted by the column strip.[3]

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[2] White, R. N., Gergely, P., and Sexsmith, R. G., 1974, *Structural Engineering*, Vol. 3: *Behavior of Members and Systems* (Hoboken, NJ: John Wiley & Sons), pp. 456–461.
[3] Ibid.

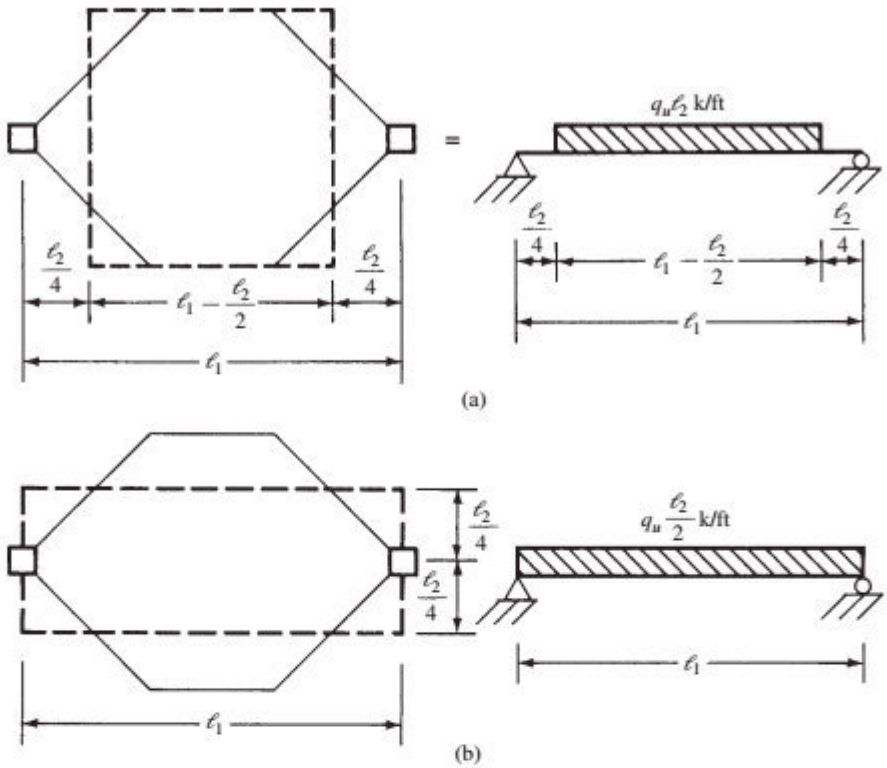


FIGURE 16.12 Load distribution.

Section 8.10.5.1 of the ACI Code states that the column strip shall be proportioned to resist the percentages of the total interior negative design moment given in Table 16.3.

In the table, α_1 is again the ratio of the stiffness of a beam section to the stiffness of a width of slab bounded laterally by the centerline of the adjacent panel, if any, on each side of the beam and equals $E_{cb}I_b/E_{cs}I_s$.

Section 8.10.5.2 of the ACI Code states that the column strip is to be assumed to resist percentages of the exterior negative design moment, as given in Table 16.4. In this table, β_t is the ratio of the torsional stiffness of an edge beam section to the flexural stiffness of a width of slab equal to the span length of the beam center to center of supports ($\beta_t = E_{cb}C/2E_{cs}I_s$). The computation of the cross-sectional constant C is described in Section 16.11 of this chapter.

The column strip (Section 8.10.5.5 of the ACI Code) is to be proportioned to resist the portion of the positive moments given in Table 16.5.

TABLE 16.3 Percentages of Interior Negative Design Moments to Be Resisted by Column Strip

$\frac{\ell_2}{\ell_1}$	0.5	1.0	2.0
$\frac{\alpha_{f1}\ell_2}{\ell_1} = 0$	75	75	75
$\frac{\alpha_{f1}\ell_2}{\ell_1} \geq 1.0$	90	75	45

TABLE 16.4 Percentages of Exterior Negative Design Moment to Be Resisted by Column Strip

$\frac{\ell_2}{\ell_1}$		0.5	1.0	2.0
$\frac{\alpha_{f1}\ell_2}{\ell_1} = 0$	$\beta_t = 0$	100	100	100
	$\beta_t \geq 2.5$	75	75	75
$\frac{\alpha_{f1}\ell_2}{\ell_1} \geq 1.0$	$\beta_t = 0$	100	100	100
	$\beta_t \geq 2.5$	90	75	75

TABLE 16.5 Percentages of Positive Design Moment to Be Resisted by Column Strip

$\frac{\ell_2}{\ell_1}$	0.5	1.0	2.0
$\frac{\alpha_{f1}\ell_2}{\ell_1} = 0$	60	60	60
$\frac{\alpha_{f1}\ell_2}{\ell_1} \geq 1.0$	90	75	45

Equations can be used instead of the two-way interpolation sometimes required by Tables 16.3 to 16.5. Instead of Table 16.3, the percentage of interior negative moment to be resisted by the column strip ($\%^-_{\text{int col}}$) can be determined by

$$\%^-_{\text{int col}} = 75 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right)$$

The percentage of exterior negative design moment resisted by the column strip ($\%^-_{\text{ext col}}$) given in Table 16.4 can be found by

$$\%^-_{\text{ext col}} = 100 - 10\beta_t + 12\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right)$$

Finally, for positive design moment in either an interior span or an exterior span (Table 16.5), the percentage resisted by the column strip ($\%^+$) is

$$\%^+ = 60 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1.5 - \frac{\ell_2}{\ell_1}\right)$$

In the preceding three equations, if $\beta_t > 2.5$, use 2.5, and if $\alpha_{f1}\ell_2/\ell_1 > 1$, use 1.

Section 8.10.5.7.1 of the ACI Code requires that the beam be allotted 85% of the column strip moment if $\alpha_{f1}(\ell_2/\ell_1) \geq 1.0$. Should $\alpha_{f1}(\ell_2/\ell_1)$ be between 1.0 and 0, the moment allotted to the beam is determined by linear interpolation from 85% to 0%. The part of the moment not given to the beam is allotted to the slab in the column strip.

Finally, Section 8.10.6.1 of the ACI Code requires that the portion of the design moments not resisted by the column strips, as previously described, is to be allotted to the corresponding half middle strip. The middle strip will be designed to resist the total of the moments assigned to its two half middle strips.

16.9 Design of an Interior Flat Plate

In this section, an interior flat plate is designed by the direct design method. The procedure specified in Chapter 8 of the ACI Code is applicable not only to flat plates but also to flat slabs, waffle slabs, and two-way slabs with beams. The steps necessary to perform the designs are briefly summarized at the end of this paragraph. The order of the steps may have to be varied somewhat for different types of slab designs. Either the direct design method or the equivalent frame method may be used to determine the design moments. The design steps are as follows:

1. Estimate the slab thickness to meet the code requirements.
2. Determine the depth required for shear.
3. Calculate the total static moments to be resisted in the two directions.
4. Estimate the percentages of the static moments that are positive and negative and proportion the resulting values between the column and middle strips.
5. Select the reinforcement.

Example 16.3 illustrates this method of design applied to a flat plate.

Example 16.3

Design an interior flat plate for the structure considered in Example 16.1. This plate is shown in Figure 16.13. Assume a service live load equal to 80 psf, a service dead load equal to 110 psf (including slab weight), $f_y = 60{,}000$ psi, $f'_c = 3000$ psi, normal-weight concrete, and column heights of 12 ft.

SOLUTION

Estimate Slab Thickness

When shear is checked, the 7-in. slab estimated in Example 16.1 is not quite sufficient. One alternative is to increase f'_c from 3000 psi, which is a fairly low strength. However, we will increase the slab thickness. The calculations for the 7-in.-thick slab are the same as those that follow for the 7.5-in. slab thickness with the exception of the slab thickness change.

$\therefore$Try $7\frac{1}{2}$-in. slab

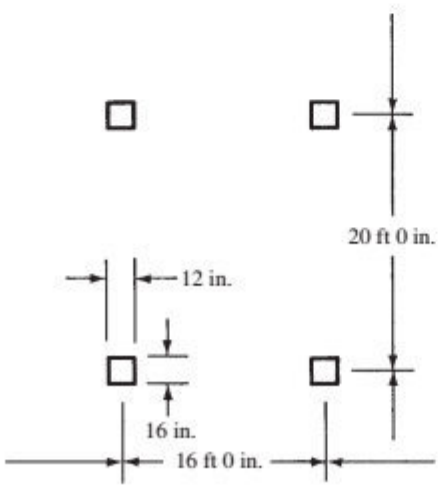


FIGURE 16.13 Interior panel of flat-plate structure of Example 16.1.

Determine Depth Required for Shear

Using d for shear equal to the estimated average of the d values in the two directions, we obtain

$$d = 7.50 \text{ in.} - \frac{3 \text{ in.}}{4} \text{ cover} - 0.50 \text{ in.} = 6.25 \text{ in.}$$

$$q_u = (1.2)\,(110 \text{ psf}) + (1.6)\,(80 \text{ psf}) = 260 \text{ psf}$$

Checking One-Way or Beam Shear (Seldom Controls in Two-Way Floor Systems)

Using the dimensions shown in Figure 16.14, we obtain

$$V_{u1} = (8.81 \text{ ft})\,(260 \text{ psf}) = 2291 \text{ lb for a 12 in. width}$$

$$\phi V_c = \phi 2\lambda\sqrt{f'_c}bd$$

$$= (0.75)\,(1.0)\,(2\sqrt{3000} \text{ psi})\,(12 \text{ in.})\,(6.25 \text{ in.})$$

$$= 6162 \text{ lb} > 2291 \text{ lb} \quad \underline{\text{OK}}$$

Checking Two-Way or Punching Shear around the Column

$$b_o = (2)\,(16 \text{ in.} + 6.25 \text{ in.}) + (2)\,(12 \text{ in.} + 6.25 \text{ in.}) = 81 \text{ in.}$$

$$V_{u2} = \left[(20 \text{ ft})\,(16 \text{ ft}) - \left(\frac{16 \text{ in.} + 6.25 \text{ in.}}{12 \text{ in./ft}}\right)\left(\frac{12 \text{ in.} + 6.25 \text{ in.}}{12 \text{ in./ft}}\right)\right] (0.260 \text{ ksf})$$

$$= 82.47 \text{ k} = 82{,}470 \text{ lb}$$

$$\phi V_c = (0.75)\,(1.0)\,(4\sqrt{3000} \text{ psi})\,(81 \text{ in.})\,(6.25 \text{ in.})$$

$$= 83{,}185 \text{ lb} > 82{,}470 \text{ lb} \quad \underline{\text{OK}}$$

$$\underline{\text{Use } h = 7\tfrac{1}{2} \text{ in.}}$$

Calculate Static Moments in the Long and Short Directions

$$M_{o\ell} = \frac{q_u \ell_2 \ell_n^2}{8} = \frac{(0.260 \text{ ksf})\,(16 \text{ ft})\left(20 \text{ ft} - \frac{16}{12} \text{ ft}\right)^2}{8} = 181.2 \text{ ft-k} \qquad \text{(ACI Equation 8.10.3.2)}$$

$$M_{os} = \frac{q_u \ell_1 \ell_n^2}{8} = \frac{(0.260 \text{ ksf})\,(20 \text{ ft})\left(16 \text{ ft} - \frac{12}{12} \text{ ft}\right)^2}{8} = 146.2 \text{ ft-k}$$

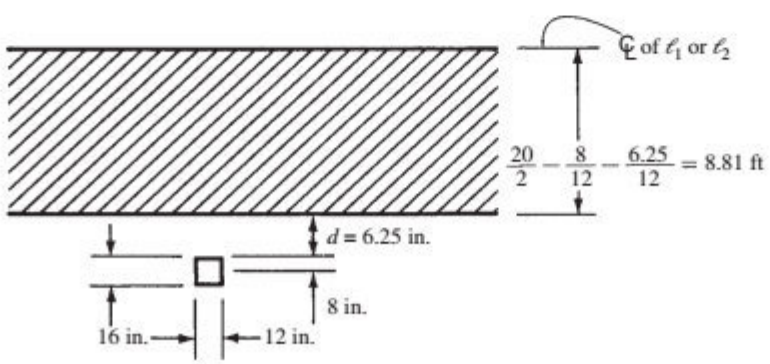


FIGURE 16.14 Dimensions for Example 16.3.

TABLE 16.6 Summary of Moments and Steel Selections for Example 16.3

	Long Span (estimate $d = 6.50$ in.)				Short Span (estimate $d = 6.00$ in.)			
	Column Strip (8 ft)		Middle Strip (8 ft)		Column Strip (8 ft)		Middle Strip (12 ft)	
	–	+	–	+	–	+	–	+
M_u	(0.65)(0.75) (181.2 ft-k) = –88.4 ft-k	(0.35)(0.60) (181.2 ft-k) = +38.1 ft-k	(0.65)(181.2 ft-k) –88.4 ft-k = –29.4 ft-k	(0.35)(181.2 ft-k) –38.1 ft-k = +25.3 ft-k	(0.65)(0.75) (146.2 ft-k) = –71.3 ft-k	(0.35)(0.60) (146.2 ft-k) = +30.7 ft-k	(0.65)(146.2 ft-k) –71.3 ft-k = –23.8 ft-k	(0.35)(146.2 ft-k) –30.7 ft-k = +20.5 ft-k
$\frac{M_u}{\phi bd^2}$	290.6 psi	125.2 psi	97.6 psi	83.2 psi	275.1 psi	118.4 psi	61.2 psi	52.7 psi
ρ*	0.00516bd	0.00214bd	0.0018bh	0.0018bh	0.00486bd	0.00202bd	0.0018bh	0.0018bh
A_s	3.22 in.^{2}	13.4 in.^{2}	1.30 in.^{2}	1.30 in.^{2}	2.80 in.^{2}	1.16 in.^{2}	1.94 in.^{2}	1.94 in.^{2}
Bars selected	17 #4	7 #4	7 #4	7 #4	15 #4	6 #4	10 #4	10 #4

*Values may not be less than the temperature and shrinkage minimum 0.0018bh.

Proportion the Static Moments to the Column and Middle Strips and Select the Reinforcing Bars

The static moment is divided into positive and negative moments in accordance with Section 8.10.4 of the ACI Code. Because this is an interior span, 65% of the total static moment is negative and 35% is positive. See also the right side of Figure 16.10(c). If this example were an end span, then the total static moment would be divided into positive and negative values in accordance with Table 16.2.

The next step is to divide the moments determined in the previous paragraph into column and middle strips. Again, because this is an interior span, Table 16.3 applies. Because $\alpha_{f1} = 0$ (no interior beams), 75% of the moment goes to the column strip. This value is independent of ℓ_2/ℓ_1 for the case where there are no interior beams. The remaining 25% of the moment is assigned to the middle strip, half on each side of the column strip. Table 16.5 is used to determine how much of the total positive moment is assigned to the column strip. In this case, because $\alpha_{f1} = 0$, 60% goes to the column strip, and the remaining 40% is assigned to the middle strip, half on each side of the column strip.

These calculations can be conveniently arranged, as in Table 16.6. This table is very similar to the one used for the design of the continuous one-way slab in Chapter 14. To assist in the interpretation of Table 16.6, the numbers in the first column will be discussed. The first is the determination of M_u. This calculation uses the 0.65 factor from ACI 8.10.4 and the 0.75 factor from Table 16.3, both applied to the total static moment of 181.2 ft-k. Dividing this value of $M_u = -88.4$ ft-k by ϕbd^2 ($\phi = 0.9$, $b = 8$ ft $= 96$ in., $d = 6.5$ in.) results in $M_u/\phi bd^2 = 290.6$ psi. From Appendix A, Table A.12, $\rho = 0.00516$ (by interpolation). *The area of reinforcing steel in the column strip is* $A_s = \rho bd = 0.00516(96 \text{ in.})(6.5 \text{ in.}) = 3.22 \text{ in.}^2$. A selection of 17 #4 bars is performed, having a total $A_s = 3.34$ in.^{2}. The remaining entries in Table 16.6 follow a similar procedure.

As the different percentages of moments are selected from the tables for the column and middle strips of this slab, it will be noted that $\alpha_f = 0$.

In the solution to Example 16.3, it will be noted that the $M_u/\phi bd^2$ values are sometimes quite small, and thus most of the ρ values do not fall within Table A.12 (see Appendix A). For such cases, the authors use the temperature and shrinkage minimum 0.0018bh.

Actually, the temperature reinforcement includes bars in the top and bottom of the slab. In the negative moment region, some of the positive steel bars have been extended into the support region and are also available for temperature and shrinkage steel. If desirable, these positive bars can be lapped instead of being stopped in the support.

The selection of the reinforcing bars is the final step taken in the design of this flat plate. The ACI Figure 8.7.4.1.3a (given as Figure 16.15 here) shows the minimum lengths of slab reinforcing bars for

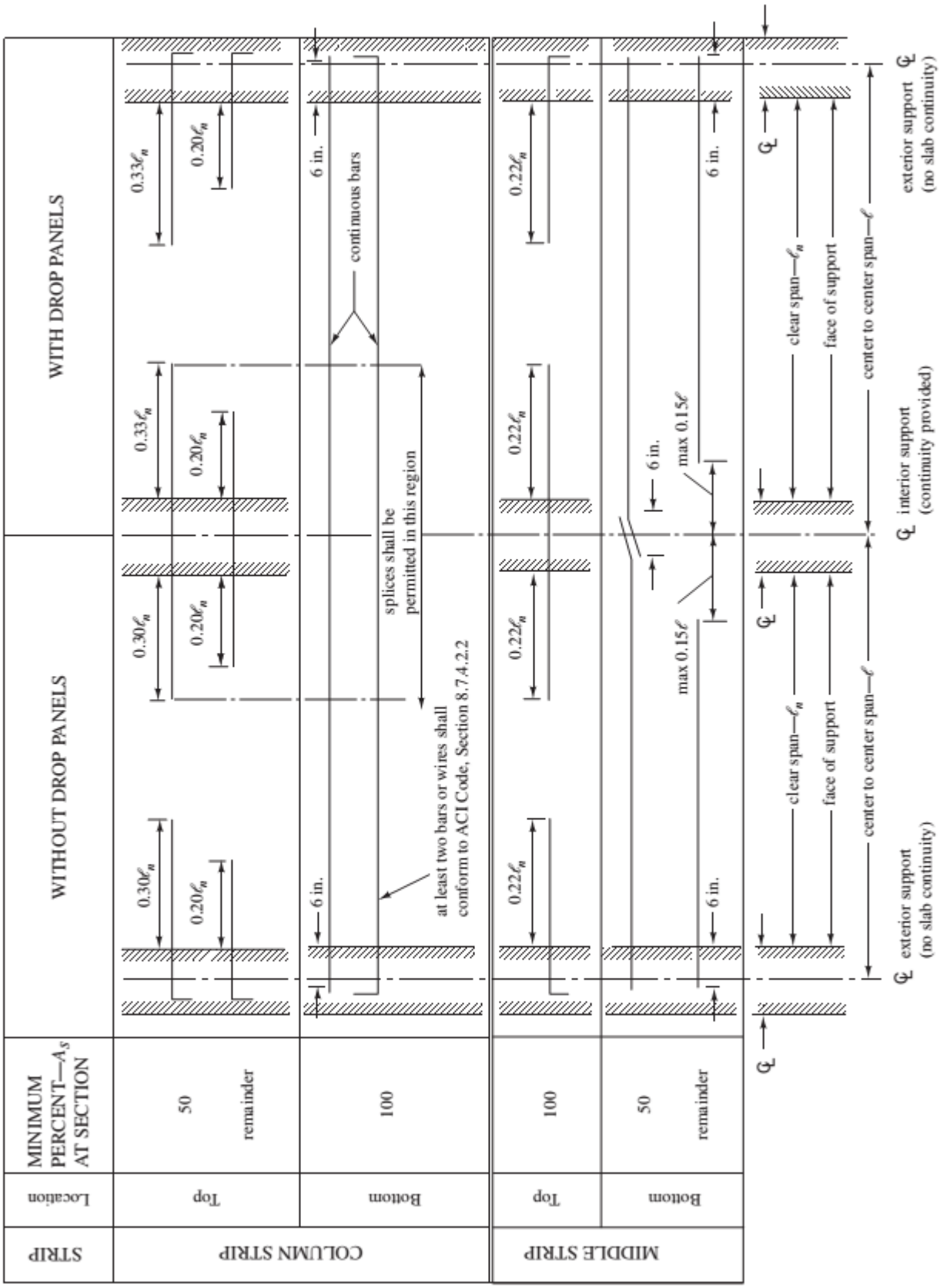


FIGURE 16.15 Minimum extensions for reinforcement in slabs without beams (see Section 9.7.3.8 of the ACI Code for reinforcement extension into supports).

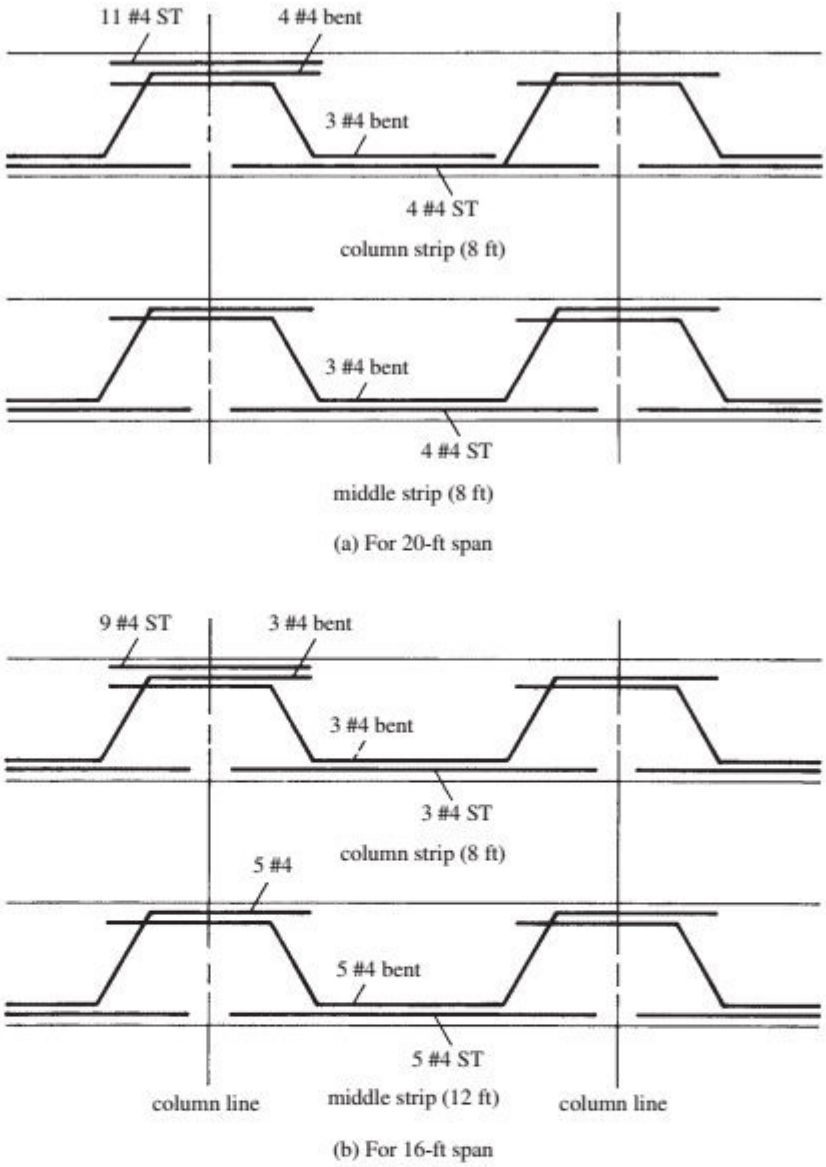


FIGURE 16.16 Bar details.

flat plates and for flat slabs with drop panels. This figure shows that some of the positive reinforcement must be run into the support area.

The bars selected for this flat plate are shown in Figure 16.16. Bent bars are used in this example, but straight bars could have been used just as well. There seems to be a trend among designers in the direction of using more straight bars in slabs and fewer bent bars.

16.10 Placing of Live Loads

The moments in a continuous floor slab are appreciably affected by different positions or patterns of the live loads. The usual procedure, however, is to calculate the total static moments, assuming that all panels are subjected to full live load. When different loading patterns are used, the moments can be changed so much that overstressing may occur in the slab.

Section 6.4.3.2 of the ACI Code states that if a variable unfactored live load does not exceed three-fourths of the unfactored dead load, or if it is of a type such that all panels will be loaded simultaneously, it is permissible to assume that full live load placed over the entire area will cause maximum moment values throughout the entire slab system.

For other loading conditions, it may be assumed (according to ACI Code, Section 6.4.3.3) that maximum positive moment at the midspan of a panel will occur when three-fourths of the full factored live load is placed on the panel in question and on alternate spans. It may further be assumed that the maximum negative moment at a support will occur when three-fourths of the full factored live load is placed only on the adjacent spans.

The ACI Code permits the use of the three-fourths factor because the absolute maximum positive and negative moments cannot occur simultaneously under a single loading condition and also because some redistribution of moments is possible before failure will occur. Although some local overstress may be the result of this procedure, it is felt that the ultimate capacity of the system after redistribution will be sufficient to resist the full factored dead and live loads in every panel.

The moment determined as described in the last paragraph may not be less than moments obtained when full factored live loads are placed in every panel (ACI Code, Section 6.4.3).

16.11 Analysis of Two-Way Slabs with Beams

In this section, the moments are determined by the direct design method for an exterior panel of a two-way slab with beams. The example problem presented in this section is about as complex as any that may arise in flat plates, flat slabs, or two-way slabs with beams, using the direct design method.

The requirements of the code are so lengthy and complex that in Example 16.4, which follows, the steps and appropriate code sections are spelled out in detail. The practicing designer should obtain a copy of the *CRSI Design Handbook*, because the tables therein will be of tremendous help in slab design.

Example 16.4

Determine the negative and positive moments required for the design of the exterior panel of the two-way slab with beam structure shown in Figure 16.17. The slab is to support a live load of 120 psf and a dead load of 100 psf, including the slab weight. The columns are 15 in. × 15 in. and 12 ft long. The slab is supported by beams along the column line with the cross section shown. Determine the slab thickness and check the shear stress if $f'_c = 3000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

1. Check ACI Code limitations (Section 8.10.2). These conditions, which are discussed in Section 16.7 of this text, are met. The first five of these criteria are easily satisfied by inspection. The sixth requires calculations that follow.
2. Minimum thickness as required by Section 8.3.1.2 of the ACI Code
   - **(a)** Assume $h = 6$ in.
   - **(b)** Effective flange projection of column line beam as specified by Section 8.4.1.8 of the ACI Code

$$= 4h_f = (4)\,(6 \text{ in.}) = 24 \text{ in.} \quad \text{or} \quad h - h_f = 20 \text{ in.} - 6 \text{ in.} = \underline{\underline{14 \text{ in.}}}$$

   - **(c)** Gross moments of inertia of T beams. The following values are the gross moments of inertia of the edge and interior beams computed, respectively, about their centroidal axes. Many designers use approximate values for these moments of inertia, I_s, with almost identical results for slab thicknesses. One common practice is to use two times the gross moment of inertia of the stem

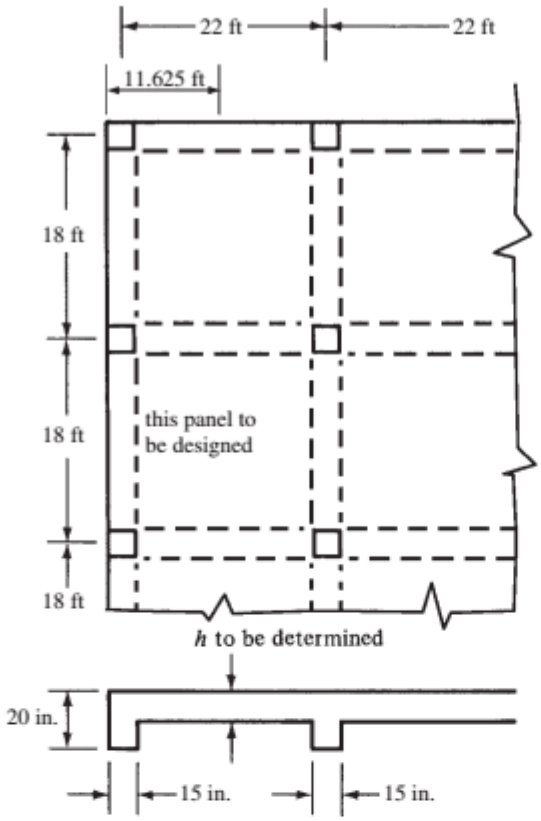


FIGURE 16.17 Two-way slab for Example 16.4.

(using a depth of stem running from top of slab to bottom of stem) for interior beams and one and a half times the stem gross moment of inertia for edge beams.

$$I \text{ for edge beams} = 13{,}468 \text{ in.}^4$$
$$I \text{ for interior beams} = 15{,}781 \text{ in.}^4$$

(d) Calculating α values (where α is the ratio of the stiffness of the beam section to the stiffness of a width of slab bounded laterally by the centerline of the adjacent panel, if any, on each side of the beam).

For edge beam $\left(\text{width} = \frac{1}{2} \times 22 \text{ ft} + \frac{7.5 \text{ in.}}{12 \text{ in./ft}} = 11.625 \text{ ft}\right)$

$$I_s = \left(\frac{1}{12}\right)(12 \text{ in./ft} \times 11.625 \text{ ft})\,(6 \text{ in.})^3 = 2511 \text{ in.}^4$$

$$\alpha_f = \frac{13{,}468 \text{ in.}^4}{2511 \text{ in.}^4} = 5.36$$

For 18-ft interior beam (with 22-ft slab width)

$$I_s = \left(\frac{1}{12}\right)(12 \text{ in./ft} \times 22 \text{ ft})\,(6 \text{ in.})^3 = 4752 \text{ in.}^4$$

$$\alpha_f = \frac{15{,}781 \text{ in.}^4}{4752 \text{ in.}^4} = 3.32$$

For 22-ft interior beam (with 18-ft slab width)

$$I_s = \left(\frac{1}{12}\right)(12 \text{ in./ft} \times 18 \text{ ft})\,(6 \text{ in.})^3 = 3888 \text{ in.}^4$$

$$\alpha_f = \frac{15{,}781 \text{ in.}^4}{3888 \text{ in.}^4} = 4.06$$

$$\text{Avg. } \alpha_f = \alpha_{fm} = \frac{5.36 + 3.32 + (2)\,(4.06)}{4} = 4.20$$

$$\beta = \text{ratio of long to short clear span} = \frac{22 \text{ ft} - \dfrac{15 \text{ in.}}{12 \text{ in./ft}}}{18 \text{ ft} - \dfrac{15 \text{ in.}}{12 \text{ in./ft}}} = 1.24$$

(e) Now that we have determined the values of α_f in the two perpendicular directions, the sixth and final limitation for use of the direct design method (ACI Code, 8.10.2.7) can be checked.

$$\frac{\alpha_{f1}\ell_2^2}{\alpha_{f2}\ell_1^2} = \frac{4.06(18 \text{ ft})^2}{3.32(22 \text{ ft})^2} = 0.818$$

Because this value is between 0.2 and 5.0, this condition is satisfied. Note that the directions that are designated as ℓ_1 and ℓ_2 are arbitrary. Had the short direction been used as ℓ_1, the preceding calculation would simply have been inverted, and the ratio would have been 1.22 instead of 0.818. This value would also have been between the limits of 0.2 and 5.0.

(f) Thickness limits by ACI Code, Section 8.3.1.2

$$\ell_n = 22.0 \text{ ft} - \frac{15 \text{ in.}}{12 \text{ in./ft}} = 20.75 \text{ ft}$$

As $\alpha_{fm} > 2.0$, use ACI Equation 8.3.1.2d

$$h = \frac{\ell_n\left(0.8 + \dfrac{f_y}{200{,}000}\right)}{36 + 9\beta}$$

$$= \frac{(12 \text{ in./ft})\,(20.75 \text{ ft})\left(0.8 + \dfrac{60{,}000 \text{ psi}}{200{,}000 \text{ psi}}\right)}{36 + (9)\,(1.24)} = 5.81 \text{ in.} \leftarrow$$

h not less than 3.5 in. as per ACI Code, Section 8.3.1.2e

Try $h = 6$ in. (shear checked later)

3. Moments for the short-span direction centered on interior column line

$$q_u = (1.2)\,(100 \text{ psf}) + (1.6)\,(120 \text{ psf}) = 312 \text{ psf}$$

$$M_o = \frac{(q_u\ell_2)\,(\ell_n)^2}{8} = \frac{(0.312 \text{ ksf})\,(22 \text{ ft})\,(16.75 \text{ ft})^2}{8} = 241 \text{ ft-k}$$

(a) Dividing this static design moment into negative and positive portions, per Section 8.10.4 of the ACI Code

$$\text{Negative design moment} = (0.65)\,(241 \text{ ft-k}) = -157 \text{ ft-k}$$

$$\text{Positive design moment} = (0.35)\,(241 \text{ ft-k}) = +84 \text{ ft-k}$$

(b) Allotting these interior moments to beam and column strips, per Section 8.10.5 of the ACI Code

$$\frac{\ell_2}{\ell_1} = \frac{22 \text{ ft}}{18 \text{ ft}} = 1.22$$

$$\alpha_{f1} = \alpha_f \text{ in direction of short span} = 3.32$$

$$\alpha_{f1}\frac{\ell_2}{\ell_1} = (3.32)\,(1.22) = 4.05$$

The portion of the interior negative moment to be resisted by the column strip, per Table 16.3 of this chapter, by interpolation is (0.68) (−157) = −107 ft-k. This result can also be obtained from the equation

$$\%^{-}_{\text{int col}} = 75 + 30\left(\frac{\alpha_{f1}\ell_2}{\ell_1}\right)\left(1 - \frac{\ell_2}{\ell_1}\right) = 75 + 30(1)\left(1 - \frac{22 \text{ ft}}{18 \text{ ft}}\right) = 68.3\%$$

$$M^{-}_{\text{int col}} = 0.683(157 \text{ ft-k}) = 107 \text{ ft-k}$$

Note that because $\alpha_{f1}\ell_2/\ell_1 = 4.05 > 1$, a value of 1 was used in the preceding equation.

This −107 ft-k is allotted 85% to the beam (ACI Table 8.10.5.7.1), or −91 ft-k, and 15% to the slab, or −16 ft-k. The remaining negative moment, 157 ft-k − 107 ft-k = 50 ft-k, is allotted to the middle strip.

The portion of the interior positive moment to be resisted by the column strip, per Table 16.5 of this chapter, by interpolation is (0.68) (+84 ft-k) = +57 ft-k. The 68% value can also be obtained from the equation for Table 16.5. This 57 ft-k is allotted 85% to the beam (ACI Code, Section 8.10.5.7.1), or +48 ft-k, and 15% to the slab, or +9 ft-k. The remaining positive moment, 84 ft-k − 57 ft-k = 27 ft-k, goes to the middle strip.

4. Moments for the short-span direction centered on the edge beam

$$M_o = \frac{(q_u\ell_2)\,(\ell_n)^2}{8} = \frac{(0.312 \text{ ksf})\,(11.625 \text{ ft})\,(16.75 \text{ ft})^2}{8} = 127 \text{ ft-k}$$

(a) Dividing this static design moment into negative and positive portions, per Section 8.10.4 of the ACI Code

$$\text{Negative design moment} = (0.65)\,(127 \text{ ft-k}) = -83 \text{ ft-k}$$

$$\text{Positive design moment} = (0.35)\,(127 \text{ ft-k}) = +44 \text{ ft-k}$$

(b) Allotting these exterior moments to beam and column strips, per Section 8.10.5.2 of the ACI Code

$$\frac{\ell_2}{\ell_1} = \frac{22 \text{ ft}}{18 \text{ ft}} = 1.22$$

$$\alpha_{f1} = \alpha_{f1} \text{ for edge beam} = 5.36$$

$$\alpha_{f1}\frac{\ell_2}{\ell_1} = (5.36)\,(1.22) = 6.54$$

5. The portion of the exterior negative moment going to the column strip, from Table 16.4 of this chapter, by interpolation is (0.68) (−83 ft-k) = −56 ft-k. This −56 ft-k is allotted 85% to the beam (ACI Code, Section 8.10.5.7.1), or −48 ft-k, and 15% to the slab, or −8 ft-k. The remaining negative moment, 83 ft-k − 56 ft-k = −27 ft-k, is allotted to the middle strip.

TABLE 16.7 Short-Span Moments (ft-k)

	Column Strip Moments		Middle Strip Slab Moments
	Beam	Slab	
Interior slab-beam strip			
Negative	−91	−16	−50
Positive	+48	+9	+30
Exterior slab-beam strip			
Negative	−48	−9	−27
Positive	+26	+5	+16

The portion of the exterior positive moment to be resisted by the column strip, per Table 16.5 or the equation for Table 16.5, is (0.68) (+44 ft-k) = +30 ft-k. This 30 ft-k is allotted 85% to the beam, or +26 ft-k, and 15% to the slab, or +5 ft-k. The remaining positive moment, 44 ft-k – 30 ft-k = +14 ft-k, goes to the middle strip.

A summary of the short-span moments is presented in Table 16.7.

6. Moments for the long-span direction

$$M_o = \frac{(q_u \ell_2)(\ell_n)^2}{8} = \frac{(0.312 \text{ ksf})(18 \text{ ft})(20.75 \text{ ft})^2}{8} = 302.3 \text{ ft-k}$$

(a) From Table 16.2 of this chapter (ACI Code, Section 8.10.4.2) for an end span with beams between all interior supports:

$$\text{Interior negative factored moment} = 0.70M_o = -(0.70)(302.3 \text{ ft-k}) = -212 \text{ ft-k}$$

$$\text{Positive factored moment} = 0.57M_o = +(0.57)(302.3 \text{ ft-k}) = 172 \text{ ft-k}$$

$$\text{Exterior negative factored moment} = 0.16M_o = -(0.16)(302.3 \text{ ft-k}) = -48 \text{ ft-k}$$

These factored moments may be modified by 10%, according to Section 8.10.4.3 of the ACI Code, but this reduction is neglected here.

(b) Allotting these moments to beam and column strips

$$\frac{\ell_2}{\ell_1} = \frac{18 \text{ ft}}{22 \text{ ft}} = 0.818$$

$$\alpha_{f1} = \alpha_{f1} \text{ (for the 22-ft beam)} = 4.06$$

$$\alpha_{f1}\frac{\ell_2}{\ell_1} = (4.08)(0.818) = 3.32$$

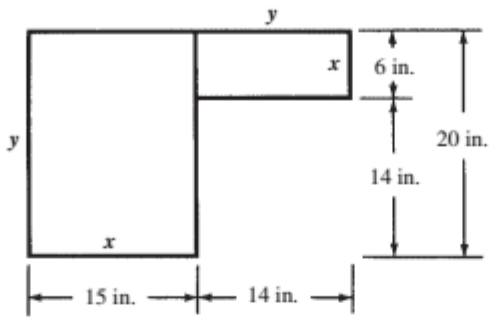
Next an expression is given for β_t. It is the ratio of the torsional stiffness of an edge beam section to the flexural stiffness of a width of slab equal to the span length of the beam measured center to center of supports.

$$\beta_t = \frac{E_{cb}C}{2E_{cs}I_s}$$

Involved in the equation is a term C, which is a property of the cross-sectional area of the torsion arm estimating the resistance to twist.

$$C = \Sigma\left(1 - 0.63\frac{x}{y}\right)\frac{x^3 y}{3}$$

where x is the length of the short side of each rectangle and y is the length of the long side of each rectangle. The exterior beam considered here is described in Section 8.4.1.8 of the ACI Code and is



$$C = \left[1 - 0.63\left(\frac{15}{20}\right)\right]\frac{(15^3)(20)}{3} + \left[1 - 0.63\left(\frac{6}{14}\right)\right]\frac{(6^3)(14)}{3} = 12{,}605 \text{ in.}^4$$

FIGURE 16.18 Evaluation of C.

shown in Figure 16.18, together with the calculation of C. The beam cross section could be divided into rectangles in other ways, but the configuration shown results in the greatest value for C.

$$\beta_t = \frac{E_{cb}C}{2E_{cs}I_s} = \frac{(E_c)\,(12{,}605 \text{ in.}^4)}{(2)\,(E_c)\,(3888 \text{ in.}^4)} = 1.62$$

The portion of the interior negative design moment allotted to the column strip, from Table 16.3, by interpolation or by equation is (0.80)(−212 ft-k) = −170 ft-k. This −170 ft-k is allotted 85% to the beam (ACI Code, Section 8.10.5.7.1), or −145 ft-k, and 15% to the slab, or −26 ft-k. The remaining negative moment, −212 ft-k + 170 ft-k = −42 ft-k, is allotted to the middle strip.

The portion of the positive design moment to be resisted by the column strip, per Table 16.5, is (0.80) (172 ft-k) = +138 ft-k. This 138 ft-k is allotted 85% to the beam, or (0.85) (138 ft-k) = 177 ft-k, and 15% to the slab, or +21 ft-k. The remaining positive moment, 172 ft-k − 138 ft-k = 34 ft-k, goes to the middle strip.

The portion of the exterior negative moment allotted to the column strip is obtained by double interpolation from Table 16.4 and is (0.86) (−48 ft-k) = −42 ft-k. This −42 ft-k is allotted 85% to the beam, or −36 ft-k, and 15% to the slab, or −6 ft-k. The remaining negative moment, −6 ft-k, is allotted to the middle strip.

A summary of the long-span moments is presented in Table 16.8.

7. Check shear strength in the slab at a distance d from the face of the beam. Shear is assumed to be produced by the load on the tributary area shown in Figure 16.19, working with a 12-in.-wide strip as shown.

$$\text{average } d = h - \text{cover} - \text{one bar diam.} = 6.00 \text{ in.} - \frac{3}{4} \text{ in.} - \frac{1}{2} \text{ in.} = 4.75 \text{ in.}$$

$$V_u = (0.312 \text{ ksf})\left(9 \text{ ft} - \frac{7.5 \text{ in.}}{12 \text{ in./ft}} - \frac{4.75 \text{ in.}}{12 \text{ in./ft}}\right) = 2.49 \text{ k} = 2490 \text{ lb}$$

$$\phi V_c = (0.75)\,(1.0)\,(2\sqrt{3000} \text{ psi})\,(12 \text{ in.})\,(4.75 \text{ in.}) = 4684 \text{ lb} > 2490 \text{ lb} \quad \underline{\text{OK}}$$

TABLE 16.8 Long-Span Moments (ft-k)

	Column Strip Moments		Middle Strip Slab Moments
	Beam	Slab	
Interior negative	−145	−26	−42
Positive	+117	+21	+34
Exterior negative	−36	−6	−6

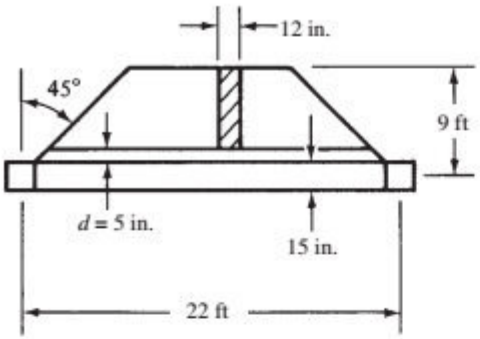


FIGURE 16.19 Load distribution to beams.

16.12 Transfer of Moments and Shears between Slabs and Columns

On many occasions, the maximum load that a two-way slab can support is dependent upon the strength of the joint between the column and the slab. Not only is the load transferred by shear from the slab to the column along an area around the column, but also there may be moments that have to be transferred. The moment situation is usually most critical at the exterior columns.

If there are moments to be transferred, they will cause shear stresses of their own in the slabs, as will be described in this section. Furthermore, shear forces resulting from moment transfer must be considered in the design of the lateral column reinforcement (i.e., ties and spirals), as stated in Section 15.2.2 of the ACI Code.

When columns are supporting slabs without beams (i.e., flat plates or flat slabs), the load transfer situation between the slabs and columns is extremely *critical*. If we don't have the exact areas and positions of the flexural reinforcement designed just right throughout the slab, inelastic redistribution of moments (ACI Code, Section 8.10.4.3) may still allow the system to perform adequately; however, if we handle the shear strength situation incorrectly, the results may very well be disastrous.

The serious nature of this problem is shown in Figure 16.20, where it can be seen that if there is no spandrel beam, all of the total exterior slab moment has to be transferred to the column. The transfer is made by both flexure and eccentric shear, the latter being located at a distance of about $d/2$ from the column face.

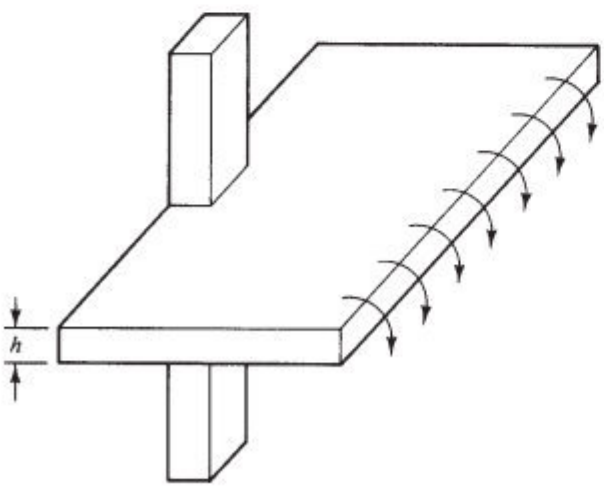


FIGURE 16.20 Illustration showing need for transfer of moments to columns.

Section 8.10.7.3 of the ACI Code states that for moment transfer between the slab and edge column, the gravity load moment to be transferred shall be $0.3M_o$ (where M_o is the factored static moment).

When gravity loads, wind or earthquake loads, or other lateral forces cause a transfer of an unbalanced moment, M_{sc}, between a slab and a column, a part of the moment equal to $\gamma_f M_{sc}$ shall be transferred by flexure, according to Section 8.4.2.3.2 of the ACI Code. Based on both tests and experience, this transfer is to be considered to be made within an effective slab width between lines that are located one and a half times the slab or drop panel thickness outside opposite faces of the column or capital. The value γ_f is to be taken as

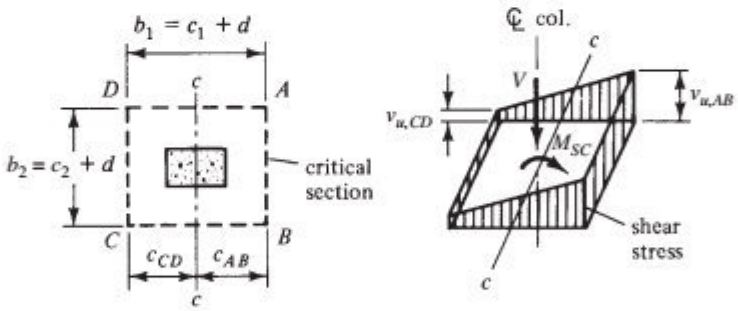
$$\gamma_f = \frac{1}{1 + \frac{2}{3}\sqrt{\frac{b_1}{b_2}}} \qquad \text{(ACI Equation 8.4.2.3.2)}$$

In Figure 16.21, b_1 is the length of the shear perimeter perpendicular to the axis of bending $(c_1 + d)$ and b_2 is the length of the shear perimeter parallel to the axis of bending $(c_2 + d)$. Also, c_1 is the width of the column perpendicular to the axis of bending and c_2 is the column width parallel to the axis of bending.

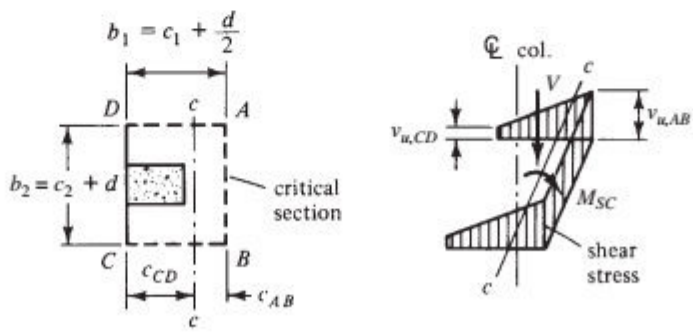
The remainder of the unbalanced moment, referred to as $\gamma_v M_{sc}$ by the code, is to be transferred by eccentricity of shear about the centroid of the critical section.

$$\gamma_v = 1 - \gamma_f \qquad \text{(ACI Equation 8.4.4.2.2)}$$

From this information, the shear stresses due to moment transfer by eccentricity of shear are assumed to vary linearly about the centroid of the critical section described in the last paragraph and are to be added to the usual factored shear forces. (In other words, there is the usual punching shear situation plus a twisting because of the moment transfer that increases the shear.) The resulting shear



(a) Interior column



(b) Edge column

FIGURE 16.21 Assumed distribution of shear stress (ACI Figure R8.4.4.2.3).
Source: Permission of American Concrete Institute.

stresses, $v_u = V_u/b_o d$, may not exceed ϕv_c (ACI Code, Section 22.6.1.2) for members without shear reinforcement, and $v_u \le \phi(v_c + v_s)$ for members with shear reinforcement other than shear heads. v_c in the two previous equations is the lesser of ACI Equation 22.6.5.2 (Section 12.6 of this text).

The combined stresses are calculated by the expressions to follow, with reference being made to Figure 16.21 and ACI Commentary R8.4.4.2.3:

$$v_u \text{ along } AB = \frac{V_u}{A_c} + \frac{\gamma_v M_{sc} c_{AB}}{J_c}$$

$$v_u \text{ along } CD = \frac{V_u}{A_c} - \frac{\gamma_v M_{sc} c_{CD}}{J_c}$$

In these expressions, A_c is the area of the concrete along the assumed critical section. For instance, for the interior column of Figure 16.21(a), it would be equal to $(2a + 2b)d$, and for the edge column of Figure 16.21(b), it would equal $(2a + b)d$.

J_c is a property analogous to the polar moment of inertia about the z–z axis of the shear areas located around the periphery of the critical section. First, the centroid of the shear area A_f is located by taking moments. The centroid is shown with the distances c_{AB} and c_{CD} in both parts (a) and (b) of Figure 16.21. The value of J_c is then computed for the shear areas. For the interior column of part (a), it is

$$J_c = d\left(\frac{a^3}{6} + \frac{ba^2}{2}\right) + \frac{ad^3}{6}$$

and for the edge column of part (b), it is

$$J_c = d\left[\frac{2a^3}{3} - (2a + b)\ (c_{AB})^2\right] + \frac{ad^3}{6}$$

Example 16.5 shows the calculations involved for shear and moment transfer for an exterior column.

Section 22.6.4.1 of the ACI Code states that the critical section described for two-way action for slabs located at $d/2$ from the perimeter of the column is appropriate for the calculation of shear stresses caused by moment transfer even when shearheads are used. Thus, the critical sections for direct shear and shear resulting from moment transfer will be different from each other.

The total reinforcement that will be provided in the column strip must include additional reinforcement concentrated over the column to resist the part of the bending moment transferred by flexure $= \gamma_f M_{sc}$.

Example 16.5

For the flat slab of Figure 16.22, compute the negative steel required in the column strip for the exterior edge indicated. Also check the slab for moment and shear transfer at the exterior column; $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, and $LL = 100$ psf. An 8-in. slab has already been selected with $d = 6.75$ in.

SOLUTION

1. Compute w_u and $M_{o\ell}$.

$$w_d = \left(\frac{8 \text{ in.}}{12 \text{ in./ft}}\right)(150 \text{ pcf}) = 100 \text{ psf}$$

$$w_\ell = 100 \text{ psf}$$

$$w_u = (1.2)\ (100 \text{ psf}) + (1.6)\ (100 \text{ psf}) = 280 \text{ psf}$$

$$M_{o\ell} = \frac{q_u \ell_2 \ell_n^2}{8} = \frac{(0.280 \text{ ksf})\ (18 \text{ ft})\ (18.75 \text{ ft})^2}{8} = 221.5 \text{ ft-k}$$

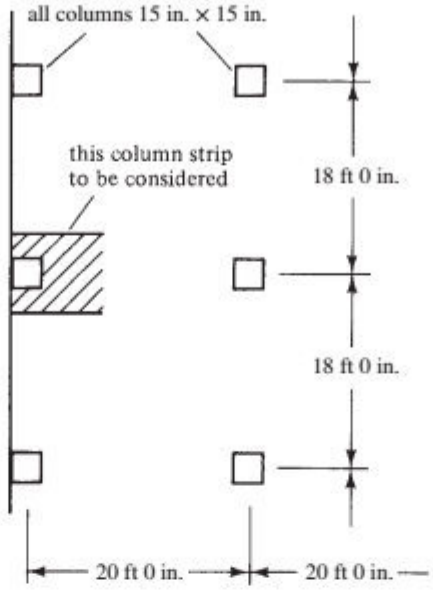


FIGURE 16.22 Dimensions of flat slab in Example 16.5.

2. Determine the exterior negative moment as per Section 8.10.4.2 of the ACI Code.

$$-0.26M_{o\ell} = -(0.26)\ (221.5\ \text{ft-k}) = 57.6\ \text{ft-k}$$

$$\text{Width of column strip} = (0.50)\ (18\ \text{ft}) = 9\ \text{ft} - 0\ \text{in.} = 108\ \text{in.}$$

3. Section 8.10.5.2 of the ACI Code shows that 100% of the exterior negative moment is to be resisted by the column strip.

4. Design of steel in column strip

$$\frac{M_u}{\phi bd^2} = \frac{(12\ \text{in./ft})\ (57{,}600\ \text{ft-lb})}{(0.9)\ (108\ \text{in.})\ (6.75\ \text{in.})^2} = 156.1\ \text{psi}$$

$$\rho = 0.0027\ \text{(from Appendix A, Table A.12)}$$

$$A_s = \rho bd = (0.0027)\ (108\ \text{in.})\ (6.75\ \text{in.}) = 1.97\ \text{in.}^2 \qquad \underline{\text{Use 10 \#4 bars}}$$

5. Moment transfer design

(a) Section 8.4.2.3.3 of the ACI Code states that additional bars must be added over the column in a width = column width + (2) (1.5h) = 15 in. + (2) (1.5 × 8 in.) = 39 in.

(b) The additional reinforcement needed over the columns is to be designed for a moment = $\gamma_f M_{sc}$. In ACI Equation 8.4.2.3.2 below, b_1 and b_2 are the side dimensions of the perimeter b_o (ACI Code, Section 22.6.4.1) (see Figure 16.21). In this case, $b_1 = c_1 + d/2 = 15$ in. + 6.75 in./2 = 18.375 in. In the perpendicular direction, $b_2 = c_2 + d = 15$ in. + 6.75 in. = 21.75 in.

$$\gamma_f = \frac{1}{1 + \frac{2}{3}\sqrt{\frac{b_1}{b_2}}} = \frac{1}{1 + \frac{2}{3}\sqrt{\frac{18.375\ \text{in.}}{21.75\ \text{in.}}}} = 0.62 \qquad \text{(ACI Equation 8.4.2.3.2)}$$

$$\gamma_f M_{sc} = (0.62)\ (57.6\ \text{ft-k}) = 35.7\ \text{ft-k}$$

(c) Add 4 #4 bars in the 39 in. width and check to see whether the moment transfer situation is satisfactory. To resist the 35.7 ft-k, we now have the 4 #4 bars just added plus 4 #4. This number of bars is obtained by taking the ratio of 39 in./108 in. times 10 bars to get 3.6 bars and rounding to 4. The total number of bars put in for the column strip design is 8 #4 bars (1.60 in.^{2}).

$$a = \frac{A_s f_y}{0.85 f'_c b} = \frac{(1.57 \text{ in.}^2)\ (60 \text{ ksi})}{(0.85)\ (3 \text{ ksi})\ (39 \text{ in.})} = 0.947 \text{ in.}$$

$$\phi M_n = M_u = \phi A_s f_y \left(d - \frac{a}{2}\right)$$

$$= \frac{(0.9)\ (1.57 \text{ in.}^2)\ (60 \text{ ksi}) \left(6.75 \text{ in.} - \dfrac{0.947 \text{ in.}}{2}\right)}{12}$$

$$= 44.3 \text{ ft-k} > 35.7 \text{ ft-k} \quad \underline{\text{OK}}$$

6. Compute combined shear stress at exterior column due to shear and moment transfer.

(a) Section 8.10.4.6 of the ACI Code requires that a moment of $(0.3M_o)\ (\gamma_v)$ be transferred from the slab to the column by eccentricity of shear. The total moment to be transferred is $0.3(M_o) = 0.3(221.5 \text{ ft-k}) = 66.45 \text{ ft-k}$.

(b) Fraction of unbalanced moment carried by eccentricity of shear $= \gamma_v(0.3M_o)$.

$$\gamma_v = 1 - \frac{1}{1 + \dfrac{2}{3}\sqrt{\dfrac{18.375 \text{ in.}}{21.75 \text{ in.}}}} = 0.38$$

$$\gamma_v M_n = (0.38)\ (66.45 \text{ ft-k}) = 25.25 \text{ ft-k}$$

(c) Compute properties of critical section for shear (Figure 16.23).

$$A_c = (2a + b)d = (2 \times 18.375 \text{ in.} + 21.75 \text{ in.})\ (6.75 \text{ in.}) = 394.875 \text{ in.}^2$$

$$c_{AB} = \frac{(2)\ (18.375 \text{ in.})\ (6.75 \text{ in.}) \left(\dfrac{18.375 \text{ in.}}{2}\right)}{394.875 \text{ in.}^2} = 5.77 \text{ in.}$$

$$J_c = d\left[\frac{2a^3}{3} - (2a + b)\ (c_{AB})^2\right] + \frac{ad^3}{6}$$

$$= 6.75 \text{ in.} \left[\frac{(2)\ (18.375 \text{ in.})^3}{3} - (2 \times 18.375 \text{ in.} + 21.75 \text{ in.})\ (5.77 \text{ in.})^2\right]$$

$$+ \frac{(18.375 \text{ in.})\ (6.75 \text{ in.})^3}{6}$$

$$= 15{,}714 \text{ in.}^4$$

(d) Compute gravity load shear to be transferred at the exterior column.

$$V_u = \frac{q_u \ell_1 \ell_2}{2} = \frac{(0.280 \text{ ksf})\ (18 \text{ ft})\ (20 \text{ ft})}{2} = 50.4 \text{ k}$$

(e) Combined stresses

$$v_u = \frac{V_u}{A_c} + \frac{\gamma_v M_n c_{AB}}{J_c}$$

$$= \frac{50{,}400 \text{ lb}}{394.875 \text{ in.}^2} + \frac{(12 \text{ in./ft} \times 25{,}250 \text{ ft-lb})\ (5.77 \text{ in.})}{15{,}714 \text{ in.}^4}$$

$$= 128 \text{ psi} + 111 \text{ psi} = 239 \text{ psi}$$

$$> 4\sqrt{3000} \text{ psi} = 219 \text{ psi} \quad \underline{\text{No good}}$$

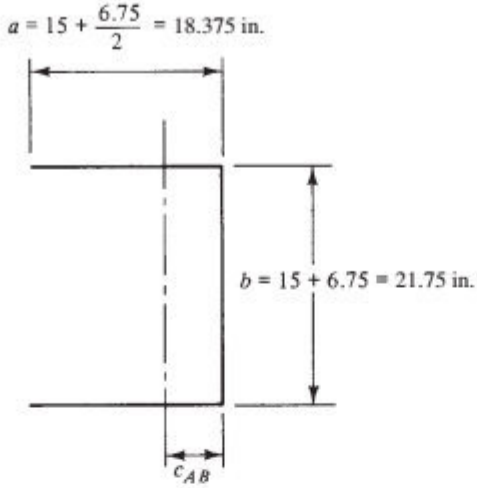


FIGURE 16.23 Dimensions of critical section for shear.

∴ It is necessary to do one or more of the following: increase depth of slab, use higher-strength concrete, use drop panel, or install shearhead reinforcement.

Factored Moments in Columns and Walls

If there is an unbalanced loading of two adjoining spans, the result will be an additional moment at the connection of walls and columns to slabs. Section 8.10.7.2 of the ACI Code provides the approximate equation listed at the end of this paragraph to consider the effects of such situations. This particular equation was derived for two adjoining spans, one longer than the other. It was assumed that the longer span was loaded with dead load plus one-half live load and that only dead load was applied to the shorter span.

$$M_{sc} = 0.07[(q_{Du} + 0.5q_{Lu})\ell_2\ell_n^2 - q_{Du'}\ell'_2(\ell'_n)^2] \qquad \text{(ACI Equation 8.10.7.2)}$$

In this expression, q'_{Du}, ℓ'_2, and ℓ'_n are for the shorter spans. The resulting approximate value should be used for unbalanced moment transfer by gravity loading at interior columns unless a more theoretical analysis is used.

16.13 Openings in Slab Systems

According to the code (8.5.4), openings can be used in slab systems if adequate strength is provided and if all serviceability conditions of the ACI, including deflections, are met.

1. If openings are located in the area common to intersecting middle strips, it will be necessary to provide the same total amount of reinforcement in the slab that would have been there without the opening.
2. For openings in intersecting column strips, the width of the openings may not be more than one-eighth the width of the column strip in either span. An amount of reinforcement equal to that interrupted by the opening must be placed on the sides of the opening.
3. Openings in an area common to one column strip and one middle strip may not interrupt more than one-fourth of the reinforcement in either strip. An amount of reinforcement equal to that interrupted shall be placed around the sides of the opening.
4. If an opening is located within a column strip or closer than **10*h*** from a concentrated load or reaction area, ACI Code, Section 22.6.4.3 for slabs without shearheads or ACI Code, Section 22.6.9.9 for slabs with shearheads shall be satisfied.

16.14 Computer Example

Example 16.6

Use the Chapter 16 Excel spreadsheet to solve Example 16.4.

SOLUTION

Open the Excel spreadsheet provided for Chapter 16 and open the worksheet Two-Way Slabs. Enter values only for cells highlighted in yellow (only in the Excel spreadsheets, gray in the printed example). Note that values for β_t and α_{f1} must be entered. To obtain β_t, open the worksheet C Torsional Constant. Enter values for cells highlighted in shaded.

torsional constant C and β_1

$$C = \left(1 - \frac{0.63x}{y}\right)\left(\frac{x^3y}{3}\right) =$$

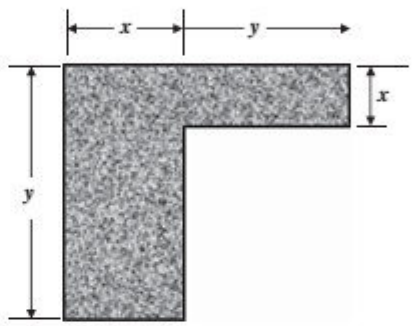
x	y
15	20
11,868.75	
6	14
735.84	
0.0	0.0
0.00	

12,605 = C

ℓ_2 (ft)	t (in.)
18	6
3888	

$$I_s = \frac{\ell_2 t^3}{12} =$$ in.^{4}

$$\beta_t = \frac{C}{2I_s} =$$ 1.621



If there are flanges on both sides of the web, a third value of x and y can be entered. In this case, their values are zero.

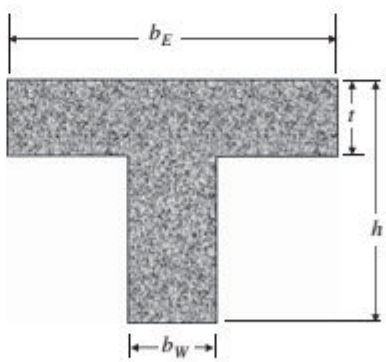
The value of α_f is obtained from the worksheet Alpha T Beam. Results of the α_f calculations in step 2(d) of Example 16.4 are shown below.

Interior Beam (flange on both sides)

b_E =	43	in.
b_W =	15	in.
t =	6	in.
h =	20	in.
ℓ_2 =	22	ft
k =	1.578	
I_b =	15,781	in.^{4}
I_s =	4752	in.^{4}
α_{f1} =	3.321	

$\frac{b_E}{b_W}$ = 2.867

$\frac{t}{h}$ = 0.3

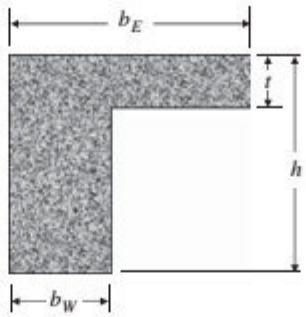


Edge Beam

b_E =	29	in.
b_W =	15	in.
t =	6	in.
h =	20	in.
ℓ_2 =	22	ft
k =	1.347	
I_b =	13,468	in.^{4}
I_s =	2511	in.^{4}
α_{f1} =	5.363	

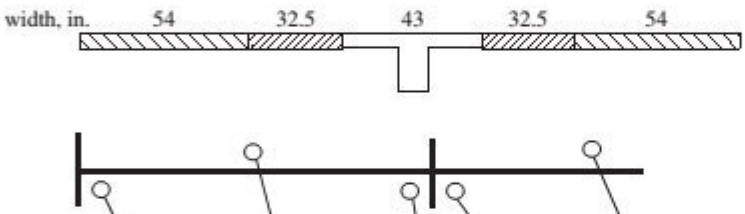
$\frac{b_E}{b_W}$ = 1.933

$\frac{t}{h}$ = 0.3



The other α_f values are obtained simply by changing ℓ_2 from 22 ft to 18 ft. Return to the Example 16.6 worksheet and enter $\beta_t = 1.62$ and $\alpha_{f1} = 3.32$. Now determine from Table 16.2 which case (1 through 5) applies to your example. In this situation, case (2) applies because there are beams between all supports. Enter 0.16, 0.57, and 0.70 in the highlighted cells in row 11. In addition to the information determined in Example 16.4, the spreadsheet also determines the required area of reinforcing steel throughout the slab.

Two-way slab



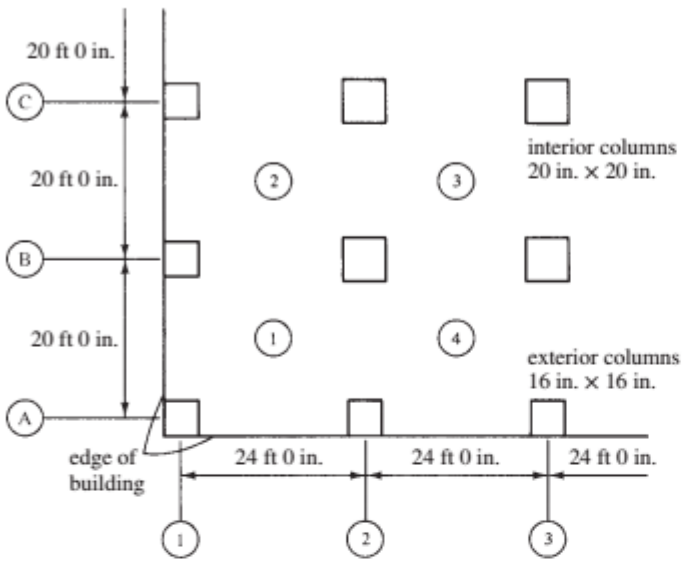
	Exterior Span Moments (ft-k)			Interior Span (ft-k)	
% of M_O from Table 16.2	16.00	57.00	70.00	65.0	35.0
Total moment at …(ft-k)	48.36	172.29	211.58	196.47	105.79
Column strip moment	42.24	138.61	170.22	158.07	85.11
Beam portion	35.90	117.82	144.69	134.36	72.35
Column strip slab	6.34	20.79	25.53	23.71	12.77
Middle strip moment	6.13	33.67	41.35	38.40	20.68
Beam $R_n = M_u/\phi_{bd}^2$	104.20	119.29	419.96	389.97	73.25
ρ	0.00177	0.00204	0.00770	0.00709	0.00124
$A_s = \rho_{bd}$ in.^{2}	0.47	1.53	2.02	1.86	0.93
Col. strip slab R_n =	57.60	189.03	232.14	215.56	116.07
ρ	0.00097	0.00328	0.00406	0.00376	0.00198
A_s-in.^{2} (half on each side of beam in col. strip slab)	0.702	1.012	1.255	1.161	0.702
Mid. strip slab R_n =	33.52	184.26	226.28	210.12	113.14
ρ	0.00056	0.00319	0.00396	0.00366	0.00193
A_s-in.^{2} (half in each middle strip)	1.167	1.637	2.029	1.877	1.167

ℓ_1 =	22	ft (centerline span)
ℓ_2 =	18	ft (transverse span)
c_1 =	15	in. column
c_2 =	15	in. dimension
t =	6	in. (slab thickness)
d_s =	4.75	in. (slab d)
ℓ_n =	20.75	ft (clear span)
b_w =	15	in. (beam web width)
h =	20	in. (beam depth)
b_{eff} =	43	in.
y =	7.487	in.
I_g =	15,781	in.^{4}
I_s =	3888	in.^{4}
d_b =	17.5	in. (beam d)
f'_c =	3	ksi
f_y =	60	ksi
m =	23.53	
w_D =	100	psf
w_L =	120	psf
w_u =	312	psf
M_{ol} =	302.3	ft-k
β_t =	1.62	not more than 2.5
α_{f1} =	4.06	
$\alpha_{f12}1/\ell_1$ =	1.00	
T & S %	0.0018	

The edge span part of the spreadsheet is set up for edge spans (part 4 of Example 16.4). Enter information the same way as for the upper part of the worksheet.

PROBLEMS

Problem 16.1 Using the ACI Code, determine the minimum thickness required for panels ① and ③ of the flat plate floor shown. Edge beams are not used along the exterior floor edges. $f'_c = 3000$ psi and $f_y = 60{,}000$ psi. (*Ans*. 9.07 in., 8.12 in.)



Problem 16.2 Assume that the floor system of Problem 16.1 is to support a service live load of 80 psf and a service dead load of 60 psf in addition to its own weight. If the columns are 10 ft long, determine depth required for an interior flat plate for panel ③.

Problem 16.3 Repeat Problem 16.2 for panel ① in the structure of Problem 16.1. Include depth required for one-way and two-way shear. (*Ans*. Use 10-in. slab)

Problem 16.4 Determine the required reinforcement in the column strip and middle strips for column line Ⓑ in Problem 16.1. Use a slab thickness of 10 in. and #6 bars. Determine theoretical and practical spacings in the exterior span (from column line ① to ②) and the interior span (from column line ② to ③).

Problem 16.5 Use the Flat Plate worksheet of Chapter 16 spreadsheet to work Problem 16.4. (*Ans*. $A_s = 6.53$ in.^{2} in column strip at column line ② ∴ use #6 bars @ 8 in. in top of slab)

CHAPTER 17 Two-Way Slabs, Equivalent Frame Method

17.1 Moment Distribution for Nonprismatic Members

Most of the moment distribution problems the student has previously faced have dealt with prismatic members for which carryover factors of $\frac{1}{2}$, fixed-end moments for uniform loads of $w_u\ell^2/8$, stiffness factors of I/ℓ, and so on were used. Should nonprismatic members be encountered, such as the continuous beam of Figure 17.1, none of the preceding values applies.

Carryover factors, fixed-end moments, and so forth can be laboriously obtained by various methods, such as the moment-area and column-analogy methods.[1] There are various tables available, however, from which many of these values can be obtained. The tables numbered A.16 through A.20 of Appendix A cover most situations encountered with the equivalent frame method.

Before the equivalent frame method is discussed, an assumed set of fixed-end moments is balanced in Example 17.1 for the nonprismatic beam of Figure 17.1. The authors' purpose in presenting this example is to show the reader how moment distribution can be applied to the analysis of structures consisting of nonprismatic members.

Example 17.1

The carryover factors (C.O.), stiffness factors (K), and fixed-end moments (FEM) shown in Figure 17.2 have been assumed for the continuous nonprismatic member of Figure 17.1. Balance these moments by moment distribution. It will be shown later, in Example 17.2, how to determine these factors for two-way slab systems. The purpose of this example is to demonstrate the moment distribution method when nonprismatic members are involved.

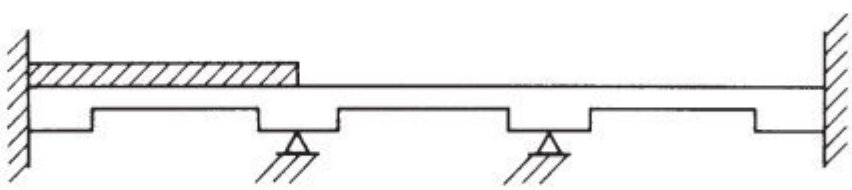
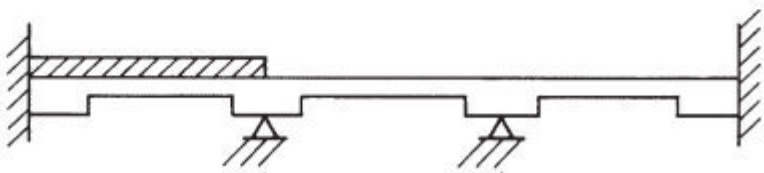


FIGURE 17.1 Nonprismatic beam.

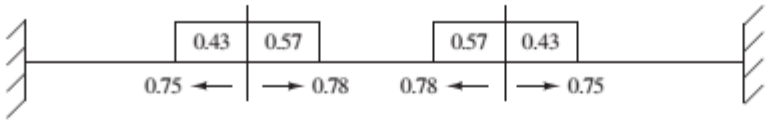


C.O.	← 0.75	0.78 →	← 0.78	0.75 →
K	17.1	22.8	22.8	17.1
FEM (ft-k)	−260.6	+260.6		

FIGURE 17.2 Moment distribution.

[1] McCormac, J. C., 1984, *Structural Analysis*, 4th ed. (New York: Harper & Row), pp. 333–334, 567–582.

SOLUTION



FEM (ft-k)	−260.6	+260.6				
	− 84.1	−112.1	− 148.5	−115.8		
		+ 51.5		+ 66.0	+ 49.8	+ 37.3
	− 16.6	− 22.1	− 29.4	− 22.9		
		+ 10.2		+ 13.1	+ 9.8	+ 7.4
	− 3.3	− 4.4	− 5.8	− 4.5		
		+ 2.0		+ 2.6	+ 1.9	+ 1.4
	− 0.7	− 0.9	− 1.1	− 0.9		
				+ 0.5	+ 0.4	+ 0.3
final moments (ft-k)	−365.3	+121.1	− 121.1	− 61.9	+ 61.9	+ 46.4

17.2 Introduction to the Equivalent Frame Method

With the preceding example, the authors hoped to provide the reader with a general idea of the kinds of calculations he or she will face with the equivalent frame method. A part of a two-way slab building will be taken out by itself and analyzed by moment distribution. The slab-beam members of this part of the structure will be nonprismatic because of the columns, beams, drop panels, and so on of which they consist. As a result, it will be quite similar to the beam of Figure 17.1, and we will analyze it in the same manner.

The only difference between the direct design method and the equivalent frame method is in the determination of the longitudinal moments in the spans of the equivalent rigid frame. The direct design method involves a one-cycle moment distribution, whereas the equivalent frame method involves a normal moment distribution of several cycles. The design moments obtained by either method are distributed to the column and middle strips in the same manner.

It will be remembered that the range in which the direct design method can be applied is limited by a maximum 2-to-1 ratio of live-to-dead load and a maximum ratio of the longitudinal span length to the transverse span length of 2 to 1. In addition, the columns may not be offset by more than 10% of the span length in the direction of the offset from either axis between centerlines of successive columns. There are no such limitations on the equivalent frame method. This is a very important matter because so many floor systems do not meet the limitations specified for the direct design method.

Analysis by either method will yield almost the same moments for those slabs that meet the limitations required for application of the direct design method. For such cases, it is simpler to use the direct design method.

The equivalent frame method involves the elastic analysis of a structural frame consisting of a row of equivalent columns and horizontal slab members that are each one panel long and have a transverse width equal to the distance between centerlines of the panels on each side of the columns in question. For instance, the hatched strip of the floor system in Figure 17.3 can be extracted and the combined slab

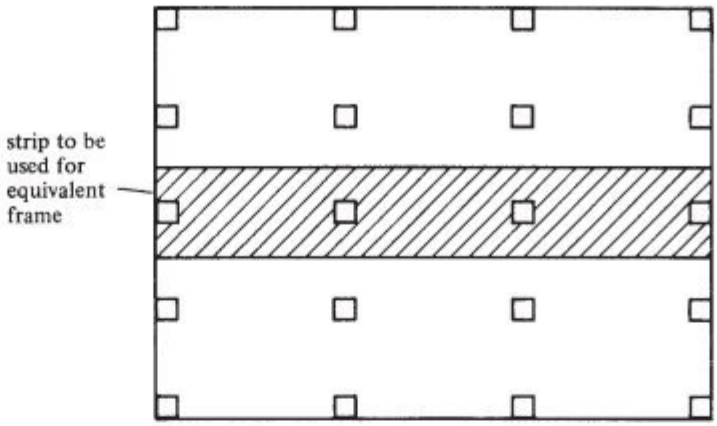


FIGURE 17.3 Slab beam.

and beam analyzed to act as a beam element as part of a structural frame (see also Figure 17.4). This assumption approximately models the actual behavior of the structure. It is a reasonably accurate way of calculating moments in the overall structural frame, which then may be distributed to the slab and beams. This process is carried out in both directions. That is, it is done to the hatched strip in Figure 17.3 and all other strips parallel to it. Then it is carried out on strips that are perpendicular to these strips, hence the term two-way floor system.

For vertical loads, each floor, together with the columns above and below, is analyzed separately. For such an analysis, the far ends of the columns are considered fixed. Figure 17.4 shows a typical equivalent slab beam as described in this chapter.

Should there be a large number of panels, the moment at a particular joint in a slab beam can be satisfactorily obtained by assuming that the member is fixed two panels away. This simplification is permissible because vertical loads in one panel only appreciably affect the forces in that panel and in the one adjacent to it on each side.

For lateral loads, it is necessary to consider an equivalent frame that extends for the entire height of the building, because the forces in a particular member are affected by the lateral forces

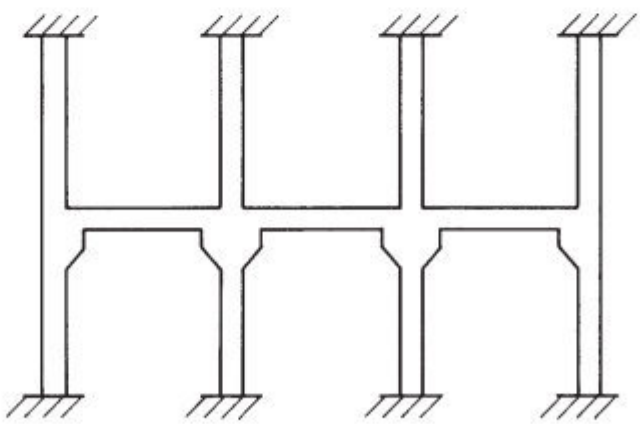


FIGURE 17.4 Equivalent frame for the crosshatched strip of Figure 17.3 for vertical loading.

on all the stories above the floor being considered. When lateral loads are considered, it will be necessary to analyze the frame for them and combine the results with analyses made for gravity loads (ACI Code, Section 8.4.1.9).

The equivalent frame is made up of the horizontal slab, any beams spanning in the direction of the frame being considered, the columns or other members that provide vertical support above and below the slab, and any parts of the structure that provide moment transfer between the horizontal and vertical members. You can see there will be quite a difference in moment transfer from the case where a column provides this transfer and where there is a monolithic reinforced concrete wall extending over the full length of the frame. For cases in between, the stiffnesses of the torsional members such as edge beams will be estimated.

The same minimum ACI slab thicknesses must be met as in the direct design method. The depths should be checked for shear at columns and other supports, as specified in Section 8.5.1.1d of the ACI Code. Once the moments have been computed, it will also be necessary to check for moment shear transfer at the supports.

The analysis of the frame is made for the full design live load applied to all spans, unless the actual unfactored live load exceeds 0.75 times the unfactored dead load (ACI Code, Section 6.4.3.3). When the live load is greater than 0.75 times the dead load, a pattern loading with three-fourths times the live load is used for calculating moments and shears.

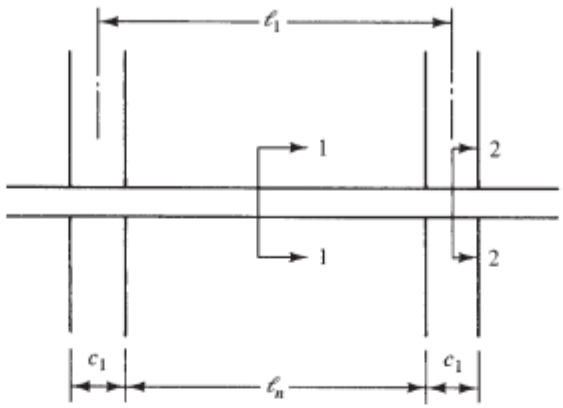
The maximum positive moment in the middle of a span is assumed to occur when three-fourths of the full design load is applied in that panel and in alternate spans. The maximum negative moment in the slab at a support is assumed to occur when three-fourths of the full design live load is applied only to the adjacent panels. The values so obtained may not be less than those calculated, assuming full live loads in all spans.

17.3 Properties of Slab Beams

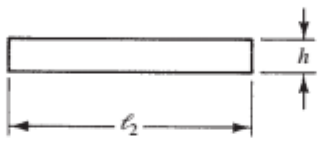
The parts of the frame are the slabs, beams, drop panels, columns, and so on. Our first objective is to compute the properties of the slab beams and the columns (i.e., the stiffness factors, distribution factors, carryover factors, and fixed-end moments). To simplify this work, the properties of the members of the frame are permitted by Section 8.11.3.3 of the ACI Code to be based on their gross moments of inertia rather than their transformed or cracked sections. Despite the use of the gross dimensions of members, the calculations involved in determining the properties of nonprismatic members still represent a lengthy task, and we will find the use of available tables very helpful.

Figures 17.5 and 17.6 present sketches of two-way slab structures, together with the equivalent frames that will be used for their analysis. A flat slab with columns is shown in Figure 17.5(a). Cross sections of the structure are shown through the slab in part (b) of the figure and through the column in part (c). In part (d), the equivalent frame that will be used for the actual numerical calculations is shown. In this figure, E_{cs} is the modulus of elasticity of the concrete slab. With section 2–2, a fictitious section is shown that will have a stiffness approximately equivalent to that of the actual slab and column. An expression for I for the equivalent section is also given. In this expression, c_2 is the width of the column in a direction perpendicular to the direction of the span, and ℓ_2 is the width of the slab beam. The gross moment of inertia at the face of the support is calculated and is divided by $(1 - c_2/\ell_2)^2$. This approximates the effect of the large increase in depth provided by the column for the distance in which the slab and column are in contact.

In Figure 17.6, similar sketches and I values are shown for a slab with drop panels. Figure 13.7.3 of the 1983 ACI Commentary showed such information for slab systems with beams and for slab systems with column capitals.

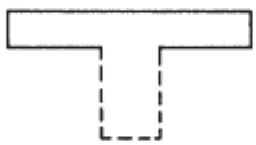


(a) Actual floor system



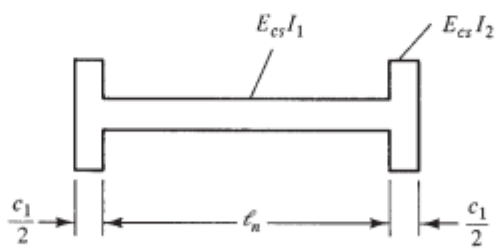
$$I_1 = \left(\frac{1}{12}\right)(\ell_2)(h)^3$$

(b) Section 1–1



$$I_2 = \frac{I_1}{\left(1 - \frac{c_2}{\ell_2}\right)^2}$$

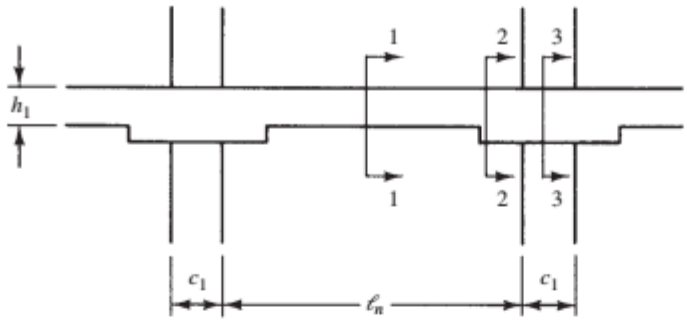
(c) Equivalent Section 2–2



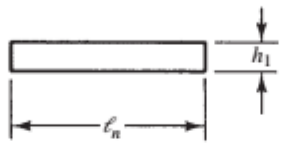
(d) Equivalent slab beam stiffness diagram

FIGURE 17.5 Slab system without beams.

With the equivalent slab beam stiffness diagram, it is possible, using the conjugate beam method, column analogy, or some other method, to compute stiffness factors, distribution factors, carryover factors, and fixed-end moments for use in moment distribution. Tables A.16 through A.20 of Appendix A of this text provide tabulated values of these properties for various slab systems. The numerical examples of this chapter make use of this information.

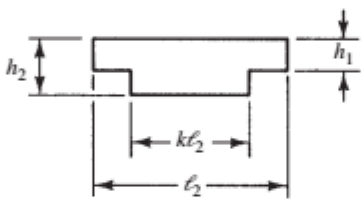


(a) Actual floor system



$$I_1 = \left(\frac{1}{12}\right)(\ell_2)(\ell_n)^3$$

(b) Section 1–1

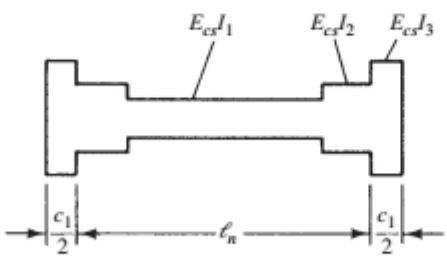


$$I_2 = \frac{I_1}{\left(1 - \frac{c_2}{\ell_2}\right)^2}$$

(c) Section 2–2

$$I_3 = \frac{I_2}{\left(1 - \frac{c_2}{\ell_2}\right)^2}$$

(d) Equivalent Section 3–3



(e) Equivalent slab beam stiffness diagram

FIGURE 17.6 Slab systems with drop panels.

17.4 Properties of Columns

The length of a column is assumed to run from the middepth of the slab on one floor to the middepth of the slab on the next floor. For stiffness calculations, the moments of inertia of columns are based on their gross dimensions. Thus, if capitals are present, the effect of their dimensions must be used for those parts of the columns. Columns are assumed to be infinitely stiff for the depth of the slabs.

Figure 17.7 shows a sample column, together with its column stiffness diagram. Similar diagrams are shown for other columns (where there are drop panels, capitals, etc.) in Figure 13.7.4 of the 1983 ACI Commentary.

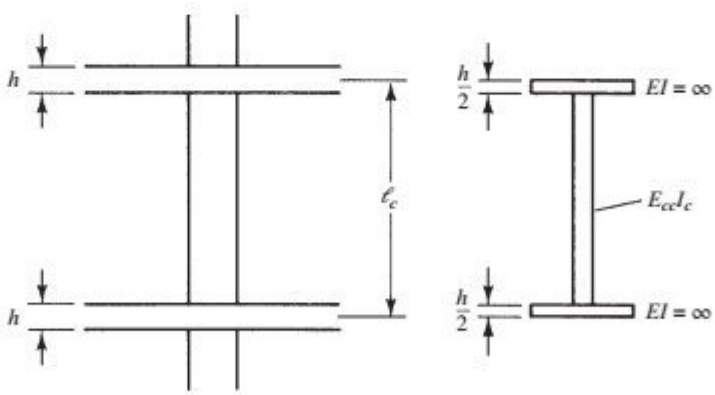
With a column stiffness diagram, the column flexural stiffness, K_c, can be determined by the conjugate beam procedure or other methods. Tabulated values of K_c are given in Table A.20 of the Appendix A of this text for typical column situations.

In applying moment distribution to a particular frame, we need the stiffnesses of the slab beam, the torsional members, and the equivalent column so that the distribution factors can be calculated. For this purpose, the equivalent column, the equivalent slab beam, and the torsional members are needed at a particular joint.

For this discussion, see Figure 17.8, where it is assumed that there is a column above and below the joint in question. Thus, the column stiffness (K_c) here is assumed to include the stiffness of the column above (K_{ct}) and the one below (K_{cb}). Thus, $\Sigma K_c = K_{ct} + K_{cb}$. In a similar manner, the total torsional stiffness is assumed to equal that of the torsional members on both sides of the joint ($\Sigma K_t = K_{t1} + K_{t2}$). For an exterior frame, the torsional member will be located on one side only.

The following approximate expression for the stiffness (K_t) of the torsional member was determined using a three-dimensional analysis for various slab configurations (ACI Code, Section 8.11.5).

$$K_t = \Sigma \frac{9E_{cs}C}{\ell_2\left(1 - \frac{c_2}{\ell_2}\right)^3}$$



(a) Actual column and slab, no beams

(b) Column stiffness diagram

FIGURE 17.7 Equivalent column.

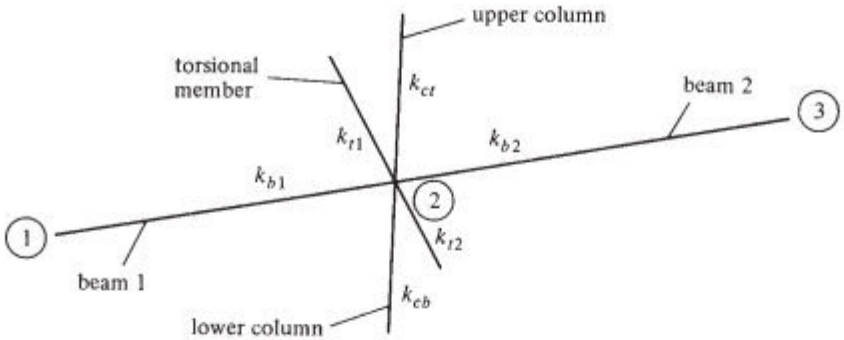


FIGURE 17.8 Equivalent frame model.

In this formula, C is to be determined with the following expression by dividing the cross section of the torsional member into rectangular parts and summing the C values for the different parts.

$$C = \Sigma \left(1 - 0.63\frac{x}{y}\right) \frac{x^3 y}{3} \qquad \text{(ACI Equation 8.10.5.2b)}$$

If there is no beam framing into the column in question, a part of the slab equal to the width of the column or capital shall be used as the effective beam. If a beam frames into the column, a T beam or L beam will be assumed, with flanges of widths equal to the projection of the beam above or below the slab but not more than four times the slab thickness.

The flexibility of the equivalent column is equal to the reciprocal of its stiffness, as follows:

$$\frac{1}{K_{ec}} = \frac{1}{\Sigma K_c} + \frac{1}{\Sigma K_t}$$

$$\frac{1}{K_{ec}} = \frac{1}{K_{ct} + K_{cb}} + \frac{1}{K_c + K_t}$$

Solving this expression for the equivalent column stiffness and multiplying through by K_c

$$K_{ec} = \frac{(K_{ct} + K_{cb})(K_t + K_t)}{(K_{ct} + K_{cb}) + (K_t + K_t)}$$

An examination of this brief derivation shows that the torsional flexibility of the slab column joint reduces the joint's ability to transfer moment.

After the value of K_{ec} is obtained, the distribution factors can be computed as follows (see Figure 17.8):

$$\text{DF for beam 2–1} = \frac{K_{b1}}{K_{b1} + K_{b2} + K_{ec}}$$

$$\text{DF for beam 2–3} = \frac{K_{b2}}{K_{b1} + K_{b2} + K_{ec}}$$

$$\text{DF for column above} = \frac{K_{ec}/2}{K_{b1} + K_{b2} + K_{ec}}$$

17.5 Example Problem

Example 17.2 illustrates the determination of the moments in a flat-plate structure by the equivalent frame method.

Example 17.2

Using the equivalent frame method, determine the design moments for the hatched strip of the flat-plate structure shown in Figure 17.9 if $f'_c = 4000$ psi, $f_y = 60{,}000$ psi, and (unfactored dead load) $q_D = 120$ psf and (unfactored live load) $q_L = 82.5$ psf. Column length = 9 ft 6 in.

SOLUTION

1. Determine the depth required for ACI depth limitations (Section 8.3.1.1). Assume that this has been done and that a preliminary slab $h = 8$ in. ($d = 6.75$ in.) has been selected.
2. Check beam shear for exterior column

$$q_u = 1.2q_D + 1.6q_L = 1.2(120 \text{ psf}) + 1.6(82.5 \text{ psf}) = 276 \text{ psf}$$

$$V_u \text{ for 12 in. width} = (0.276 \text{ ksf})\left(11.0 \text{ ft} - \frac{7.5 \text{ in.}}{12 \text{ in./ft}} - \frac{6.75 \text{ in.}}{12 \text{ in./ft}}\right) = 2.708 \text{ k/ft}$$

$$\phi V_c = \frac{(0.75)(2)(1.0)(\sqrt{4000} \text{ psi})(12 \text{ in.})(6.75 \text{ in.})}{1000} = 7.684 \text{ k/ft} > 2.708 \text{ k/ft} \quad \underline{\text{OK}}$$

3. Check two-way shear around interior columns

$$V_u = \left[(18 \text{ ft})(22 \text{ ft}) - \left(\frac{15 \text{ in.} + 6.75 \text{ in.}}{12 \text{ in./ft}}\right)^2\right](0.276 \text{ ksf}) = 108.39 \text{ k} = 108{,}390 \text{ lb}$$

$$v_u = V_u/b_o d = 108{,}390 \text{ lb}/[(4)(15 \text{ in.} + 6.75 \text{ in.})(6.75 \text{ in.})] = 184 \text{ psi}$$

$$\phi v_c = (0.75)(4)(1.0)(\sqrt{4000} \text{ psi})$$

$$= 190 \text{ psi} > 184 \text{ psi} \quad \underline{\text{OK}}$$

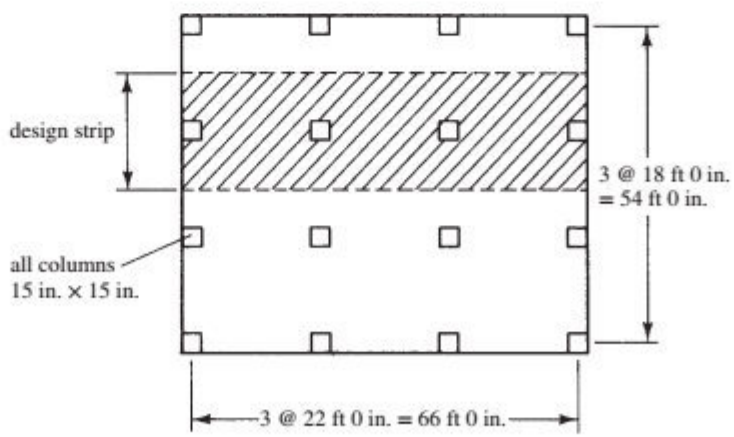


FIGURE 17.9 Flat plate for Example 17.2.

4. Using tables in Appendix A, determine stiffness factor and fixed-end moments for the 22-ft spans

$$I_s = \frac{\ell_2 h^3}{12} = \frac{(12 \text{ in./ft} \times 18 \text{ ft})(8 \text{ in.})^3}{12} = 9216 \text{ in.}^4$$

$$E_{cs} = 3.64 \times 10^6 \text{ psi (from Appendix A, Table A.1)}$$

See Table A.16, notice that *C* values are column dimensions as shown in the figures accompanying Tables A.16 to A.19. The tables are rather difficult to read.

$$C_{1A} = C_{2A} = C_{1B} = C_{2B} = 15 \text{ in.} = 1.25 \text{ ft}$$

$$\frac{C_{1A}}{\ell_1} = \frac{1.25 \text{ ft}}{22.0 \text{ ft}} = 0.057$$

$$\frac{C_{1B}}{\ell_1} = \frac{1.25 \text{ ft}}{22.0 \text{ ft}} = 0.057$$

By interpolation in the table (noting that *A* is for near end and *B* is for far end), the values from the table are very rough.

$$k_{AB} = 4.17$$

$$k_{AB} = \frac{4.17 E_{cs} I_s}{\ell_1} = \frac{(4.17)(3.64 \times 10^6 \text{ psi})(9216 \text{ in.}^4)}{(12 \text{ in./ft})(22 \text{ ft})}$$

$$= 529.9 \times 10^6 \text{ in.-lb}$$

$$\text{FEM}_{AB} = \text{FEM}_{BA} = 0.084 q_u \ell_2 \ell_1^2$$

$$= (0.084)(0.276 \text{ ksf})(18 \text{ ft})(22 \text{ ft})^2 = 202 \text{ ft-k}$$

$$C_{AB} = C_{BA} = \text{carryover factor}$$

$$= 0.503$$

5. Determine column stiffness

$$I_c = \left(\frac{1}{12}\right)(15 \text{ in.})(15 \text{ in.})^3 = 4219 \text{ in.}^4$$

$$E_{cc} = 3.64 \times 10^6 \text{ psi}$$

Using Appendix A, Table A.20

$$\ell_n = 9 \text{ ft } 6 \text{ in.} = 9.50 \text{ ft}$$

$$\ell_c = 9.50 \text{ ft} - \frac{8 \text{ in.}}{12 \text{ in./ft}} = 8.833 \text{ ft}$$

$$\frac{\ell_n}{\ell_c} = \frac{8.833 \text{ ft}}{9.50 \text{ ft}} = 0.930 = \frac{\ell_u}{\ell_c}$$

With reference to the figure given with Table A.20,

$$\frac{a}{b} = \frac{4 \text{ in.}}{4 \text{ in.}} = 1.00$$

$$k_{AB} = 4.81 \text{ by interpolation}$$

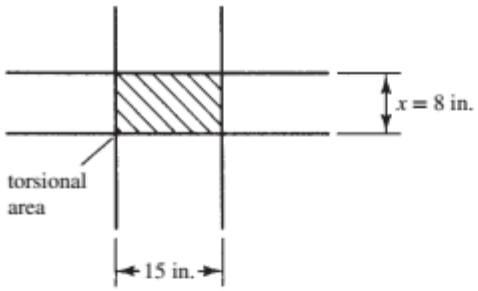


FIGURE 17.10 Torsional member.

$$K_c = \frac{4.81E_{cc}I_c}{H} = \frac{(4.81)(3.64 \times 10^6 \text{ psi})(4219 \text{ in.}^4)}{(9.5 \text{ ft})(12 \text{ in./ft})} = 648 \times 10^6 \text{ in.-lb}$$

$C_{AB} = 0.55$ by interpolation

6. Determine the torsional stiffness of the slab section (see Figure 17.10)

$$C = \Sigma\left(1 - 0.63\frac{x}{y}\right)\left(\frac{x^3y}{3}\right) = \left(1 - \frac{0.63 \times 8 \text{ in.}}{15 \text{ in.}}\right)\left[\frac{(8 \text{ in.})^3 \times 15 \text{ in.}}{3}\right] = 1700 \text{ in.}^4$$

$$K_t = \frac{9E_{cs}C}{\left[\ell_2\left(1 - \frac{c_2}{\ell_2}\right)^3\right]} = \frac{(9)(3.64 \times 10^6 \text{ in.-lb})(1700 \text{ in.}^4)}{\left\{(12 \text{ in./ft})(18 \text{ ft})\left[1 - \frac{15 \text{ in.}}{(12 \text{ in./ft})(22 \text{ ft})}\right]^3\right\}} = 307.3 \times 10^6 \text{ in.-lb}$$

7. Compute K_{ec}, the stiffness of the equivalent column

$$K_{ec} = \frac{\Sigma K_c \Sigma K_t}{\Sigma K_c + \Sigma K_t} = \frac{(2 \times 648.0)(2 \times 307.3)}{2 \times 648.0 + 2 \times 307.3}$$
$$= 416.9 \times 10^6 \text{ in.-lb}$$

A summary of the stiffness values is shown in Figure 17.11.

8. Computing distribution factors and balancing moments (see Figure 17.12): The authors do not show moments at tops and bottoms of columns, but this could easily be done by multiplying the balanced column moments at the joints with the slabs by the carryover factor for the columns, which is 0.55.

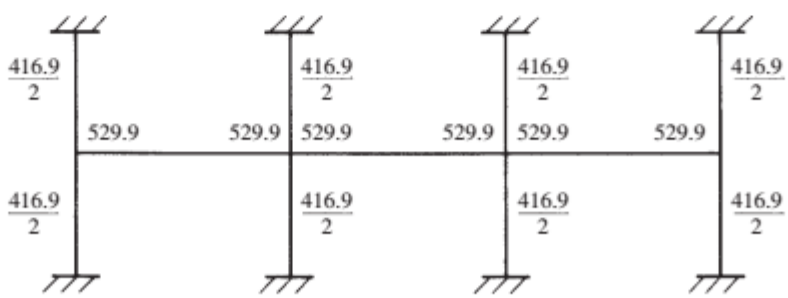


FIGURE 17.11 Equivalent frame with stiffnesses (in.-lb).

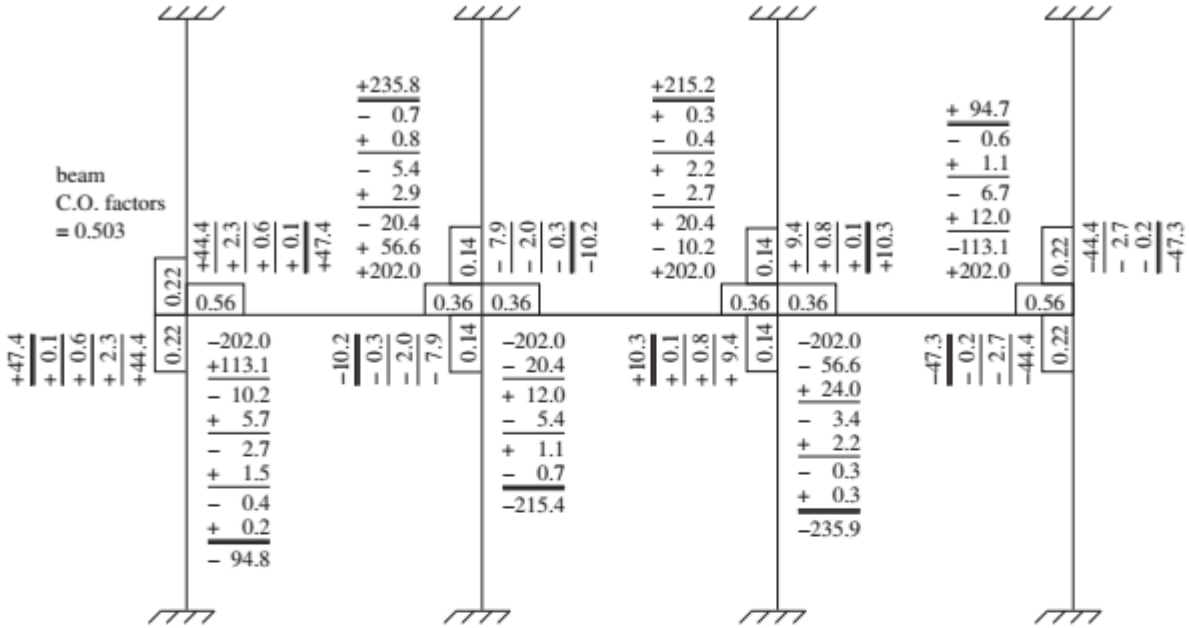


FIGURE 17.12 Results of moment distribution (moments, ft-k).

A summary of the moment values for Example 17.2 is given in Figure 17.13. The positive moments shown in each span are assumed to be equal to the simple beam centerline moments plus the average of the end negative moments. This is correct if the end moments in a particular span are equal and is approximately correct if the end moments are unequal. For span 1,

$$^{+}M = \frac{(0.276 \text{ ksf})(18 \text{ ft})(22 \text{ ft})^2}{8} - \left(\frac{94.8 \text{ ft-k} + 235.8 \text{ ft-k}}{2}\right) = 135.3 \text{ ft-k}$$

The negative moments shown in Figures 17.12 and 17.13 were calculated at the centerlines of the supports. At these supports, the cross section of the slab beam is very large because of the presence of the column. At the face of the column, however, the cross section is far smaller, and Section 8.11.6.1 of the ACI Code specifies that negative reinforcement be designed for the moment there. (If the column is not rectangular, it is replaced with a square column of the same total area and the moment is computed at the face of that fictitious column.) Because the ratio of unfactored dead to live load is less than 0.75, Section 6.4.3.2 of the ACI Code permits a single analysis with live load for all spans. No pattern load analysis is required.

The design moments shown in Figure 17.14 were determined by drawing the shear diagram and computing the area of that diagram between the centerline of each support and the face of the column.

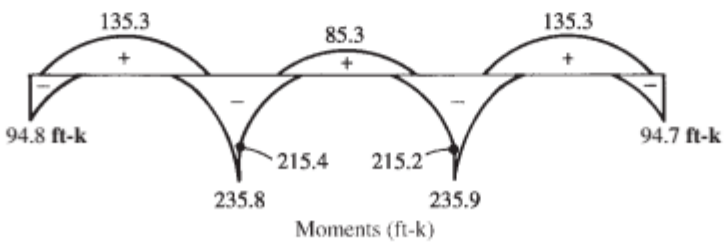


FIGURE 17.13 Results of Example 17.2.

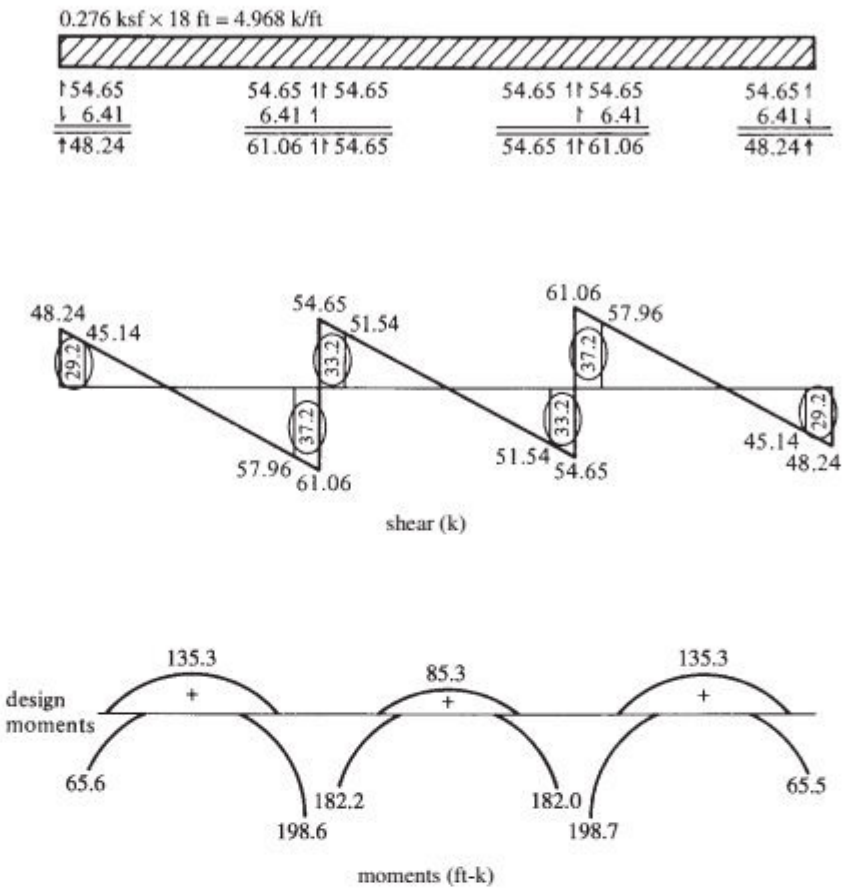


FIGURE 17.14 Final shear and moment diagrams for Example 17.2.

For interior columns, the critical section (for both column and middle strips) is to be taken at the face of the supports, but not at a distance greater than $0.175\ell_1$ from the center of the column. At exterior supports with brackets or capitals, the moment used in the span perpendicular to the edge shall be computed at a distance from the face of the support element not greater than one-half of the projection of the bracket or capital beyond the face of the supporting element.

Sometimes the total of the design moments (i.e., the positive moment plus the average of the negative-end moments) obtained by the equivalent frame method for a particular span may be greater than $M_o = q_u\ell_2\ell_n^2/8$, as used in Chapter 16. Should this happen, Section 8.11.6.5 of the ACI Code permits a reduction in those moments proportionately, so their sum does equal M_o.

17.6 Computer Analysis

The equivalent frame method was developed with the intention that the moment distribution method was to be used for the structural analysis. Truthfully, the method is so involved that it is not satisfactory for hand calculations. It is possible, however, to use computers and plane frame analysis programs if the structure is especially modeled. (In other words, we must establish various nodal points in the structure so as to account for the changing moments of inertia along the member axes.) There are also some computer programs on the market especially written for these frames. One of the best known is called ADOSS (analysis and design of concrete floor systems) and was prepared by the Portland Cement

Association. The moments shown in Figure 17.14 can be entered in the Chapter 16 Excel spreadsheet, Two-Way Slab, in cells 12C to 12G for interior spans and 45C to 45G for edge spans.

17.7 Computer Example

Example 17.3

Use the Excel spreadsheet provided for Chapter 17 to solve Example 17.2.

SOLUTION

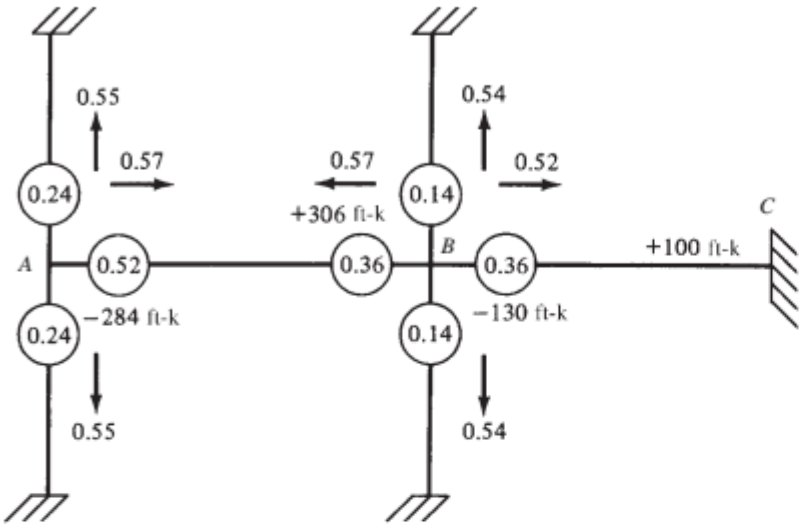
Open the Chapter 17 Excel spreadsheet, and open the worksheet Moment Distribution. Enter values only for cells highlighted in yellow (only in the Excel spreadsheets, not in the printed example). Results of the moment distribution are shown at the bottom of each column. Note that they are in agreement with those in Example 17.2.

Moment Distribution for User Input Distribution and Carryover Factors and Fixed-End Moments

	A	*B*		*C*		*D*
Joint member	*AB*	*BA*	*BC*	*CB*	*CD*	*DC*
Distribution factor	0.56	0.36	0.36	0.36	0.36	0.56
Fixed-end moment	−202	202	−202	202	−202	202
Carryover	0.503	0.503	0.503	0.503	0.503	0.503
Balance	113.12	0	0	0	0	−113.12
Carryover	0.000	56.899	0.000	0.000	−56.899	0.000
Balance	0.000	−20.484	−20.484	20.484	20.484	0.000
Carryover	−10.303	0.000	10.303	−10.303	0.000	10.303
Balance	5.770	−3.709	−3.709	3.709	3.709	−5.770
Carryover	−1.866	2.902	1.866	−1.866	−2.902	1.866
Balance	1.045	−1.716	−1.716	1.716	1.716	−1.045
Carryover	−0.863	0.526	0.863	−0.863	−0.526	0.863
Balance	0.483	−0.500	−0.500	0.500	0.500	−0.483
Carryover	−0.252	0.243	0.252	−0.252	−0.243	0.252
Balance	0.141	−0.178	−0.178	0.178	0.178	−0.141
Carryover	−0.090	0.071	0.090	−0.090	−0.071	0.090
Balance	0.050	−0.058	−0.058	0.058	0.058	−0.050
Carryover	−0.029	0.025	0.029	−0.029	−0.025	0.029
Balance	0.016	−0.020	−0.020	0.020	0.020	−0.016
Final moments, ft-k	−94.78	236.00	−215.26	215.26	−236.00	94.78

PROBLEMS

Problem 17.1 Determine the end moments for the beams and columns of the frame shown for which the fixed-end moments, carryover factors, and distribution factors (circled) have been computed. Use the moment distribution method. (*Ans*. 44.9 ft-k, 21.0 ft-k at column bases)



Problem 17.2 Repeat Problem 16.4 using the equivalent frame method instead of the direct design method.

Problem 17.3 Use Chapter 17 spreadsheet to determine the end moments in members *AB* and *BC* of Problem 17.1. (*Ans*. $M_{AB} = -163.6$ ft-k, $M_{BA} = 307.2$ ft-k, $M_{BC} = -229.7$ ft-k, $M_{CB} = 48.1$ ft-k)

Walls

CHAPTER 18

18.1 Introduction

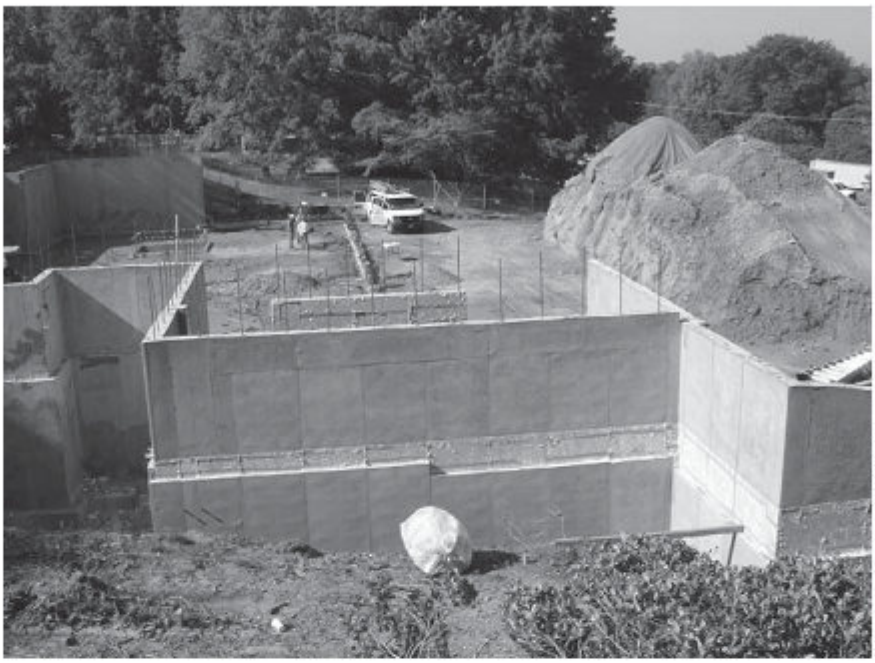
Before the advent of frame construction during the nineteenth century, most walls were of a load-bearing type. Since the late 1800s, however, the non-load-bearing wall has become quite common because other members of the structural frame can be used to provide stability. As a result, today we have walls that serve all sorts of purposes, such as retaining walls, basement walls, partition walls, fire walls, and so on. These walls may or may not be of a load-bearing type.

In this chapter, the following kinds of concrete walls will be considered: non-load-bearing, load-bearing, and shear walls (the latter being either load-bearing or non–load-bearing).

18.2 Non-Load-Bearing Walls

Non-load-bearing walls are those that support only their own weight and perhaps some lateral loads. Falling into this class are retaining walls, façade-type walls, and some basement walls. For non-load-bearing walls, the ACI Code provides several specific limitations, which are listed at the end of this paragraph. The values given for minimum reinforcement quantities and wall thicknesses do not have to be met if lesser values can be proved satisfactory by structural analysis (11.3.1.1). The numbers given in parentheses are ACI section numbers.

1. The thickness of a non-load-bearing wall cannot be less than 4 in. or $\frac{1}{30}$ times the least distance between members that provide lateral support (11.3.1.1).
2. The minimum amount of longitudinal reinforcement as a percentage of gross concrete area is 0.0012 for deformed bars #5 or smaller with f_y = at least 60,000 psi, 0.0015 for other deformed bars, and 0.0012 for plain or deformed welded wire fabric not larger than $W31$ or $D31$—that is, $\frac{5}{8}$ in. in diameter (11.6.1).
3. The longitudinal reinforcement does not have to be enclosed by ties unless the percentage of longitudinal reinforcement is greater than 0.01 times the gross concrete area or where the longitudinal reinforcement is not required as compression reinforcement (11.7.4.1).
4. The minimum amount of transverse reinforcement as a percentage of gross concrete area is 0.0020 for deformed bars #5 or smaller with $f_y \geq$ 60,000 psi, 0.0025 for other deformed bars, and 0.0020 for plain or deformed welded wire fabric not larger than $W31$ or $D31$ (11.6.1).
5. The spacing of longitudinal and transverse reinforcement may not exceed three times the wall thickness, or 18 in. (11.7.2.1).
6. Reinforcement for walls more than 10 in. thick (not including basement walls) must be placed in two layers as follows: one layer containing from one-half to two-thirds of the total reinforcement placed in the exterior surface not less than 2 in. nor more than one-third times the wall thickness from the exterior surface; the other layer placed not less than $\frac{3}{4}$ in. nor more than one-third times the wall thickness from the interior surface (11.7.2.3).



Retaining wall with stepped wall thickness.

7. For walls less than 10 in. thick, the code (11.7.2.3) does not specify two layers of steel, but to control shrinkage, it is probably a good practice to put one layer on the face of walls exposed to view and one on the nonstressed side of foundation walls 10 ft or more in height.
8. In addition to the reinforcement specified in the preceding paragraphs, at least two #5 bars in walls having two layers of reinforcement in both directions, and one #5 bar in walls having a single layer of reinforcement in both directions, must be provided around all window, door, and similar-sized openings. These bars must be anchored to develop f_y in tension at the corners of the openings (11.7.5.1).
9. For cast-in-place walls, the reinforcement area across the interface between a wall and a footing must not be less than the minimum longitudinal wall reinforcement as given in the ACI 318 Code (Sections 11.6.1 and 16.3.4.2).
10. For precast, non-prestressed walls, the reinforcement must be designed in accordance with the preceding requirements on this list, except that the area of the transverse and longitudinal reinforcement must not be less than 0.001 times the gross cross-sectional area of the wall (ACI Table 11.6.1). In addition, the spacing of the reinforcement may not be greater than five times the wall thickness or 30 in. for interior walls, or 18 in. for exterior ones (11.7.2.2).

18.3 Load-Bearing Concrete Walls—Empirical Design Method

Most of the concrete walls in buildings are load-bearing walls that support not only vertical loads but also some lateral moments. As a result of their considerable in-plane stiffnesses, they are quite important in resisting wind and earthquake forces.

Load-bearing walls with solid rectangular cross sections may be designed as were columns subject to axial load and bending, or they may be designed by an empirical method given in Section 11.5.3 of the ACI Code. The empirical method may be used only if the resultant of all the factored loads falls within the middle third of the wall (i.e., the eccentricity must be equal to or less than one-sixth the thickness of the wall). *Whichever of the two methods is used, the design must meet the minimum requirements given in the preceding section of this chapter for non-load-bearing walls.*

Courtesy of EFCO Corp.

Pump station construction in Belle Chasse, Louisiana is part of the Gulf Intracoastal Waterway-West Closure Complex construction project to protect New Orleans against flooding due to potential storm surge.

This section is devoted to the empirical design method, which is applicable to relatively short vertical walls with approximately concentric loads. Section 11.5.3.1 of the ACI Code provides an empirical formula for calculating the design axial load strength of solid rectangular cross-sectional walls with e less than one-sixth of wall thicknesses. Should walls have nonrectangular cross sections (such as ribbed wall panels) and/or should e be greater than one-sixth of wall thicknesses, the rational design procedure for columns subject to axial load and bending (Chapter 10 of this text) must be followed.

The practical use of the empirical wall formula, which is given at the end of this paragraph, is for relatively short walls with small moments. When lateral loads are involved, e will quickly exceed one-sixth of wall thicknesses. The number 0.55 in the equation is an eccentricity factor that causes the equation to yield a strength approximately equal to that which would be obtained by the axial load and bending procedure in Section 11.5.2 of the ACI Code if the eccentricity is $h/6$.

$$\phi P_{nw} = 0.55\phi f'_c A_g \left[1 - \left(\frac{k\ell_c}{32h}\right)^2\right] \qquad \text{(ACI Equation 11.5.3.1)}$$

where

$\phi = 0.65$

A_g = gross area of the wall section (in.^{2})

ℓ_c = vertical distance between supports (in.)

h = overall thickness of member (in.)

k = effective length factor determined in accordance with the values given in Table 18.1

Other ACI requirements for load-bearing concrete walls designed by the empirical formula follow. The numbers given in parentheses are ACI section numbers.

1. The thickness of the walls may not be less than $\frac{1}{25}$ the supported height or length, whichever is smaller, or less than 4 in. (11.3.1.1b).
2. The thickness of exterior basement walls and foundation walls may not be less than $7\frac{1}{2}$ in. (11.3.1.1e).

TABLE 18.1 Effective Length Factors for Load-Bearing Walls (ACI Table 11.5.3.2)

1. Walls braced top and bottom against lateral translation and	
(a) Restrained against rotation at one or both ends (top and/or bottom)	0.80
(b) Not restrained against rotation at either end	1.0
2. For walls not braced against lateral translation	2.0

3. The horizontal length of a wall that can be considered effective for each concentrated load may not exceed the smaller of the center-to-center distance between loads or the bearing width plus four times the wall thickness. This provision may be waived if a larger value can be proved satisfactory by a detailed analysis (ACI 11.2.3.1).
4. Load-bearing walls must be anchored to intersecting elements, such as floors or roofs, or they should be anchored to columns, pilasters, footings, buttresses, and intersecting walls (ACI 11.2.4.1).

The empirical method is quite easy to apply because only one calculation has to be made to determine the design axial strength of a wall. Example 18.1, which follows, illustrates the design of a bearing wall with a small moment.[1]

Example 18.1

Design a concrete-bearing wall using the ACI empirical equation 11.5.3.1 to support a set of precast concrete roof beams 7 ft 0 in. on center, as shown in Figure 18.1. The bearing width of each beam is 10 in. The wall is considered to be laterally restrained top and bottom and is further assumed to be restrained against rotation at the footing; thus $k = 0.8$. Neglect wall weight. Other data: $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, beam reaction, $D = 30$ k, $L = 18$ k.

SOLUTION

1. Determine Minimum Wall Thickness (14.5.3.1)

(a) $h = \left(\frac{1}{25}\right)(12 \text{ in./ft} \times 16 \text{ ft}) = 7.68 \text{ in.} \leftarrow$

(b) $h = 4$ in.

Try 8 in.

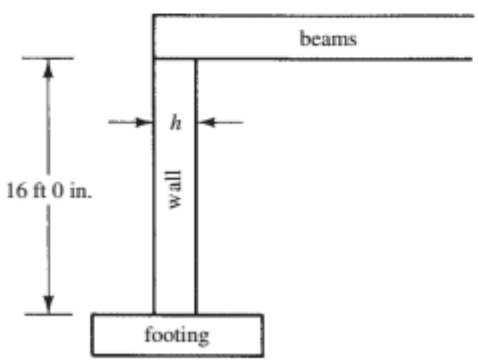


FIGURE 18.1 Empirically designed wall for Example 18.1.

[1] For the example problems presented in this chapter, the authors have followed the general procedures used in B. G. Rabbat, et al., ed., *Notes on ACI 318–08 Building Code Requirements for Structural Concrete*, 2008 (Skokie, IL: Portland Cement Association), pp. 21-17 to 21-18.

Compute factored beam reactions

$$P_u = (1.2)(30\text{ k}) + (1.6)(18\text{ k}) = 64.8\text{ k}$$

2. Is the Bearing Strength of Wall Concrete Satisfactory Under Beam Reactions (10.17.1)?

$$\phi(0.85f'_cA_1) = (0.65)(0.85)(3\text{ ksi})(8\text{ in.} \times 10\text{ in.})$$
$$= 132.6\text{ k} > 64.8\text{ k} \quad \underline{\text{OK}}$$

3. Horizontal Length of Wall to be Considered as Effective in Supporting Each Concentrated Load (14.2.4)

(a) Center-to-center spacing of beams 7 ft 0 in. = 84 in.

(b) Width of bearing + $4h$ = 10 in. + (4)(8 in.) = 42 in. ←

4. Design Strength of Wall

$$\phi P_{nw} = 0.55\phi f'_cA_g\left[1 - \left(\frac{k\ell_c}{32h}\right)^2\right] \qquad \text{(ACI Equation 11.5.3.1)}$$

$$= (0.55)(0.65)(3\text{ ksi})(8\text{ in.} \times 42\text{ in.})\left[1 - \left(\frac{0.80 \times 12\text{ in./ft} \times 16\text{ ft}}{32 \times 8\text{ in.}}\right)^2\right]$$

$$= 230.6\text{ k} > 64.8\text{ k} \quad \underline{\text{OK}}$$

5. Select Reinforcing (14.3.5, 14.3.2, and 14.3.3)

Maximum spacing = (3)(8 in.) = 24 in. or <u>18 in.</u>

Vertical $A_s = (0.0012)(12\text{ in.})(8\text{ in.}) = 0.115\text{ in.}^2\text{/ft}$ <u>#4 bars@18 in.</u>

Horizontal $A_s = (0.0020)(12\text{ in.})(8\text{ in.}) = 0.192\text{ in.}^2\text{/ft}$ <u>#4 bars@12 in.</u>

Although the code is not specific on this issue, it would be prudent to provide continuity of the vertical wall reinforcement into the footing. This is usually accomplished by using a hooked bar embedded in the wall footing that is lap spliced with the vertical wall bars. If a value of $k = 1$ had been used in this example, there would have been no assumption of continuity at the base of the wall. It would still be necessary to ensure that adequate shear capacity is provided at the base of the wall.

18.4 Load-Bearing Concrete Walls—Rational Design

Reinforced concrete-bearing walls may be designed as columns by the rational method of Chapter 10 of the text whether the eccentricity is smaller or larger than $h/6$ (they must be designed rationally if $e > h/6$). The minimum longitudinal and transverse reinforcement requirements of Section 11.6.1 of the ACI Code must be met.

The design of walls as columns is difficult unless design aids are available. Various wall design aids are available from the Portland Cement Association, but the designer can prepare his or her own aids, such as axial load and bending interaction diagrams. The designs may be complicated by the fact that walls often will be classified as "long columns" with the result that the slenderness requirements of Section 6.6.4 of the ACI Code will have to be met. An alternative procedure for slender walls is presented in Section 11.8 of the ACI Code.

Very slender walls are rather common, particularly in tilt-up wall construction. The Portland Cement Association has made available a design aid that is particularly useful for such cases.[2]

[2] Portland Cement Association, 2005. *The Tilt-up Construction and Engineering Manual*, 6th ed. (Skokie, IL), 28 pages.

The interaction diagrams discussed in Section 10.6 of this text can be used to design walls with steel in two layers and subject to out-of-plane bending combined with axial loads. However, the reinforcement ratio is limited to 0.01, unless the compression reinforcement is laterally tied (ACI Code, Section 11.7.4.1), which is impractical in many cases. Graphs 2 through 5 in Appendix A are applicable to walls that have two layers of steel. The values of γ for these graphs may be too large, however, especially for thinner walls. Example 18.2 illustrates how to use these design aids when designing walls.

Example 18.2

Design the reinforced concrete foundation wall shown in Figure 18.2 that has the following conditions: $\ell = 15$ ft between lateral supports, backfill height is also 15 ft, $P_D = 520$ plf at $e = 2$ in., and $P_L = 250$ plf at $e = 2$ in. Assume the base is pinned.

$f'_c = 4000$ psi, normal-weight aggregate concrete, Grade 60 reinforcing steel

Soil properties, $k_a = 0.40$, $\gamma = 100$ pcf

SOLUTION

Assuming that the wall is simply supported at both the top and bottom, the soil pressure exerted on the wall increases linearly from zero at the top of backfill to $k_a \gamma \ell$ at the base. The maximum bending moment[3] is

$$M_H = 0.064 k_a \gamma \ell^3 = (0.064)(0.4)(100 \text{ pcf})(15 \text{ ft})^3$$
$$= 8640 \text{ ft-lb/ft} = 103{,}680 \text{ in.-lb/ft}$$

Maximum moment occurs at $0.577\ell = 8.66$ ft from top of wall or 6.34 ft from the bottom. Note that the eccentric axial loads cause a reduction in the moment. In the case of dead axial load, the moment at the top of the wall is $P \times e = 520$ plf(2 in.) = 1040 in.-lb/ft. This moment varies linearly to zero at the base of the wall, so at the location of maximum soil moment, its value is 1040 in.-lb(6.34 ft)/15 ft = 440 in.-lb/ft. The same analysis applied to the live load results in a moment of 212 in.-lb/ft. In this case, the reduction in moment from eccentric axial load is small (less than 3%) and could be ignored. At the location of maximum bending moment, the axial loads are

$$P_D = 520 \text{ plf} + (8.66 \text{ ft})(150 \text{ pcf})(8 \text{ in.})(12 \text{ in.})/144 \text{ in.}^2/\text{ft}^2$$
$$= 1386 \text{ plf (assuming an 8 in. wall thickness)}$$
$$P_L = 250 \text{ plf}$$

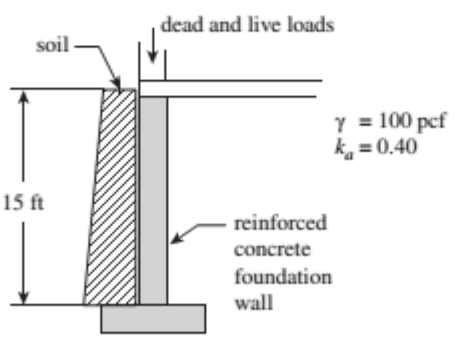


FIGURE 18.2 Foundation wall for Example 18.2.

[3] McCormac, J. C., 2006, *Structural Analysis: Using Classical and Matrix Methods*, 4th ed. (Hoboken, NJ: John Wiley & Sons), pp. 104–106.

Three load combinations applicable to this situation are shown in the table below along with K_n and R_n values used in conjunction with Graph 2 in Appendix A.

	P_D	P_L	P_H	M_D	M_L	M_H	P_U	M_U	$K_n = \frac{P_U}{\phi f'_c A_g}$	$R_n = \frac{M_U}{\phi f'_c A_g h}$	ρ_t
Unfactored	1386	250	0	600	288	103,680					
$U = 1.4D$	1.4			1.4			1940	840	0.0056	0.0003	< 0.01
$U = 1.2D + 1.6L + 1.6H$	1.2	1.6	1.6	1.2	1.6	1.6	2063	166,877	0.0060	0.0604	0.01
$U = 0.9D + 1.6H$	0.9		1.6	0.9		1.6	1247	166,284	0.0036	0.0602	0.01

There is so little difference between the second and third load case that the difference in ρ_t is indistinguishable when reading the graph. Also, a value of $\phi = 0.9$ was used because when reading the graph, it is obvious that the controlling points are below the line for $\epsilon_t = 0.005$. Using the resulting value of ρ_t, $A_{st} = \rho_t bh = 0.010(12 \text{ in.})(8 \text{ in.}) = 0.96 \text{ in.}^2/\text{ft}$, half in each face ($0.48 \text{ in.}^2/\text{ft}$). From Appendix A, Table A.6, select #5 bars at $7\frac{1}{2}$ in. o.c. vertically in both faces. The #5 bar size was picked because the required cover increases for larger bars from $1\frac{1}{2}$ to 2 in. Horizontal reinforcement must be provided in accordance with Section 11.6.1 of the ACI Code, $A_s \geq 0.0025bh = 0.0025(12 \text{ in.})(8 \text{ in.}) = 0.24 \text{ in.}^2/\text{ft}$. Choose #4 bars at 18 in. o.c. horizontally in both faces. Because this wall is less than 10 in. thick, Section 11.7.2.3 of the ACI Code permits the reinforcement to be placed in a single layer. However, Graph 2 is based on having steel in two layers separated by a distance of $\gamma h = 0.6(8) = 4.8$ in. Actually, a value of $\gamma = 0.5$ would have been a better choice in this problem in order to provide sufficient cover, but it is not available.

Because the axial load is so small in this case, the wall could have been designed as a vertical beam with compression steel. The beam width would be 12 in., $h = 8$ in., $d = 6.4$ in., and the moment taken as 167 in.-k. The authors carried out this design approach using the Chapter 5 spreadsheet and calculated a required area of reinforcing steel of $0.98 \text{ in.}^2/\text{ft}$ (compared with $0.96 \text{ in.}^2/\text{ft}$ from Graph 2).

The wall exerts reactions at the bottom against the footing and at the top against the floor. The floor system must be designed to resist this horizontal force, which is called a "diaphragm force." At the bottom, the interface of the wall and the footing must be designed for shear transfer between these elements. Shear friction (Section 8.12 of this text) is the best way to accommodate this transfer.

Note that this wall has a slenderness ratio of $kl_u/r = 1.0(15 \text{ ft})12 \text{ in./ft}/(0.3)(8 \text{ in.}) = 75 > 34 - 12(0/M_2) = 34$, so slenderness must be considered. However, because the axial loads are so small, the moment magnifier is not greater than 1.0; hence the moment is not magnified.

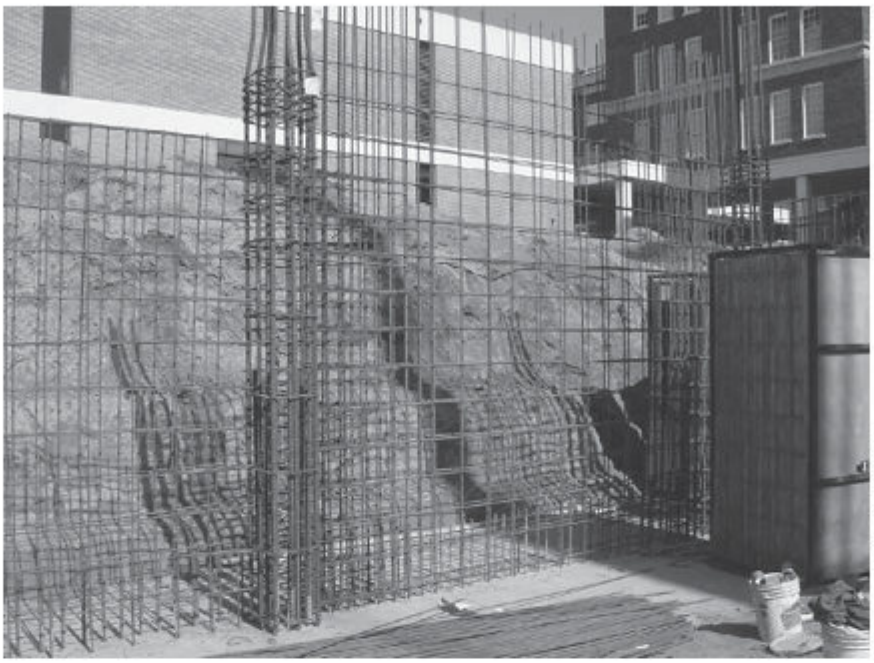
If this wall were not laterally supported at the top by the floor shown in Figure 18.2, it would be a retaining wall. The bending moment from soil pressure on the retaining wall would be $M_H = k_a \gamma \ell^3/6$, which is 260% of the moment in Example 18.2. Be sure to use the correct moment for the boundary conditions that apply to your type of construction.

If the reinforcement ratio had exceeded 0.01, this method of solution would not have been valid. It would have been necessary to use mechanics to determine the wall capacity without using steel in compression. The design aids used in this example use the compression force in the reinforcing steel, hence the steel has to be laterally tied for this solution to be valid.

18.5 Shear Walls

For tall buildings, it is necessary to provide adequate stiffness to resist the lateral forces caused by wind and earthquake. When such buildings are not properly designed for these forces, there may be very high stresses, vibrations, and sidesway when the forces occur. The results may include not only severe damages to the buildings but also considerable discomfort for their occupants.

When reinforced concrete walls with their very large in-plane stiffnesses are placed at certain convenient and strategic locations, often they can be economically used to provide the needed resistance



Shear wall with integral end columns—Rhodes Annex.

to horizontal loads. Such walls, called *shear walls*, are in effect deep vertical cantilever beams that provide lateral stability to structures by resisting the in-plane shears and bending moments caused by the lateral forces.

Because the strength of shear walls is almost always controlled by their flexural resistance, their name is something of a misnomer. It is true, however, that on some occasions they may require some shear reinforcement to prevent diagonal tension failures. Indeed, one of the basic requirements of shear walls designed for high seismic forces is to ensure flexure rather than shear-controlled design.

The usual practice is to assume that the lateral forces act at the floor levels. The stiffnesses of the floor slabs horizontally are quite large as compared to the stiffnesses of the walls and columns. Thus, it is assumed that each floor is displaced in its horizontal plane as a rigid body.

Figure 18.3 shows the plan of a building that is subjected to horizontal forces. The lateral forces, usually from wind or earthquake loads, are applied to the floor and roof slabs of the building, and those slabs, acting as large beams lying on their sides or diaphragms, transfer the loads primarily to the shear walls A and B. Should the lateral forces be coming from the other (perpendicular) direction, they would be resisted primarily by the shear walls C and D.

The walls must be sufficiently stiff so as to limit deflections to reasonable values.

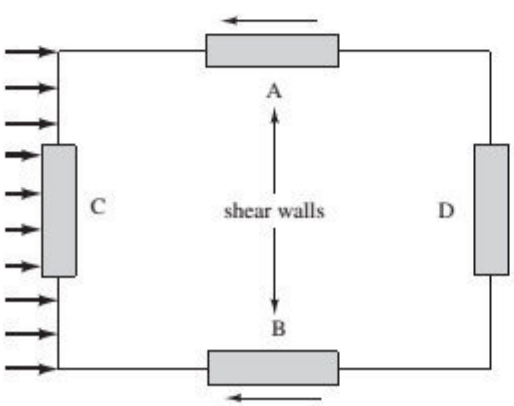


FIGURE 18.3 Plan view of a floor supported by shear walls.

Courtesy of Professor Larry Kahn, Georgia Institute of Technology.

Buttress shear wall, Grace Building, New York, NY.

Shear walls are commonly used for buildings with flat-plate floor slabs. In fact, this combination of slabs and walls is the most common type of construction used today for tall apartment buildings and other residential buildings.

Shear walls span the entire vertical distances between floors. If the walls are carefully and symmetrically placed in plan, they will efficiently resist both vertical and lateral loads and do so without interfering substantially with the architectural requirements. Reinforced concrete buildings of up to 70 stories have been constructed with shear walls as their primary source of lateral stiffness. In the horizontal direction, full shear walls may be used—that is, they will run for the full panel or bay lengths. When forces are smaller, they need only run for partial bay lengths.

Shear walls may be used to resist lateral forces only, or they may be used in addition as bearing walls. Furthermore, they may be used to enclose elevators, stairwells, and perhaps restrooms, as shown in Figure 18.4. These box-type structures are very satisfactory for resisting horizontal forces.

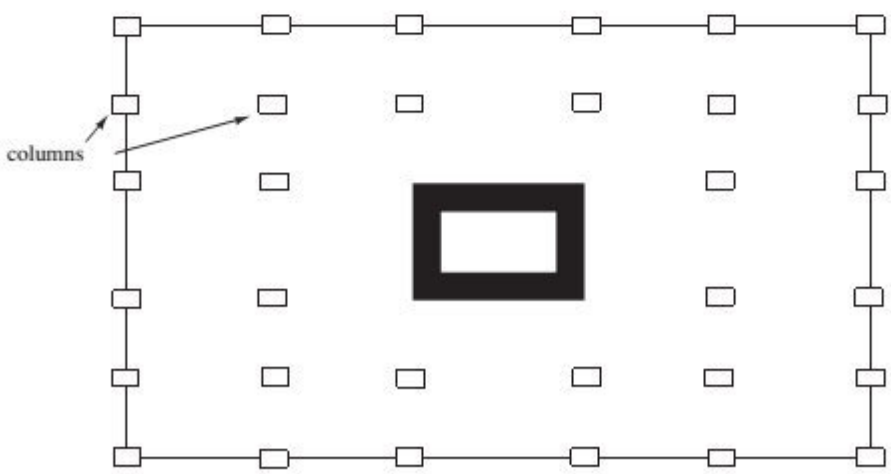


FIGURE 18.4 Shear walls around elevators and stairwells.

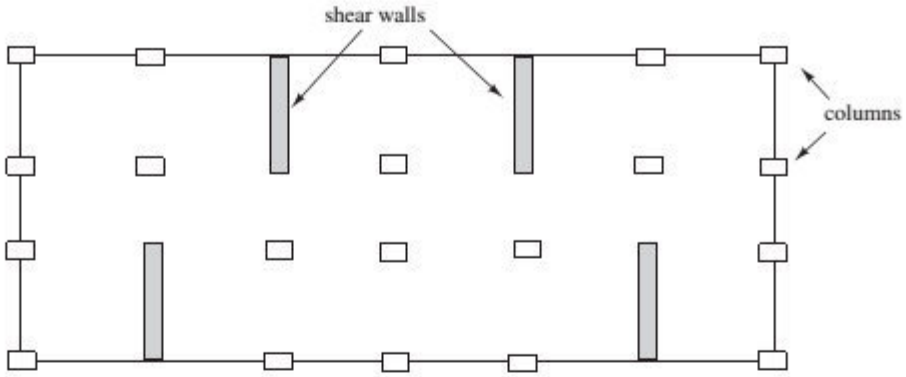


FIGURE 18.5 Building plan having shear walls in only one direction.

Another possible arrangement of shear walls is shown in Figure 18.5. Although shear walls may also be needed in the long direction of this building, they are not included in this figure.

On most occasions, it is not possible to use shear walls without some openings in them for doors, windows, and penetrations for mechanical services. Usually it is possible, however, with careful planning to place these openings so they do not seriously affect stiffnesses or stresses in the walls. When the openings are small, their overall effect is minor, but this is not the case when large openings are present.

Usually the openings (windows, doors, etc.) are placed in vertical and symmetrical rows in the walls throughout the height of the structure. The wall sections on the sides of these openings are tied together by beams enclosed in the walls, by the floor slabs, or by a combination of both. As you can see, the structural analysis for such a situation is extremely complicated. Although shear wall designs are often handled with empirical equations, they can be appreciably affected by the designer's previous experience.

When earthquake-resistant construction is being considered, note that the relatively stiff parts of a structure will attract much larger forces than will the more flexible parts. A structure with reinforced concrete shear walls is going to be quite stiff and, thus, will attract large seismic forces. If the shear walls are brittle and fail, the rest of the structure may not be able to take the shock. If the shear walls are ductile, however (and they will be if properly reinforced), they will be very effective in resisting seismic forces.

Tall reinforced concrete buildings are often designed with shear walls to resist seismic forces, and such buildings have performed quite well in recent earthquakes. During an earthquake, properly designed shear walls will decidedly limit the amount of damage to the structural frame. They will also minimize damages to the nonstructural parts of a building, such as the windows, doors, ceilings, and partitions.

Figure 18.6 shows a shear wall subjected to a lateral force, V_u. The wall is in actuality a cantilever beam of width h and overall depth ℓ_w. In part (a) of the figure, the wall is being bent from left to right by V_u, with the result that tensile bars are needed on the left or tensile side. If V_u is applied from the right side as shown in part (b) of the figure, tensile bars will be needed on the right-hand end of the wall. Thus, it can be seen that a shear wall needs tensile reinforcement on both sides because V_u can come from either direction. For horizontal shear calculations, the depth of the beam from the compression end of the wall to the center of gravity of the tensile bars is estimated to be about 0.8 times the wall length, ℓ_w, as per Section 11.5.4.2 of the ACI Code. (If a larger value of d is obtained by a proper strain compatibility analysis, it may be used.)

The shear wall acts as a vertical cantilever beam. In providing lateral support, it is subjected to both bending and shear forces. For such a wall, the maximum shear, V_u, and the maximum moment, M_u, can be calculated at the base. If flexural stresses are calculated, their magnitude will be affected by the design axial load, N_u, and thus its effect should be included in the analysis.

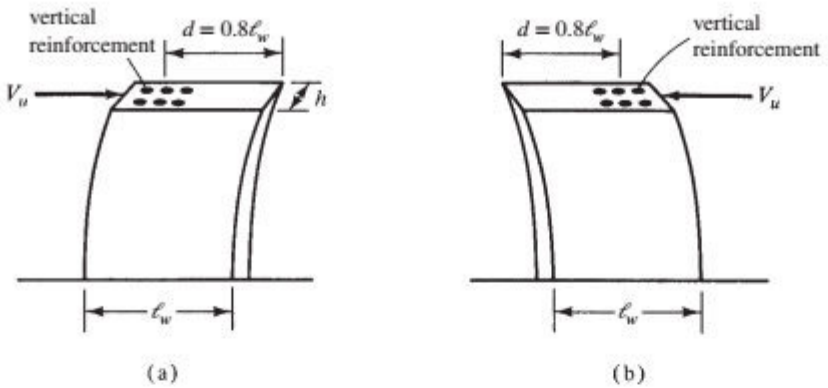


FIGURE 18.6 Shear walls loaded in opposite directions.

Shear is more important in walls with small height-to-length ratios. Moments will be more important for higher walls, particularly those with uniformly distributed reinforcement.

It is necessary to provide both horizontal and vertical shear reinforcement for shear walls. The commentary (R11.6.2) says that in low walls, the horizontal shear reinforcement is less effective, and the vertical shear reinforcement is more effective. For high walls, the situation is reversed. This situation is reflected in ACI Equation 22.6.5.2c, which is presented in the next section. The vertical shear reinforcement contributes to the shear strength of a wall by shear friction.

Reinforcing bars are placed around all openings, regardless of whether structural analysis indicates a need for them. Such a practice is deemed necessary to prevent diagonal tension cracks, which tend to develop radiating from the corners of openings.

18.6 ACI Provisions for Shear Walls

The numbers given in parentheses are ACI section numbers.

1. The factored beam shear must be equal to or less than the design shear strength of the wall.

$$V_u \le \phi V_n$$

2. The design shear strength of a wall is equal to the design shear strength of the concrete plus that of the shear reinforcement.

$$V_u \le \phi V_c + \phi V_s$$

3. The nominal shear strength, V_n, at any horizontal section in the plane of the wall may not be taken greater than $10\sqrt{f'_c}hd$ (11.5.4.3).

4. In designing for the horizontal shear forces in the plane of a wall, d is to be taken as equal to $0.8\ell_w$, where ℓ_w is the horizontal wall length between faces of the supports, unless it can be proved to be larger by a strain compatibility analysis (11.5.4.2).

5. Section 11.5.4.5 states that unless a more detailed calculation is made (as described in the next paragraph), the value of the nominal shear strength, V_c, used may not be larger than $2\lambda\sqrt{f'_c}hd$ for walls subject to a factored axial compressive load, N_u. Should a wall be subject to a tensile load, N_u, the value of V_c may not be larger than the value obtained with the following equation:

$$V_c = 2\left(1 + \frac{N_u}{500A_g}\right)\lambda\sqrt{f'_c}b_w d \ge 0 \qquad \text{(ACI Table 11.5.4.6b)}$$

6. Using a more detailed analysis, the value of V_c is to be taken as the smaller value obtained by substituting into the two equations that follow, in which N_u is the factored axial load normal to the cross section occurring simultaneously with V_u. N_u is to be considered positive for compression and negative for tension (11.5.4.6).

$$V_c = 3.3\lambda\sqrt{f'_c}hd + \frac{N_u d}{4\ell_w} \qquad \text{(ACI Table 11.5.4.6d)}$$

or

$$V_c = \left[0.6\lambda\sqrt{f'_c} + \frac{\ell_w\left(1.25\lambda\sqrt{f'_c} + 0.2N_u/\ell_w h\right)}{(M_u/V_u) - (\ell_w/2)}\right]hd \qquad \text{(ACI Table 11.5.4.6e)}$$

The first of these equations was developed to predict the inclined cracking strength at any section through a shear wall corresponding to a principal tensile stress of about $4\lambda\sqrt{f'_c}$ at the centroid of the wall cross section. The second equation was developed to correspond to an occurrence of a flexural tensile stress of $6\lambda\sqrt{f'_c}$ at a section $\ell_w/2$ above the section being investigated. Should $M_u/V_u - \ell_w/2$ be negative, the second equation will have no significance and will not be used.

In SI units, these last three equations are as follows:

$$V_c = \left(1 + \frac{0.3N_u}{A_g}\right)\frac{\lambda\sqrt{f'_c}}{6}b_w d \geq 0 \qquad \text{(ACI Table 11.5.4.6b)}$$

$$V_c = \frac{1}{4}\lambda\sqrt{f'_c}hd + \frac{N_u d}{4\ell_w} \qquad \text{(ACI Table 11.5.4.6d)}$$

$$V_c = \left[\frac{1}{2}\lambda\sqrt{f'_c} + \frac{\ell_w\left(\lambda\sqrt{f'_c} + 2N_u/\ell_w h\right)}{(M_u/V_u) - (\ell_w/2)}\right]\frac{hd}{10} \qquad \text{(ACI Table 11.5.4.6e)}$$

7. The values of V_c computed by the two preceding equations at a distance from the base equal to $\ell_w/2$ or $h_w/2$ (whichever is less) are applicable for all sections between this section and one at the wall base (11.5.4.7).

8. Should the factored shear, V_u, be less than $\phi V_c/2$ computed as described in the preceding two paragraphs, minimum longitudinal and transverse reinforcement must still be provided in accordance with Table 18.2 (ACI Table 11.6.1).

9. Should V_u be greater than ϕV_c, shear wall reinforcement must be designed as described in Section 11.5.4 of the ACI Code.

10. If the factored shear force, V_u, exceeds the shear strength, ϕV_c, the value of V_s is to be determined from the following expression, in which A_v is the area of the horizontal (transverse) shear reinforcement and s is the spacing of the shear or torsional reinforcement in a direction perpendicular to the transverse reinforcement (11.5.4.8).

$$V_s = \frac{A_v f_{yt} d}{s} \qquad \text{(ACI Equation 11.5.4.8)}$$

TABLE 18.2 Minimum Reinforcement for Walls with In-Plane $V_u \leq 0.5\phi V_c$ (ACI Table 11.6.1)

Wall type	Type of non-prestressed reinforcement	Bar/wire size	f_y, psi	Minimum longitudinal†, ρ_ℓ	Minimum transverse, ρ_t
Cast-in-place	Deformed bars	≤ No. 5	≥ 60,000	0.0012	0.0020
			< 60,000	0.0015	0.0025
		> No. 5	Any	0.0015	0.0025
	Welded wire reinforcement	≤ W31 or D31	Any	0.0012	0.0020
Precast*	Deformed bars or welded wire reinforcement	Any	Any	0.0010	0.0010

*In one-way precast, prestressed walls not wider than 12 ft. and not mechanically connected to cause restraint in the transverse direction, the minimum reinforcement requirement in the direction normal to the flexural reinforcement need not be satisfied.
†Prestressed walls with an average effective compressive stress of at least 225 psi need not meet the requirement for minimum longitudinal reinforcement, ρ_t.
Source: Reproduced with permission from the American Concrete Institute.

11. The maximum spacing of transverse shear reinforcement, s_2, shall not be greater than $\ell_w/5$, $3h$, or 18 in. (11.7.2.1).

12. The amount of longitudinal reinforcement, ρ_ℓ (as a percentage of the gross horizontal concrete area) shall not be less than the value given by the following equation, in which h_w is the total height of the wall (11.6.2).

$$\rho_\ell = 0.0025 + 0.5\left(2.5 - \frac{h_w}{\ell_w}\right)(\rho_t - 0.0025) \qquad \text{(ACI Equation 11.6.2)}$$

It shall not be less than 0.0025 but need not be greater than the required transverse shear reinforcement, ρ_t.

For high walls, the longitudinal reinforcement is much less effective than it is in low walls. This fact is reflected in the preceding equation, where for walls with a height/length ratio less than half, the amount of longitudinal reinforcement required equals the transverse reinforcement required. If the ratio is larger than 2.5, only a minimum amount of longitudinal reinforcement is required (i.e., $0.0025sh$).

13. The maximum spacing of vertical shear reinforcement shall not be greater than $\ell_w/3$, $3h$, or 18 in. (11.7.2).

14. Special Structural Walls are shear walls designed to comply with Section 18.10 of the ACI Code. Such walls are designed using the principles set forth in Section 18.5 of the text. However, to ensure the ductile behavior required of structural elements in higher seismic design categories, more restrictive requirements are imposed. Limitations on the amount of longitudinal reinforcement to ensure ductile performance are imposed. Design for shear is modified to make certain that a shear failure does not occur before the wall reaches its flexural capacity, as might occur if the wall's flexural strength exceeds its design flexural strength. Detailed design of special structural walls is beyond the scope of this text.

Example 18.3 illustrates the design of a reinforced concrete shear wall for both shear and flexure. In this example no axial load is applied to the shear wall.

Example 18.3

Design the reinforced concrete shear wall shown in Figure 18.7 if $f'_c = 3000$ psi and $f_y = 60{,}000$ psi.

SOLUTION

1. Is the Wall Thickness Satisfactory?

$$V_u = \phi 10\sqrt{f'_c}hd \qquad \text{(ACI Section 11.5.4.3)}$$

$$d = 0.8\ell_w = (0.8)(12 \times 10 \text{ ft}) = 96 \text{ in.} \qquad \text{(ACI Section 11.5.4.2)}$$

$$V_u = (0.75)(10)(\sqrt{3000} \text{ psi})(8 \text{ in.})(96 \text{ in.})$$

$$V_u = 315{,}488 \text{ lb} = 315.5 \text{ k} > 240 \text{ k} \quad \underline{\text{OK}}$$

2. Compute V_c for Wall (Lesser of Two Values)

(a) $$V_c = 3.3\lambda\sqrt{f'_c}hd + \frac{N_u d}{4\ell_w} = (3.3)(1.0)(\sqrt{3000} \text{ psi})(8 \text{ in.})(96 \text{ in.}) + 0$$

$$= 138{,}815 \text{ lb} = 138.8 \text{ k} \leftarrow \text{controls} \qquad \text{(ACI Equation 11.5.4.6d)}$$

(b) $$V_c = \left[0.6\lambda\sqrt{f'_c} + \frac{\ell_w(1.25\lambda\sqrt{f'_c} + 0.2\,N_u/\ell_w h)}{(M_u/V_u) - (\ell_w/2)}\right] hd \qquad \text{(ACI Equation 11.5.4.6e)}$$

Computing V_u and M_u at the lesser of $\ell_w/2 = 10/2 = 5$ ft or $h_w/2 = 14/2 = 7$ ft from base (ACI Code, Section 11.5.4.7):

$$V_u = 240 \text{ k}$$

$$M_u = 240 \text{ k}(14 \text{ ft} - 5 \text{ ft}) = 2160 \text{ ft-k} = 25{,}920 \text{ in.-k}$$

$$V_c = \left[(0.6)(1.0)(\sqrt{3000} \text{ psi}) + \frac{(12 \text{ in./ft} \times 10 \text{ ft})(1.25)(1.0)(\sqrt{3000} \text{ psi}) + 0}{\dfrac{25{,}920 \text{ in.-k}}{240 \text{ k}} - \dfrac{(12 \text{ in./ft})(10 \text{ ft})}{2}}\right](8 \text{ in.})(96 \text{ in.})$$

$$= 156{,}692 \text{ lb} = 156.7 \text{ k}$$

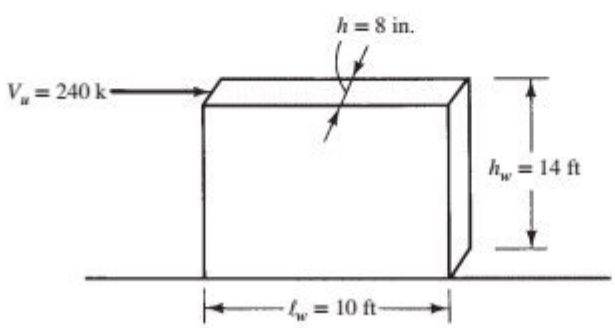


FIGURE 18.7 Shear wall for Example 18.3.

3. Is Shear Reinforcing Needed?

$$\frac{\phi V_c}{2} = \frac{(0.75)(1.0)(138.8\text{ k})}{2} = 52.05\text{ k} < 240\text{ k} \quad \underline{\text{Yes}}$$

4. Select Horizontal (Transverse) Shear Reinforcing

$$V_u = \phi V_c + \phi V_s$$

$$V_u = \phi V_c + \phi \frac{A_v f_y d}{s}$$

$$\frac{A_v}{s} = \frac{V_u - \phi V_c}{\phi f_y d} = \frac{240\text{ k} - (0.75)(138.8\text{ k})}{(0.75)(60\text{ ksi})(96\text{ in.})} = 0.0315\text{ in.}$$

Try different-sized horizontal bars with A_v = two-bar cross-sectional areas. Two layers of horizontal bars will be placed at the calculated spacing, hence A_v = twice the bar area. Compute s_2 = vertical spacing of horizontal shear reinforcement.

$$\text{Try \#3 bars: } s = \frac{(2)(0.11\text{ in.}^2)}{0.0315\text{ in.}} = 6.98\text{ in.}$$

$$\text{Try \#4 bars: } s = \frac{(2)(0.20\text{ in.}^2)}{0.0315\text{ in.}} = 12.70\text{ in.}$$

Maximum vertical spacing of horizontal shear reinforcement

$$\frac{\ell_w}{5} = \frac{(12\text{ in./ft})(10\text{ ft})}{5} = 24\text{ in.}$$

$$3h = (3)(8\text{ in.}) = 24\text{ in.}$$

$$18\text{ in.} = 18\text{ in.} \leftarrow \qquad \underline{\text{Use \#4 bars @12 in.}}$$

$$\rho_t = \frac{A_v}{A_s}$$

where A_g = wall thickness times the vertical spacing of the horizontal shear reinforcement

$$\rho_t = \frac{(2)(0.20)}{(8)(12)} = 0.00417$$

which is greater than the minimum ρ_ℓ of 0.0025 required by Section 11.6.2 of the ACI Code.

$$\underline{\text{Use 2 \#4 horizontal bars 12 in. o.c. vertically}}$$

5. Design Vertical Shear Reinforcing

$$\min\rho_\ell = 0.0025 + 0.5\left(2.5 - \frac{h_w}{\ell_w}\right)(\rho_t - 0.0025) \qquad \text{(ACI Equation 11.6.2)}$$

$$= 0.0025 + 0.5\left(2.5 - \frac{12\text{ in./ft} \times 14\text{ ft}}{12\text{ in./ft} \times 10\text{ ft}}\right)(0.00417 - 0.0025)$$

$$= 0.00342$$

Assume #4 vertical bars with A_v = two-bar cross-sectional areas and with s = horizontal spacing of vertical bars.

$$s = \frac{(2)(0.20\text{ in.}^2)}{(8\text{ in.})(0.00342)} = 14.62\text{ in.}$$

Maximum horizontal spacing of vertical shear reinforcement

$$\frac{\ell_w}{3} = \frac{(12 \text{ in./ft})(10 \text{ ft})}{3} = 40 \text{ in.}$$

$$3h = (3)(8 \text{ in.}) = 24 \text{ in.}$$

$$18 \text{ in.} = 18 \text{ in.} \leftarrow$$

Use 2 #4 vertical bars 14 in. o.c. horizontally

6. Design Vertical Flexural Reinforcement

$$M_u = (240 \text{ k})(14 \text{ ft}) = 3360 \text{ ft-k at base of wall}$$

$$\frac{M_u}{\phi b d^2} = \frac{(12 \text{ in./ft})(3360 \text{ ft-k})(1000)}{(0.9)(8 \text{ in.})(96 \text{ in.})^2} = 607.6 \text{ psi}$$

$$\rho = 0.0118 \text{ (from Appendix A, Table A.12)}$$

$$A_s = \rho b d$$

where b is wall thickness and d is approximated by $0.80\ell_w = (0.8)(12 \text{ in./ft} \times 10 \text{ ft}) = 96$ in.

$$A_s = (0.0118)(8 \text{ in.})(96 \text{ in.}) = 9.06 \text{ in.}^2$$

Use 10 #9 bars each end (assuming V_u could come from either direction)

7. A sketch of the wall cross section is given in Figure 18.8. If this same wall were subjected to significant axial load, the method used to calculate A_s for flexure would have to be revised to include its effect. Spreadsheets to calculate the coordinates of the interaction diagram using the assumptions in Chapter 10 can be developed for this purpose.

The centroid of the bar group at either end of the wall is approximately 7 in. from the wall end. Assuming all of the tension bars are yielding, the resultant tension force is also located at 7 in. from the wall end. The assumed value of $d = 0.8\ell_w$ was overly conservative. It can be taken as 120 – 7 = 113 in. Revising the calculation for A_s using this value of d results in a new $A_s = 7.32$ in.^{2} As a result, the bar size can be reduced to #8 with the same number of bars (10 #8 bars at each end).

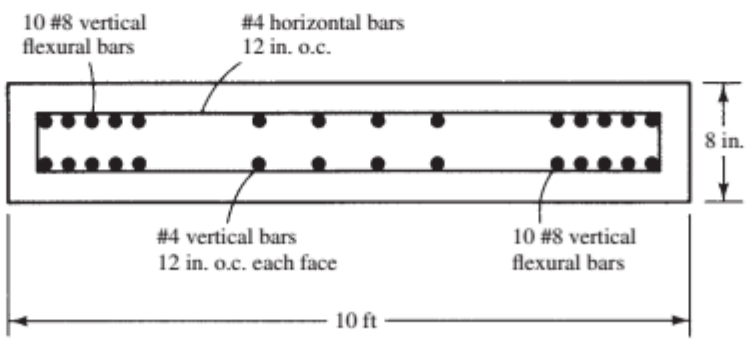


FIGURE 18.8 Final reinforcement details for Example 18.3.

18.7 Economy in Wall Construction

To achieve economical reinforced concrete walls, it is necessary to consider such items as wall thicknesses, openings, footing elevations, and so on.

The thicknesses of walls should be sufficient to permit the proper placement and vibration of the concrete. All of the walls in a building should have the same thickness if practical. Such a practice will permit the reuse of forms, ties, and other items. Furthermore, it will reduce the possibilities of field mistakes.

As few openings as possible should be placed in concrete walls. Where openings are necessary, it is desirable to repeat the sizes and positions of openings in different walls rather than using different sizes and positions. Furthermore, a few large openings are more economical than a larger number of smaller ones.

Much money can be saved if a footing elevation can be kept constant for any given wall. Such a practice will appreciably simplify the use of wall forms. If steps are required in a footing, their number should be kept to the minimum possible.[4]

18.8 Computer Example

Concrete walls can be designed using the Excel spreadsheet provided for Chapters 9 and 10 with appropriate input values and interpretation of results. A value of $b = 12$ in. is used for walls loaded out-of-plane; thus, the loads per foot of wall length are simply the input loads. Only a value of A_{s1} is

Courtesy of EFCO Corp.

South Point water facility, Durham, North Carolina.

[4] Neville, G. B., ed., 1984, *Simplified Design Reinforced Concrete Buildings of Moderate Size* (Skokie, IL: Portland Cement Association), pp. 9-12 to 9-13.

input, as it is uncommon to have more than two layers of steel in a wall. If only one layer is used, then input a value of $\gamma = 0$. In order to avoid having to laterally tie the compression reinforcement, the total longitudinal reinforcement area is limited to 0.01 times the gross concrete area. The spreadsheet could easily be modified to neglect the contribution of the compression steel, and if this were done, the 0.01 limit would not have to be applied.

Example 18.4

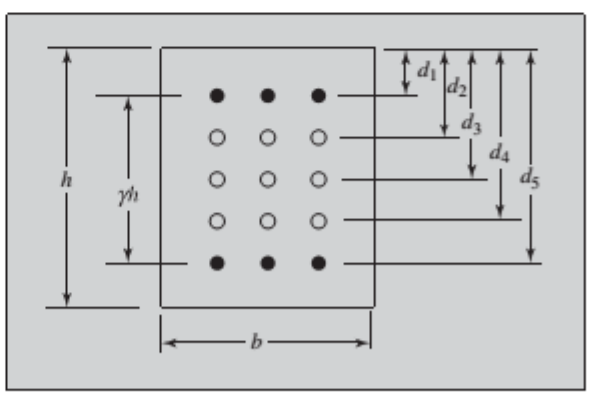
Work Example 18.2 using the spreadsheets for Chapters 9 and 10.

SOLUTION

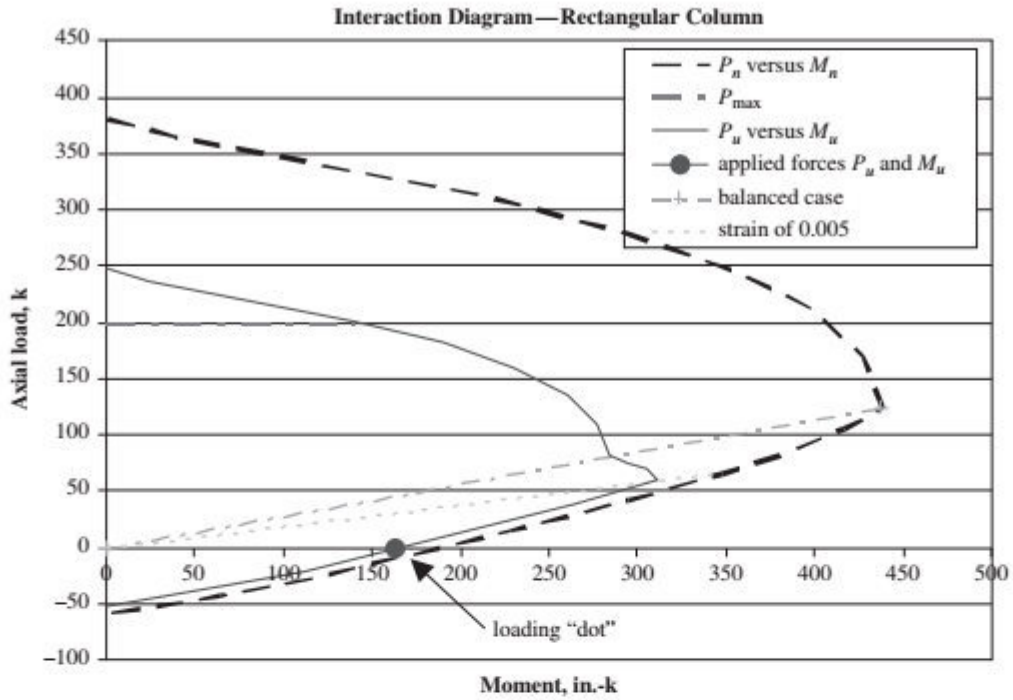
Open the Excel spreadsheet for Chapters 9 and 10, and the Rect Col Worksheet. Enter one load case at a time. Refer to the table of load combinations in Example 18.2, and look under the heading P_u and M_u. Only the values of P_u and M_u for loading combination $U = 0.9D + 1.6H$ are shown in the screenshot below. It is not possible in this example to distinguish between the value of A_{s1} required for load cases $U = 0.9D + 1.6H$ and $U = 1.2D + 1.6L + 1.6H$, so only the former is shown. Once values are entered for P_u, M_u, b (always 12 in. for walls loaded out-of-plane), h, f'_c, and f_y, enter trial values for A_{s1} (A_{s2} is always zero for walls loaded out-of-plane). Look at the interaction diagram and see if the loading "dot" is within the contour of the interaction diagram. Use the smaller diagram that has been reduced by the ϕ factor. If the dot falls well outside the contour, you may need to increase the wall thickness. In working this problem with the interaction diagrams in Appendix A, it was necessary to use $\gamma = 0.6$. For walls with two steel layers, $\gamma = 0.5$ is more realistic, and this example could be easily worked using this value. The value of A_{s1} obtained by trial and error is the total area of steel per foot of wall length. Half goes in each layer, so enter Appendix A, Table A.6, seeking a value close to and exceeding 0.48 in.^{2}/ft. Select #5 bars at $7\frac{1}{2}$ in.($A_s = 0.49$ in.^{2}/ft). If the steel is in only one layer, enter a value of $\gamma = 0$. Steel placed in this manner is less efficient, and often a thicker wall is needed. Refer to Example 18.2 for horizontal steel requirements.

Rectangular Column Capacity

P_u =	1.247	k
M_u =	13.8667	ft-k = 166.4 in.-k
b =	12	in.
h =	8	in.
γ =	0.6	
f'_c =	4000	psi
f_y =	60,000	psi
A_{s1} =	0.96	in.^{2}
A_{s2} =	0.00	in.^{2}
A_{st} =	0.96	in.^{2}
A_g =	96.0	in.^{2}
ρ_t =	0.0100	
β_1 =	0.85	
ϵ_y =	0.00207	
E_s =	29,000	ksi
c_{bal} =	3.79	in.
$c_{0.005}$ =	2.4	in.

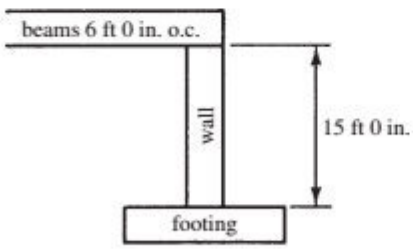


γ in cell C6 may be too large to meet cover requirements



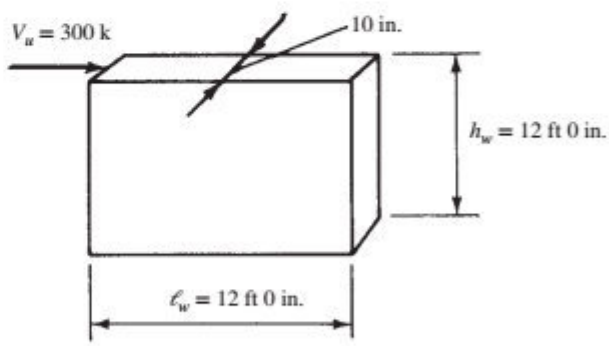
PROBLEMS

Problem 18.1 Design a reinforced concrete bearing wall using the ACI empirical formula to support a set of precast concrete roof beams 6 ft 0 in. on center as shown. The bearing width of each beam is 8 in. The wall is considered to be laterally supported at top and bottom and is further assumed to be restrained against rotation at the footing. Neglect wall weight. Other data: $f'_c = 3000$ psi, $f_y = 60{,}000$ psi, beam reaction, $D = 35$ k, and $L = 25$ k. (*Ans.* $7\frac{1}{2}$-in. wall with #3 bars @ 12-in. vertical steel)



Problem 18.2 Repeat Problem 18.1 if the wall is not restrained against rotation at top or bottom, is 8 ft 0 in. in height, and has an $f'_c = 4000$ psi.

Problem 18.3 Design the reinforced concrete wall shown if $f'_c = 4000$ psi and $f_y = 60{,}000$ psi. (*One ans.* 10-in. wall with 8 #9 flexural bars at each end)



Problem 18.4 Repeat Problem 18.3 if $h_w = 15$ ft 0 in. and $f'_c = 3000$ psi.

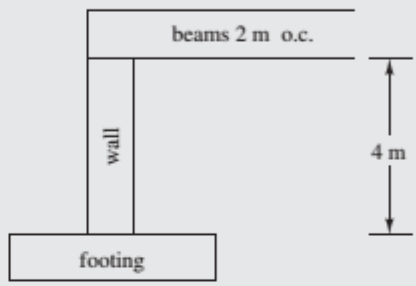
Problem 18.5 Design the wall in Problem 18.1 using the column interaction diagrams (see Example 18.2). Change the 15-ft wall height to 20 ft. Replace the beams with a solid, one-way slab. The slab exerts a dead load of 6 k/ft and a live load of 4 k/ft, both at a 3-in. eccentricity measured from the center of the wall toward the left. Soil backfill is placed to a depth of 20 ft on the right-hand side of the wall. $\gamma_s = 100$ pcf and $k_a = 0.33$. (*Ans*. 8-in. wall with #7 bars @ 5 in., 2 layers)

Problem 18.6 Repeat Problem 18.5 using Chapters 8 and 9 Excel spreadsheets.

Problem 18.7 Repeat Problem 18.6 using steel in one layer. (*Ans*. 12-in. wall with #9 @ 3 in.)

Problems in SI Units

Problem 18.1 Design a reinforced concrete bearing wall using the ACI empirical formula to support a set of precast roof beams 2 m on center as shown. The bearing width of each beam is 200 mm. The wall is considered to be laterally restrained at top and bottom and is further assumed to be restrained against rotation at the footing. Neglect wall weight. Other data: $f'_c = 21$ MPa, $f_y = 420$ MPa, beam reaction, $D = 120$ kN, and $L = 100$ kN. (*Ans*. 160-mm-thick wall with #10 bars @ 200 mm o.c. horizontal reinforcing bars)



Prestressed Concrete

CHAPTER 19

19.1 Introduction

Prestressing can be defined as the imposition of internal stresses into a structure that are of opposite character to those that will be caused by the service or working loads. A common method used to describe prestressing is shown in Figure 19.1, where a row of books has been squeezed together by a person's hands. The resulting "beam" can carry a downward load as long as the compressive stress from squeezing at the bottom of the "beam" is greater than the tensile stress there from the moment produced by the weight of the books and the superimposed loads. Such a beam has no tensile strength and, thus, no moment resistance until it is squeezed together or prestressed. You might very logically now expand your thoughts to a beam consisting of a row of concrete blocks squeezed together and then to a plain concrete beam with its negligible tensile strength similarly prestressed.

The theory of prestressing is quite simple and has been used for many years in various kinds of structures. For instance, wooden barrels have long been made by putting tightened metal bands around them, thus compressing the staves together and making a tight container with resistance to the outward pressures of the enclosed liquids. For concrete beams, prestressing is primarily used to counteract tension stresses caused by the weight of the members and the superimposed loads. Should these loads cause a positive moment in a beam, it is possible by prestressing to introduce a negative moment that can counteract part or all of the positive moment. An ordinary beam has to have sufficient strength to support itself as well as the other loads, but it is possible with prestressing to produce a negative loading that will eliminate the effect of the beam's weight, thus producing a "weightless beam."

From the preceding discussion, it is easy to see why prestressing has captured the imagination of so many persons and why it has all sorts of possibilities now and in the future.

In the earlier chapters of this book, only a portion of the concrete cross sections of members in bending could be considered effective in resisting loads because a large part of those cross sections were in tension, and thus the concrete cracked. If, however, concrete flexural members can be prestressed so that their entire cross sections are kept in compression, then the properties of the entire sections are available to resist the applied forces.

For a more detailed illustration of prestressing, see Figure 19.2. It is assumed that the following steps have been taken with regard to this beam:

1. Steel strands (represented by the dashed lines) were placed in the lower part of the beam form.
2. The strands were tensioned to a very high stress.

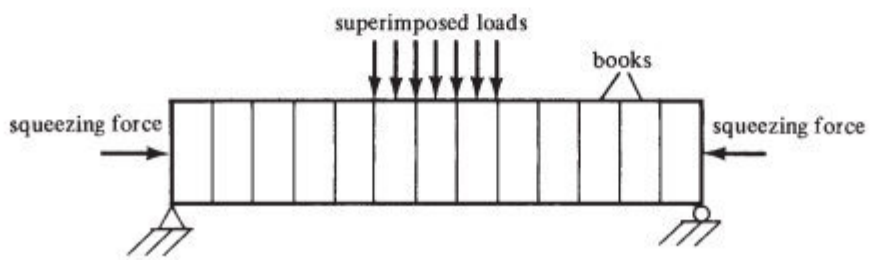


FIGURE 19.1 Prestressing.

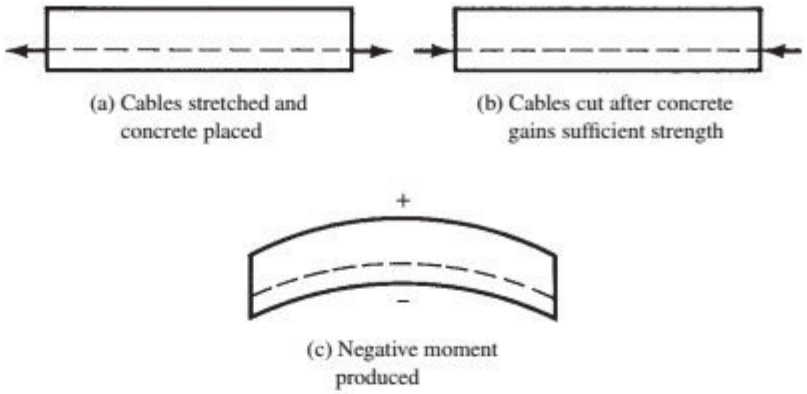
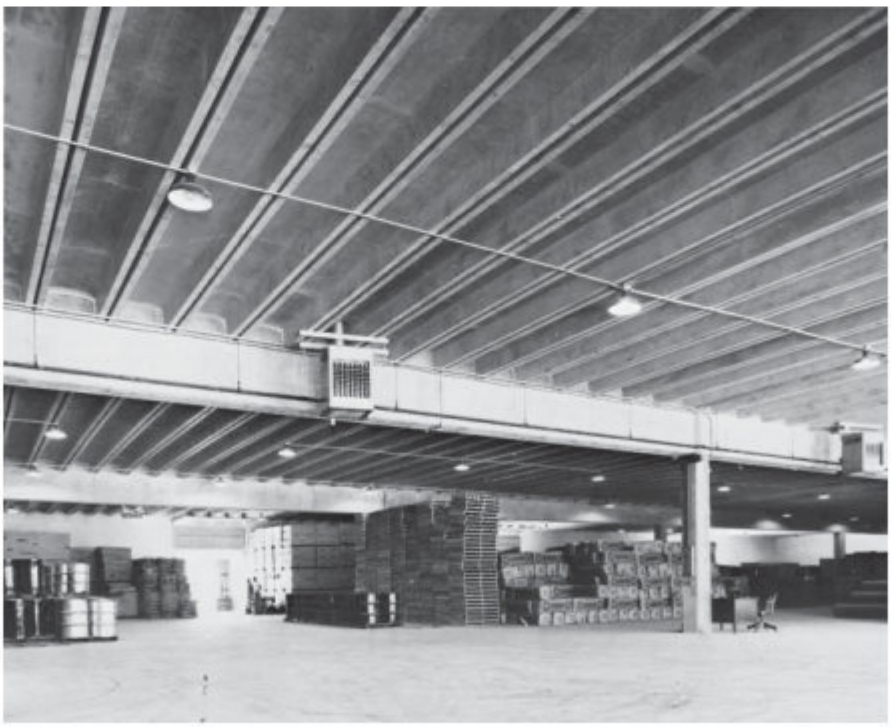

(a) Cables stretched and concrete placed
(b) Cables cut after concrete gains sufficient strength
+
−
(c) Negative moment produced

FIGURE 19.2 Sequence showing effect of prestressing force at different stages.

3. The concrete was placed in the form and allowed to gain sufficient strength for the prestressed strands to be cut.
4. The strands were cut.

The cut strands tend to resume their original length, thus compressing the lower part of the beam and causing a negative bending moment. The positive moment caused by the beam weight and any superimposed gravity loads is directly opposed by the negative moment. Another way of explaining this is to say that a compression stress has been produced in the bottom of the beam opposite in character to the tensile stress that is caused there by the working loads.



Courtesy of BASF Corporation.

Prestressed concrete channels, John A. Denies Son Company Warehouse #4, Memphis, Tennessee.

19.2 Advantages and Disadvantages of Prestressed Concrete

Advantages

As described in Section 19.1, it is possible with prestressing to utilize the entire cross sections of members to resist loads. Thus, smaller members can be used to support the same loads, or the same-size members can be used for longer spans. This is a particularly important advantage because member weights make up a substantial part of the total design loads of concrete structures.

Prestressed members are crack-free under working loads and, as a result, look better and are more watertight, providing better corrosion protection for the steel. Furthermore, crack-free prestressed members require less maintenance and last longer than cracked reinforced concrete members. Therefore, for a large number of structures, prestressed concrete provides the lowest first-cost solution, and when its reduced maintenance is considered, it provides the lowest overall cost for many additional cases.

The negative moments caused by prestressing produce camber in the members, with the result that total deflections are reduced. Other advantages of prestressed concrete include the following: reduction in diagonal tension stresses, sections with greater stiffnesses under working loads, and increased fatigue and impact resistance compared to ordinary reinforced concrete.

Disadvantages

Prestressed concrete requires the use of higher-strength concretes and steels and the use of more complicated formwork, with resulting higher labor costs. Other disadvantages include the following:

1. Closer quality control required in manufacture.
2. Losses in the initial prestressing forces. When the compressive forces from prestressing are applied to the concrete, it will shorten somewhat, partially relaxing the cables. The result is some reduction in cable tension with a resulting loss in prestressing forces. Shrinkage and creep of the concrete add to this effect.
3. Additional stress conditions must be checked in design, such as the stresses occurring when prestress forces are first applied and then after prestress losses have taken place, as well as the stresses occurring for different loading conditions.
4. Cost of end anchorage devices and end-beam plates that may be required.

19.3 Pretensioning and Posttensioning

The two general methods of prestressing are pretensioning and posttensioning. *Pretensioning* was illustrated in Section 19.1, where the prestress tendons were tensioned before the concrete was placed. After the concrete had hardened sufficiently, the tendons were cut and the prestress force was transmitted to the concrete by bond. This method is particularly well suited for mass production because the casting beds can be constructed several hundred feet long. The tendons can be run for the entire bed lengths and used for casting several beams in a line at the same time, as shown in Figure 19.3.

In *posttensioned* construction (see Figure 19.4), the tendons are stressed after the concrete is placed and has gained the desired strength. Plastic or metal tubes, conduits, sleeves, or similar devices

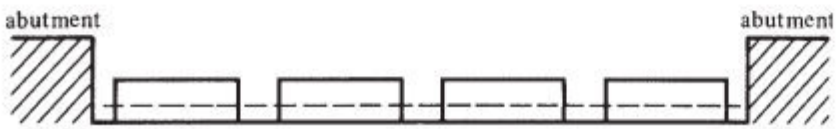


FIGURE 19.3 Prestress bed.

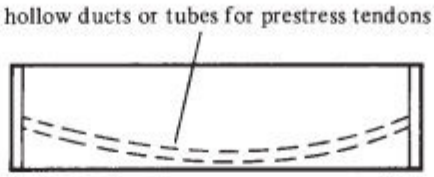


FIGURE 19.4 Posttensioned beam.

with unstressed tendons inside (or later inserted) are located in the form and the concrete is placed. After the concrete has sufficiently hardened, the tendons are stretched and mechanically attached to end anchorage devices to keep the tendons in their stretched positions. Thus, by posttensioning, the prestress forces are transferred to the concrete not by bond but by end bearing.

It is actually possible in posttensioning to have either bonded or unbonded tendons. If bonded, the conduits are often made of aluminum, steel, or other metal sheathing. In addition, it is possible to use steel tubing or rods or rubber cores that are cast in the concrete and removed a few hours after the concrete is placed. After the steel is tensioned, cement grout is injected into the duct for bonding. The grout is also useful in protecting the steel from corrosion. If the tendons are to be unbonded, they should be greased to facilitate tensioning and to protect them from corrosion.[1]

19.4 Materials Used for Prestressed Concrete

The materials ordinarily used for prestressed concrete are concrete and high-strength steels. The concrete used is normally of a higher strength than that used for reinforced concrete members, for several reasons, including the following:

1. The modulus of elasticity of such concretes is higher, with the result that the elastic strains in the concrete are smaller when the tendons are cut. Thus, the relaxations or losses in the tendon stresses are smaller.
2. In prestressed concrete, the entire members are kept in compression, and thus all the concrete is effective in resisting forces. Hence, it is reasonable to pay for a more expensive but stronger concrete if all of it is going to be used. (In ordinary reinforced concrete beams, more than half of the cross sections are in tension and, thus, assumed to be cracked. As a result, more than half of a higher-strength concrete used there would be wasted.)
3. Most prestressed work in the United States is of the precast, pretensioned type done at the prestress yard, where the work can be carefully controlled. Consequently, dependable higher-strength concrete can readily be obtained.
4. For pretensioned work, the higher-strength concretes permit the use of higher bond stresses between the cables and the concrete.

High-strength steels are necessary to produce and keep satisfactory prestress forces in members. The strains that occur in these steels during stressing are much greater than those that can be obtained with ordinary reinforcing steels. As a result, when the concrete elastically shortens in compression and also shortens because of creep and shrinkage, the losses in strain in the steel (and thus stress) represent a smaller percentage of the total stress. Another reason for using high-strength steels is that a large prestress force can be developed in a small area.

Early work with prestressed concrete using ordinary-strength bars to induce the prestressing forces in the concrete resulted in failure because the low stresses that could be put into the bars were completely lost due to the concrete's shrinkage and creep. Should a prestress of 20,000 psi be put into such rods, the resulting strains would be equal to $20{,}000/(29 \times 10^6) = 0.00069$. This value is less than the long-term creep and shrinkage strain normally occurring in concrete, roughly 0.0008,

[1] Nawy, E. G., 2005, *Prestressed Concrete*, 5th ed. (Upper Saddle River, NJ: Prentice Hall), pp. 62–69.

CC&A Collection, Courtesy of The Concrete Society.

Prestressed concrete segmental bridge over the River Trent near Scunthorpe, Lincolnshire, England.

which would completely relieve the stress in the steel. Should a high-strength steel be stressed to about 150,000 psi and have the same creep and shrinkage, the stress reduction will be of the order of $(0.0008)\ (29 \times 10^6) = 23{,}000$ psi, leaving $150{,}000 - 23{,}200 = 126{,}800$ psi in the steel (a loss of only 15.47% of the steel stress).[2]

Three forms of prestressing steel are used: single wires, wire strands, and bars. The greater the diameter of the wires, the smaller their strengths and bond to the concrete. As a result, wires are manufactured with diameters from 0.192 in. up to a maximum of 0.276 in. (about $\frac{9}{32}$ in.). In posttensioning work, large numbers of wires are grouped in parallel into tendons. Strands that are made by twisting wires together are used for most pretensioned work. They are of the seven-wire type, where a center wire is tightly surrounded by twisting the other six wires helically around it. Strands are manufactured with diameters from $\frac{1}{4}$ in. to $\frac{1}{2}$ in. Sometimes large, high-strength, heat-treated alloy steel bars are used for posttensioned sections. They are available with diameters running from $\frac{3}{4}$ in. to $1\frac{3}{8}$ in.

© Ron Buskirk/Alamy Limited.

Precast prestressed highway bridge girders at production facility in central Florida.

[2] Winter, G. and Nilson, A. H., 1991, *Design of Concrete Structures*, 11th ed. (New York: McGraw-Hill), pp. 759–760.

High-strength prestressing steels do not have distinct yield points (see Figure 19.10) as do the structural carbon reinforcing steels. The practice of considering yield points, however, is so firmly embedded in the average designer's mind that high-strength steels are normally given an arbitrary yield point anyway. The yield stress for wires and strands is usually assumed to be the stress that causes a total elongation of 1% to occur in the steel. For high-strength bars, the yield stress is assumed to occur when a 0.2% permanent strain occurs.

19.5 Stress Calculations

For a consideration of stresses in a prestressed rectangular beam, refer to Figure 19.5. For this example, the prestress tendons are assumed to be straight, although it will later be shown that a curved shape is more practical for most beams. The tendons are assumed to be located an eccentric distance, e, below the centroidal axis of the beam. As a result, the beam is subjected to a combination of direct compression and a moment because of the eccentricity of the prestress. In addition, there will be a moment from the external load, including the beam's own weight. The resulting stress at any point in the beam caused by these three factors can be written as follows, where P is the prestressing force:

$$f = -\frac{P}{A} \pm \frac{Pec}{I} \pm \frac{Mc}{I}$$

In this expression, P is the prestress force, e is the eccentricity of the prestress force with respect to the centroid of the cross section, c is the distance from the centroidal axis to the extreme fiber (top or bottom depending on where the stresses are being determined), M is the applied moment from unfactored loads at the stage at which stresses are being calculated, A is the uncracked concrete cross-sectional area, and I is the moment of inertia of the gross concrete cross section. In Figure 19.5, a stress diagram is drawn for each of these three items, and all three are combined to give the final stress diagram.

The usual practice is to base the stress calculations in the elastic range on the properties of the gross concrete section. The gross section consists of the concrete external dimensions with no additions

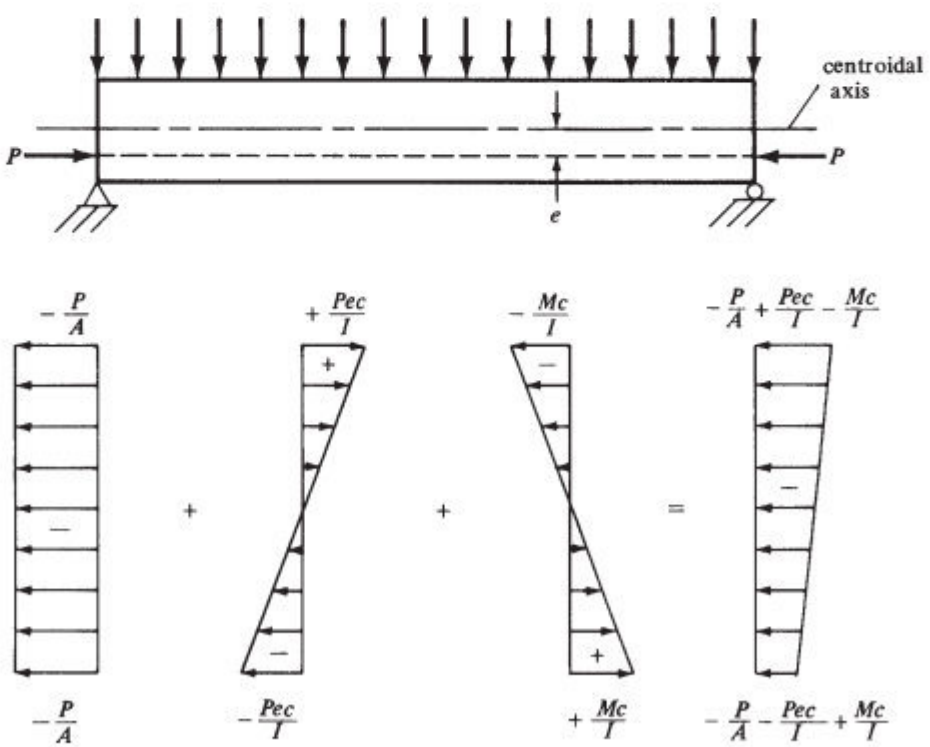


FIGURE 19.5 Concrete stress distribution from eccentric prestress force and superimposed loads.

made for the transformed area of the steel tendons or subtractions made for the duct areas in posttensioning. This method is considered to give satisfactory results because the changes in stresses obtained if net or transformed properties are used are usually not significant.

Example 19.1 illustrates the calculations needed to determine the stresses at various points in a simple-span prestressed rectangular beam. It will be noted that, as there are no moments at the ends of a simple beam from the external loads or to the beam's own weight, the Mc/I part of the stress equation becomes zero and so the equation reduces to

$$f = -\frac{P}{A} \pm \frac{Pec}{I}$$

Example 19.1

Calculate the stresses in the top and bottom fibers at the centerline and ends of the beam shown in Figure 19.6.

SOLUTION

Section Properties

$$I = \left(\frac{1}{12}\right)(12\text{ in.})(24\text{ in.})^3 = 13{,}824\text{ in.}^4$$

$$A = (12\text{ in.})(24\text{ in.}) = 288\text{ in.}^2$$

$$M = \frac{(3\text{ klf})(20\text{ ft})^2}{8} = 150\text{ ft-k}$$

Stresses at Beam Centerline

$$f_{top} = -\frac{P}{A} + \frac{Pec}{I} - \frac{Mc}{I} = -\frac{250\text{ k}}{288\text{ in.}^2} + \frac{(250\text{ k})(9\text{ in.})(12\text{ in.})}{13{,}824\text{ in.}^4} - \frac{(12\text{ in./ft})(150\text{ ft-k})(12\text{ in.})}{13{,}824\text{ in.}^4}$$

$$= -0.868\text{ ksi} + 1.953\text{ ksi} - 1.562\text{ ksi} = -0.477\text{ ksi}$$

$$f_{bottom} = -\frac{P}{A} - \frac{Pec}{I} + \frac{Mc}{I} = -0.868\text{ ksi} - 1.953\text{ ksi} + 1.562\text{ ksi} = -1.259\text{ ksi}$$

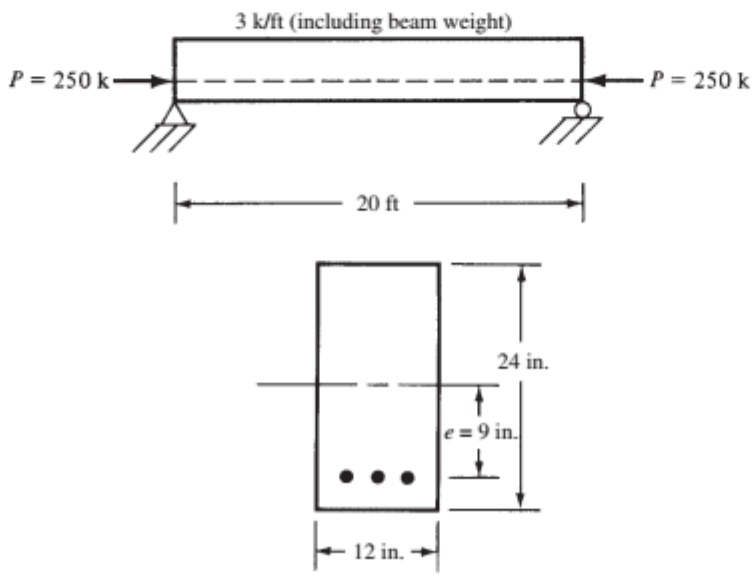


FIGURE 19.6 Information for Example 19.1.

Stresses at Beam Ends

$$f_{top} = -\frac{P}{A} + \frac{Pec}{I} = -0.868 \text{ ksi} + 1.953 \text{ ksi} = +1.085 \text{ ksi}$$

$$f_{bottom} = -\frac{P}{A} - \frac{Pec}{I} = -0.868 \text{ ksi} - 1.953 \text{ ksi} = -2.821 \text{ ksi}$$

In Example 19.1, it was shown that when the prestress tendons are straight, the tensile stress at the top of the beam at the ends will be quite high. If, however, the tendons are draped, as shown in Figure 19.7, it is possible to reduce or even eliminate the tensile stresses. Out in the span, the centroid of the strands may be below the lower kern point (see Example 19.2 for determination of the kern point for this section), but if at the ends of the beam, where there is no stress due to dead-load moment, it is below the kern point, tensile stresses in the top will be the result. If the tendons are draped so that at the ends they are located at or above this point, tension will not occur in the top of the beam.

In posttensioning, the sleeve or conduit is placed in the forms in the curved position desired. The tendons in pretensioned members can be placed at or above the lower kern points and then can be pushed down to the desired depth at the centerline or at other points. In Figure 19.7, the tendons are shown held down at the one-third points. Two alternatives to draped tendons that have been used are to use straight tendons, located below the lower kern point but that are encased in tubes at their ends or that have their ends greased. Both methods are used to prevent the development of negative moments at the beam ends.

In Section 24.5.2.1 of the ACI Code, bonded and unbonded prestressed members are designated as being Class U, T, or C members. These classifications are based on computed tensile stresses in members subject to service loads. Class U members are those that are assumed to be uncracked and have maximum tensile stresses $f_t \le 7.5\sqrt{f'_c}$. Class C members are those that are assumed to be cracked and have $f_t > 12\sqrt{f'_c}$. Class T members are assumed to be in transition between cracked and uncracked members and have maximum tensile stresses $f_t > 7.5\sqrt{f'_c} \le 12\sqrt{f'_c}$. Prestressed two-way slabs must be designed as Class U sections with $f_t \le 6\sqrt{f'_c}$.

Section 24.5.2.2 of the ACI Code states that for Class U and T members, flexural stresses may be computed using the uncracked section properties. For Class C sections, however, it is necessary to use cracked transformed section properties.

Example 19.2 shows the calculations necessary to locate the kern point for the beam of Example 19.1. In addition, the stresses at the top and bottom of the beam ends are computed. It will be noted that, according to these calculations, the kern point is 4 in. below the middepth of the beam, and it would thus appear that the prestress tendons should be located at the kern point at the beam ends and pushed down to the desired depth farther out in the beam. Actually, however, the tendons at the beam ends do not have to be as high as the kern points because the ACI Code (Table 24.5.3.2) permits some tension in the top of the beam when the tendons are cut. This value is $3\sqrt{f'_{ci}}$, where f'_{ci} is the strength of the concrete at the time the tendons are cut, as determined by testing concrete cylinders. The subscript i denotes "initial," meaning at initial release of the prestressing tendon, before the concrete gains its full 28-day strength. This permissible value equals about 40% of the cracking strength or modulus of rupture of the concrete ($7.5\sqrt{f'_{ci}}$) at that time. The stress at the bottom of the beam, which is compressive, is permitted to go as high as $0.60f'_{ci}$.

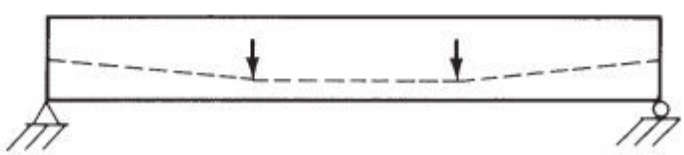


FIGURE 19.7 Draped tendons.

The code actually permits tensile stresses at the ends of simple beams to go as high as $6\sqrt{f'_{ci}}$. These allowable tensile values are applicable to the stresses that occur immediately after the transfer of the prestressing forces and after the losses occur from elastic shortening of the concrete and relaxation of the tendons and anchorage seats. It is further assumed that the time-dependent losses of creep and shrinkage have not occurred. A discussion of these various losses is presented in Section 19.7 of this chapter. If the calculated tensile stresses are greater than the permissible values, it is necessary to use some additional bonded reinforcement (prestressed or unprestressed) to resist the *total* tensile force in the concrete computed on the basis of an uncracked section.

Section 24.5.4.1 of the ACI Code provides allowable stresses at service loads for Class U and Class T members after all prestress losses have occurred. An extreme fiber compression stress equal to $0.45f'_c$ is permitted for prestress plus sustained loads. The allowable compression stress for prestress plus total loads is $0.60f'_c$. In effect, the ACI Code provides a one-third increase in allowable compression stress when a large percentage of the service loads are deemed to be transient or of short duration.

The allowable tensile stress at ends of simply supported members immediately after prestress transfer is $6\sqrt{f'_{ci}}$. Section 24.5.2.2 of the ACI Code allows higher permissible stresses than those here under certain conditions. The commentary on ACI R24.5.2.1 of the code states that it is the intent of the code's writers to permit higher stress values when justified by the development of newer and better products, materials, and prestress techniques. Approval of such increases must be in accordance with the procedures of Section 1.10 of the ACI Code.

Only compressive stresses should be allowed in prestressed sections that are to be used in severe corrosive conditions. If tension cracks occur, the result may very well be increased cable corrosion.

Example 19.2

Determine the location of the lower kern point at the ends of the beam of Example 19.1. Calculate the stresses at the top and bottom of the beam ends, assuming the tendons are placed at the kern point.

SOLUTION

Locating the Kern Point

$$f_{top} = -\frac{P}{A} + \frac{Pec}{I} = 0$$

$$-\frac{250\text{ k}}{288\text{ in.}^2} + \frac{(250\text{ k})\,(e)\,(12\text{ in.})}{13{,}824\text{ in.}^4} = 0$$

$$-0.868\text{ ksi} + 0.217e = 0$$

$$e = 4\text{ in.}$$

Computing Stresses

$$f_{top} = -\frac{P}{A} - \frac{Pec}{I} = -\frac{250\text{ k}}{288\text{ in.}^2} + \frac{(250\text{ k})\,(4\text{ in.})\,(12\text{ in.})}{13{,}824\text{ in.}^4}$$

$$= -0.868\text{ ksi} + 0.868\text{ ksi} = 0$$

$$f_{bottom} = -\frac{P}{A} - \frac{Pec}{I} = -0.868\text{ ksi} - 0.868\text{ ksi} = -1.736\text{ ksi}$$

19.6 Shapes of Prestressed Sections

For simplicity in introducing prestressing theory, rectangular sections are used for most of the examples in this chapter. From the viewpoint of formwork alone, rectangular sections are the most economical, but more complicated shapes, such as I's and T's, will require smaller quantities of concrete and prestressing steel to carry the same loads and, as a result, they frequently have the lowest overall costs.

If a member is to be made only one time, a cross section requiring simple formwork (thus, often rectangular) will probably be used. For instance, simple formwork is essential for most cast-in-place work. Should, however, the forms be used a large number of times to make many identical members, more complicated cross sections, such as I's and T's, channels, or boxes, will be used. For such sections, the cost of the formwork as a percentage of each member's total cost will be greatly reduced. Several types of commonly used prestressed sections are shown in Figure 19.8. The same general theory used for the determination of stresses and flexural strengths applies to shapes such as these, as it does to rectangular sections.

The usefulness of a particular section depends on the simplicity and reusability of the formwork, the appearance of the sections, the degree of difficulty of placing the concrete, and the theoretical properties of the cross section. The greater the amount of concrete located near the extreme fibers of a beam, the greater will be the lever arm between the C and T forces and, thus, the greater the resisting moment. Of course, there are some limitations on the widths and thicknesses of the flanges. In addition, the webs must be sufficiently large to resist shear and to allow the proper placement of the concrete and at the same time be sufficiently thick to avoid buckling.

A prestressed *single T*, such as the one shown in Figure 19.8(a), is often a very economical section because a large proportion of the concrete is placed in the compression flange, where it is quite effective in resisting compressive forces. The double T shown in Figure 19.8(b) is used for schools, office buildings, stores, and so on and is probably the most used prestressed section in the United States today. The total width of the flange provided by a double T is in the range of about 5 ft to 8 ft, and spans of 30 ft to 50 ft are common. You can see that a floor or roof system can be erected easily and quickly by placing a series of precast double T's side by side TT TT TT. The sections serve as both the beams and slabs for the floor or roof system. Single T's are normally used for heavier loads and longer spans up to as high as 100 ft or 120 ft. Double T's for such spans would be very heavy and difficult to handle. The single T is not used as much today as it was in the recent past because of stability difficulties in both shipping and erection.

David Sailors/© Corbis.

Prestressed bridge girders during erection.

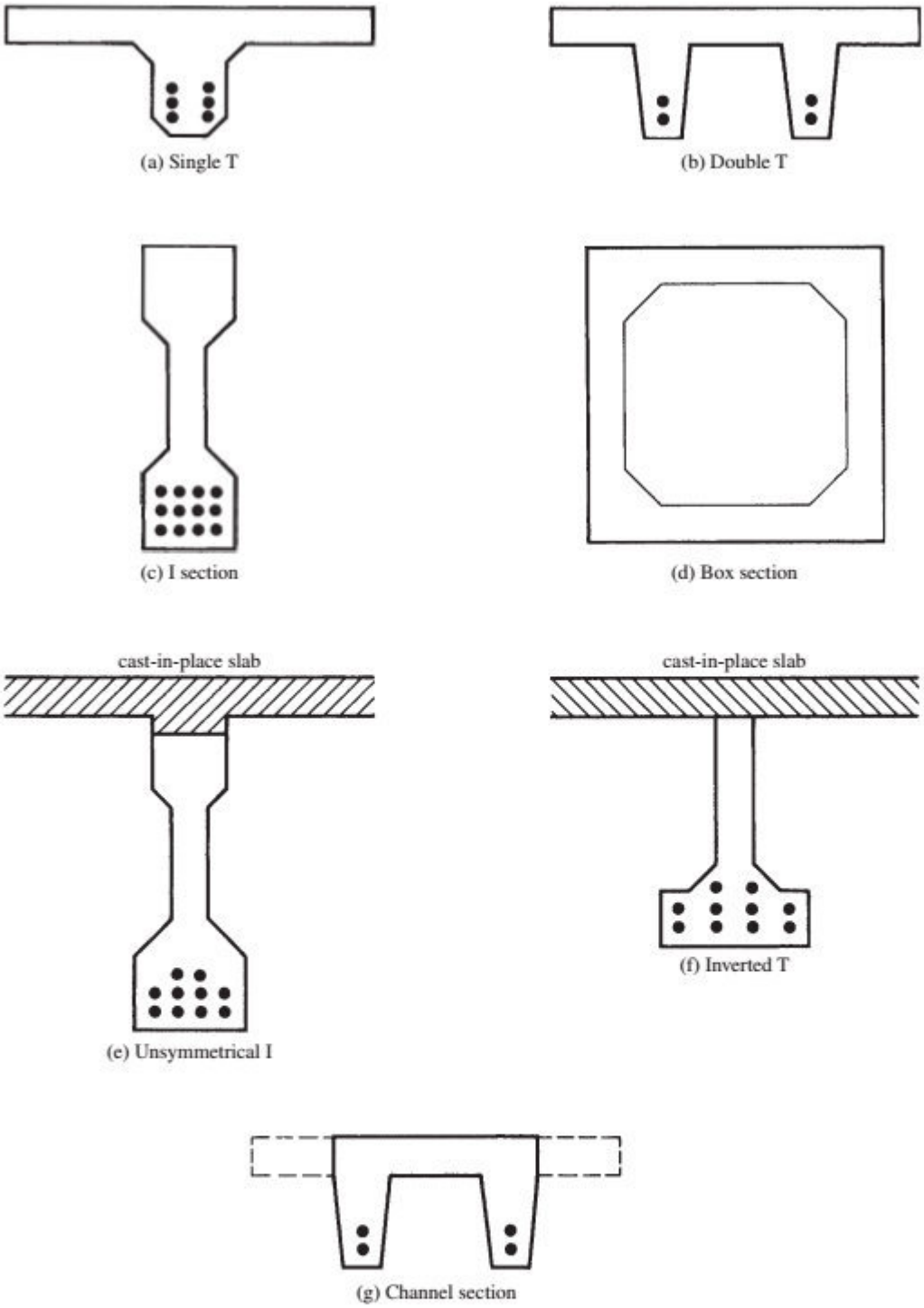
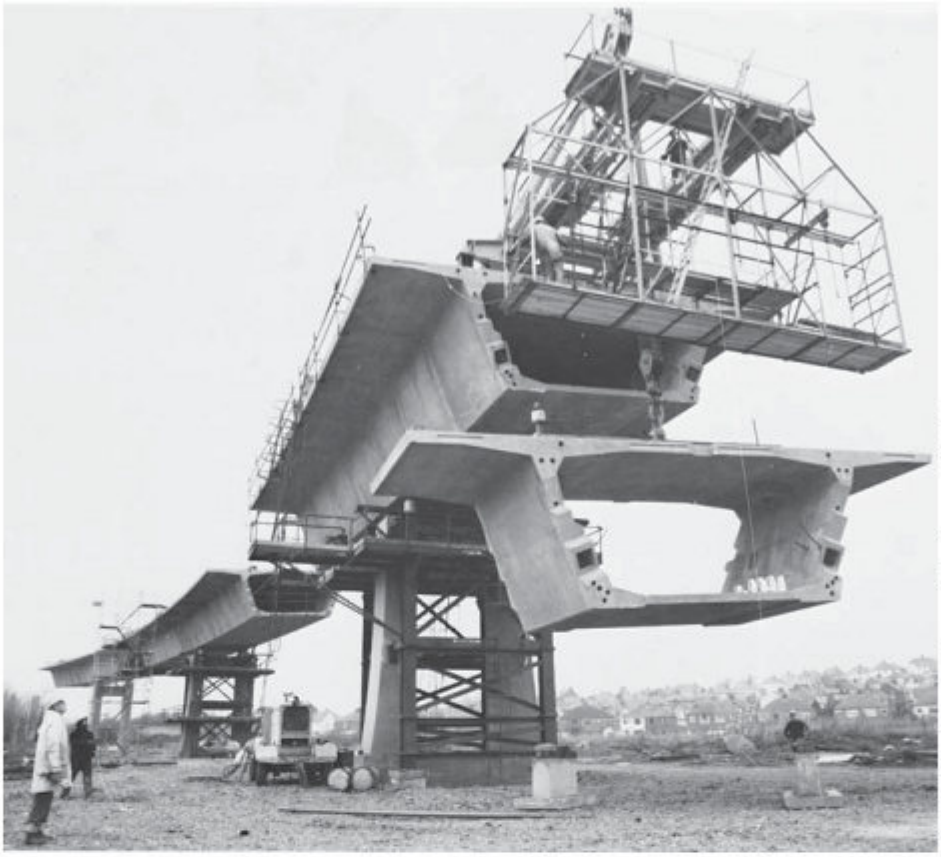


FIGURE 19.8 Commonly used prestressed sections.

The I and box sections, shown in parts (c) and (d) of Figure 19.8, have a larger proportion of their concrete placed in their flanges, with the result that larger moments of inertia are possible (as compared to rectangular sections with the same amounts of concrete and prestressing tendons). The formwork, however, is complicated, and the placing of concrete is difficult. Box girders are frequently used for bridge spans. Unsymmetrical I's [Figure 19.8(e)], with large bottom flanges to contain the tendons and small top flanges, may be economical for certain composite sections where they are used together with



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Posttensioned segmental precast concrete for East Moors Viaduct, Lanbury Way, Cardiff, South Wales.

a slab poured in place to provide the compression flange. A similar situation is shown in Figure 19.8(f), where an inverted T is used with a cast-in-place slab.

Many variations of these sections are used, such as the channel section shown in Figure 19.8(g). Such a section might be made by blocking out the flanges of a double-T form as shown, and the resulting members might be used for stadium seats or similar applications.

19.7 Prestress Losses

The flexural stresses calculated for the beams of Examples 19.1 and 19.2 were based on initial stresses in the prestress tendons. These stresses, however, become smaller with time (over a period of roughly 5 years) because of several factors. These factors, which are discussed in the paragraphs to follow, include the following:

1. Elastic shortening of the concrete
2. Shrinkage and creep of the concrete
3. Relaxation or creep in the tendons
4. Slippage in posttensioning end anchorage systems
5. Friction along the ducts used in posttensioning

Although it is possible to calculate prestress losses individually for each of the factors listed, it is usually more practical and often just as satisfactory to use single lump-sum estimates for all the items together. There are just too many interrelated factors affecting the estimates to achieve accuracy.

Such lump-sum estimates of total prestress losses are applicable only to average prestressed members made with normal concrete, construction procedures, and quality control. Should conditions be decidedly different from these and/or if the project is extremely significant, it would be well to consider making detailed loss estimates such as those introduced in the next few paragraphs.

The ultimate strength of a prestressed member is almost completely controlled by the tensile strength and cross-sectional area of the cables. Consequently, losses in prestress will have very little effect on its ultimate flexural strength. However, losses in prestress will cause more cracking to occur under working loads, with the result that deflections will be larger. Furthermore, the member's shear and fatigue strength will be somewhat reduced.

Elastic Shortening of the Concrete

When the tendons are cut for a pretensioned member, the prestress force is transferred to the concrete, with the result that the concrete is put in compression and shortens, thus permitting some relaxation or shortening of the tendons. The stress in the concrete adjacent to the tendons can be computed as described in the preceding examples. The strain in the concrete, ϵ_c, which equals f_c/E_c, is assumed due to bond to equal the steel strain, ϵ_s. Thus, the loss in prestress can be computed as $\epsilon_s E_s$. An average value of prestress loss in pretensioned members because of elastic shortening is about 3% of the initial value.

An expression for the loss of prestress because of elastic shortening of the concrete can be derived as shown in the paragraphs to follow.

It can be seen that the compressive strain in the concrete from prestress must equal the lessening of the steel strain

$$\epsilon_c = \Delta\epsilon_s$$

These values can be written in terms of stresses as follows:

$$\frac{f_c}{E_c} = \frac{\Delta f_s}{E_s}$$

thus, we can write

$$\Delta f_s = \frac{E_s f_c}{E_c} = n f_c$$

where f_c is the stress in the concrete at the level of the tendon centroid after transfer of stresses from the cables.

If we express Δf_s as being the initial tendon stress, f_{si}, minus the tendon stress after transfer, we can write

$$f_{si} - f_s = n f_c$$

Then, letting P_0 be the initial total cable stress and P_f the stress afterward, we obtain

$$P_0 - P_f = n\frac{P_f}{A_c}A_{ps}$$

$$P_0 = n\frac{P_f}{A_c}A_{ps} + P_f$$

$$P_0 = P_f\left(\frac{nA_{ps}}{A_c} + 1\right) = \frac{P_f}{A_c}(nA_{ps} + A_c)$$

Then

$$f_c = \frac{P_0}{A_c + A_{ps}} = \text{approximately } \frac{P_0}{A_g}$$

and finally

$$\Delta f_s = n f_c = \frac{nP_0}{A_g}$$

a value that can easily be calculated.

For posttensioned members, the situation is a little more involved because it is rather common to stress a few of the strands at a time and connect them to the end plates. As a result, the losses vary, with the greatest losses occurring in the first strands stressed and the least losses occurring in the last strands stressed. For this reason, an average loss may be calculated for the different strands. Losses from elastic shortening average about 0.5% for posttensioned members. It is, by the way, often possible to calculate the expected losses in each set of tendons and overstress them by that amount so the net losses will be close to zero.

Shrinkage and Creep of the Concrete

The losses in prestressing because of the shrinkage and creep in the concrete are quite variable. For one thing, the amount of shrinkage that occurs in concrete varies from almost zero up to about 0.0005 in./in. (depending on dampness and on the age of the concrete when it is loaded), with an average value of about 0.0003 in./in. being the usual approximation.

The loss in prestress from shrinkage can be said to equal $\epsilon_{sh} E_s$, where ϵ_{sh} is the shrinkage strain of the concrete. A recommended value of ϵ_{sh} is given in Zia et al.; it is to be determined by taking the basic shrinkage strain times a correction factor based on the volume (V)-to-surface (S) ratio times a relative humidity correction (H).[3]

$$\epsilon_{sh} = (0.00055)\left(1 - 0.06\frac{V}{S}\right)(1.5 - 0.15H)$$

Should the member be posttensioned, an additional multiplier is provided in Zia et al. to take into account the time between the end of the moist curing until the prestressing forces are applied.

The amount of creep in the concrete depends on several factors, which have been previously discussed in this text and can vary from one to five times the instantaneous elastic shortening. Prestress forces are usually applied to pretensioned members much earlier in the age of the concrete than for posttensioned members. Pretensioned members are normally cast in a bed at the prestress yard, where the speed of production of members is an important economic matter. The owner wants to tension the steel, place the concrete, and take the members out of the prestress bed as quickly as the concrete gains sufficient strength so that work can start on the next set of members. As a result, creep and shrinkage are larger, as are the resulting losses. Average losses are about 6% for pretensioned members and about 5% for posttensioned members.

The losses in cable stresses due to concrete creep strain can be determined by multiplying an experimentally determined creep coefficient C_t by nf_c.

$$\Delta f_s = C_t n f_c$$

In Zia et al., a value of $C_t = 2.0$ is recommended for pretensioned sections, whereas 1.6 is recommended for posttensioned ones. These values should be reduced by 20% if lightweight concrete is used. The value f_c is defined as the stress in the concrete adjacent to the centroid of the tendons due to the initial prestress ($-P/A$) and due to the permanent dead loads that are applied to the member after prestressing ($-Pec/I$), where e is measured from the centroid of the section to the centroid of the tendons.

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[3] Zia, P., Preston, H. K., Scott, N. L., and Workman, E. B., 1979, "Estimating Prestress Losses," *Concrete International: Design & Construction*, Vol. 1, No. 6 (Detroit, MI: American Concrete Institute), pp. 32–38.

Relaxation or Creep in the Tendons

The plastic flow or relaxation of steel tendons is quite small when the stresses are low, but the percentage of relaxation increases as stresses become higher. In general, the estimated losses run from about 2% to 3% of the initial stresses. The amount of these losses actually varies quite a bit for different steels and should be determined from test data available from the steel manufacturer in question. A formula is available with which this loss can be computed.

Slippage in Posttensioning End Anchorage Systems

When the jacks are released and the prestress forces are transferred to the end anchorage system, a little slippage of the tendons occurs. The amount of the slippage depends on the system used and tends to vary from about 0.10 in. to 0.20 in. Such deformations are quite important if the members and thus the tendons are short, but if they are long, the percentage is much less important.

Friction along the Ducts Used in Posttensioning

There are losses in posttensioning because of friction between the tendons and the surrounding ducts. In other words, the stress in the tendons gradually falls off as the distance from the tension points increases because of friction between the tendons and the surrounding material. These losses are the result of the so-called length and curvature effects.

The *length effect* is the friction that would have existed if the cable had been straight and not curved. Actually, it is impossible to have a perfectly straight duct in posttensioned construction, and the result is friction, called the length effect or sometimes the *wobble effect*. The magnitude of this friction is dependent on the stress in the tendons, their length, the workmanship for the particular member in question, and the coefficient of friction between the materials.

The *curvature effect* is the amount of friction that occurs in addition to the unplanned wobble effect. The resulting loss is due to the coefficient of friction between the materials caused by the pressure on the concrete from the tendons, which is dependent on the stress and the angle change in the curved tendons.

It is possible to reduce frictional losses substantially in prestressing by several methods. These include jacking from both ends, overstressing the tendons initially, and lubricating unbonded cables.

Section 20.3.2.6.2 of the ACI Code requires that frictional losses for posttensioned members be computed with wobble and curvature coefficients experimentally obtained and verified during the prestressing operation. ACI 318-08 and earlier editions provided equations for calculating frictional losses. ACI 318-14 now cites the Post-Tensioning Manual[4] for estimating such losses.

19.8 Ultimate Strength of Prestressed Sections

Considerable emphasis is given to the ultimate strength of prestressed sections, the objective being to obtain a satisfactory factor of safety against collapse. You might wonder why it is necessary in prestress work to consider *both* working-stress and ultimate-strength situations. The answer lies in the tremendous change that occurs in a prestressed member's behavior after tensile cracks occur. Before the cracks begin to form, the entire cross section of a prestressed member is effective in resisting forces, but after the tensile cracks begin to develop, the cracked part is not effective in resisting tensile forces. Cracking is usually assumed to occur when calculated tensile stresses equal the modulus of rupture of the concrete (about $7.5\sqrt{f'_c}$).

Another question that might enter your mind at this time is this: What effect do the prestress forces have on the ultimate strength of a section? The answer to the question is quite simple. An ultimate-strength analysis is based on the assumption that the prestressing strands are stressed above

[4] *Post-Tensioning Manual*, sixth edition, Post-Tensioning Institute, Farmington Hills, MI, 2006, 354 pp.

their yield point. If the strands have yielded, the tensile side of the section has cracked and the theoretical ultimate resisting moment is the same as for a non-prestressed beam constructed with the same concrete and reinforcement.

The theoretical calculation of ultimate capacities for prestressed sections is not as routine as it is for ordinary reinforced concrete members. The high-strength steels from which prestress tendons are manufactured do not have distinct yield points. Despite this fact, the strength method for determining the ultimate moment capacities of sections checks rather well with load tests as long as the steel percentage is sufficiently small as to ensure a tensile failure and as long as bonded strands are being considered.

In the expressions used here, f_{ps} is the average stress in the prestressing steel at the design load. This stress is used in the calculations because the prestressing steels usually used in prestressed beams do not have well-defined yield points (i.e., the flat portions that are common to stress–strain curves for ordinary structural steels). Unless the yield points of these steels are determined from detailed studies, their values are normally specified. For instance, Section 20.3.2.3.1 of the ACI Code states that the following approximate expression may be used for calculating f_{ps}. In this expression, f_{pu} is the ultimate strength of the prestressing steel, ρ_p is the percentage of prestress reinforcement A_{ps}/bd_p, and f_{se} is the effective stress in the prestressing steel after losses. If more accurate stress values are available, they may be used instead of the specified values. In no case may the resulting values be taken as more than the specified yield strength f_{py}, or $f_{se} + 60{,}000$. For bonded members,

$$f_{ps} = f_{pu}\left(1 - \frac{\gamma_p}{\beta_1}\left[\rho_p\frac{f_{pu}}{f'_c} + \frac{d}{d_p}\frac{f_y}{f'_c}(\rho - \rho')\right]\right) \quad \text{if } f_{se} \geq 0.5f_{pu} \qquad \text{(ACI Equation 20.3.2.3.1)}$$

where γ_p is a factor for the type of prestress tendon whose values are specified in ACI Table 20.3.2.3.1 ($\gamma_p = 0.55$ for f_{py}/f_{pu} not less than 0.80, 0.40 for f_{py}/f_{pu} not less than 0.85, and 0.28 for f_{py}/f_{pu} not less than 0.90), d_p = distance from the extreme compression fiber to the centroid of the prestress reinforcement.

If any compression reinforcement is considered in calculating f_{ps}, the terms in brackets may not be taken as less than 0.17 (see Commentary R20.3.2.3.1b). Should compression reinforcement be taken into account and if the term in brackets is small, the depth to the neutral axis will be small, and thus the compression reinforcement will not reach its yield stress. For this situation, the results obtained with ACI Equation 20.3.2.3.1 are not conservative, thus explaining why the ACI provides the 0.17 limit.

Should the compression reinforcement be neglected in using the equation, ω' will equal zero and the term in brackets may be less than 0.17. Should d' be large, the strain in the compression steel may be considerably less than the yield strain, and as a result the compression steel will not influence f_{ps} as favorably as implied by the equation. As a result, the compression reinforcement must be neglected in ACI Equation 20.3.2.3.1 for beams in which $d' \geq 0.15d_p$.

For unbonded members with span to depth ≤ 35,

$$f_{ps} = f_{se} + 10{,}000 + \frac{f'_c}{100\rho_p} \text{ but not greater than } f_{py} \text{ or } (f_{se} + 60{,}000) \qquad \text{(ACI Table 20.3.2.4.1)}$$

For unbonded members with span to depth > 35,

$$f_{ps} = f_{se} + 10{,}000 + \frac{f'_c}{300\rho_p} \qquad \text{(ACI Table 20.3.2.4.1)}$$

However, f_{ps} may not exceed f_{py}, or $f_{se} + 30{,}000$.

As in reinforced concrete members, the amount of steel in prestressed sections is limited to ensure tensile failures. The limitation rarely presents a problem except in members with very small amounts of prestressing or in members that have not only prestress strands but also some regular reinforcing bars.

Example 19.3 illustrates the calculations involved in determining the permissible ultimate capacity of a rectangular prestressed beam. Some important comments about the solution and about ultimate-moment calculations in general are made at the end of the example.

Example 19.3

Determine the permissible ultimate moment capacity, ϕM_n, of the prestressed bonded beam of Figure 19.9 if f_{py} is 240,000 psi, f_{pu} is 275,000 psi, and f'_c is 5000 psi.

SOLUTION

Approximate Value of f_{ps} from ACI Code

$$\rho_p = \frac{A_{ps}}{bd_p} = \frac{1.40 \text{ in.}^2}{(12 \text{ in.})\,(21.5 \text{ in.})} = 0.00543$$

$$\frac{f_{py}}{f_{pu}} = \frac{240{,}000 \text{ psi}}{275{,}000 \text{ psi}} = 0.873$$

$\therefore \gamma_p = 0.40$, as given immediately after the presentation of ACI Equation 20.3.2.3.1 earlier in this section.

f_{ps} = estimated stress in prestressed reinforcement at nominal strength. Note that $\beta_1 = 0.80$ for 5000 psi concrete and d, the distance from the extreme compression fiber of the beam to the centroid of any non-prestressed tension reinforcement, is zero since there is no such reinforcement in this beam.

$$f_{ps} = f_{pu}\left\{1 - \frac{\gamma_p}{\beta_1}\left[\rho_p \frac{f_{pu}}{f'_c} + \frac{d}{d_p}\frac{f_y}{f'_c}(\rho - \rho')\right]\right\} \qquad \text{(ACI Equation 20.3.2.3.1)}$$

$$= 275 \text{ ksi}\left\{1 - \frac{0.40}{0.80}\left[0.00543\left(\frac{275 \text{ ksi}}{5 \text{ ksi}}\right) + 0\right]\right\} = 233.9 \text{ ksi}$$

Moment Capacity

$$a = \frac{A_{ps} f_{ps}}{0.85 f'_c b} = \frac{(1.40 \text{ in.}^2)\,(233.9 \text{ ksi})}{(0.85)\,(5 \text{ ksi})\,(12 \text{ in.})} = 6.42 \text{ in.}$$

$$c = \frac{a}{\beta_1} = \frac{6.42 \text{ in.}}{0.80} = 8.03 \text{ in.}$$

$$\epsilon_t = \frac{d_p - c}{c}\,0.003 = \frac{21.5 \text{ in.} - 8.03 \text{ in.}}{8.03 \text{ in.}}(0.003) = 0.0050$$

$\therefore$ The member is tension controlled and $\phi = 0.9$.

$$\phi M_n = \phi A_{ps} f_{ps}\left(d - \frac{a}{2}\right) = (0.9)\,(1.40 \text{ in.}^2)\,(233.9 \text{ ksi})\left(21.5 \text{ in.} - \frac{6.42 \text{ in.}}{2}\right)$$

$$= 5390 \text{ in.-k} = \underline{\underline{449.2 \text{ ft-k}}}$$

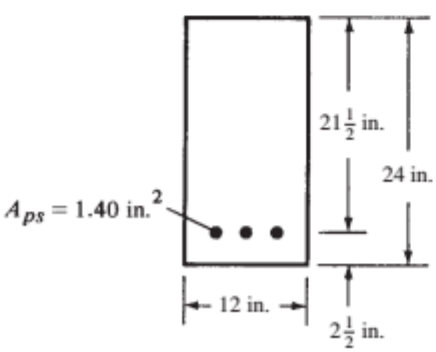

FIGURE 19.9 Beam cross section for Example 19.3.

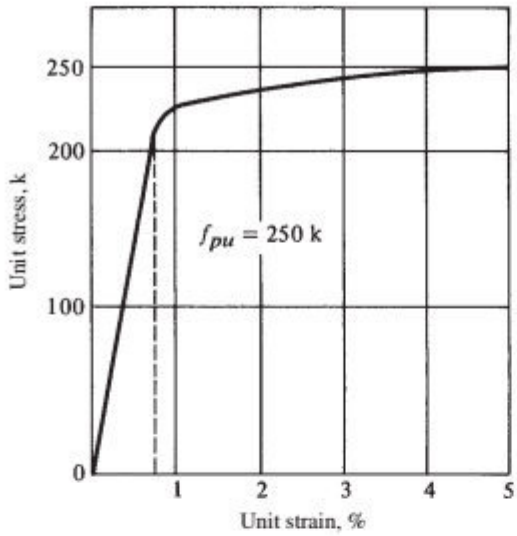

FIGURE 19.10 Typical stress–strain curve for high-tensile steel wire.

Discussion

The approximate value of f_{ps} obtained by the ACI formula is very satisfactory for all practical purposes. Actually, a slightly more accurate value of f_{ps} and, thus, of the moment capacity of the section can be obtained by calculating the strain in the prestress strands because of the prestressing and adding to it the strain from the ultimate moment. This latter strain can be determined from the values of a and the strain diagram, as frequently used in earlier chapters for checking to see if tensile failures control in reinforced concrete beams. With the total strain, a more accurate cable stress can be obtained by referring to the stress–strain curve for the prestressing steel being used. Such a curve is shown in Figure 19.10.

The analysis described herein is satisfactory for pretensioned beams or for bonded posttensioned beams but is not so good for unbonded posttensioned members. In these latter beams, the steel can slip with respect to the concrete; as a result, the steel stress is almost constant throughout the member. The calculations for M_u for such members are less accurate than for bonded members. Unless some ordinary reinforcing bars are added to these members, large cracks may form, which are not attractive and which can lead to some corrosion of the prestress strands.

If a prestressed beam is satisfactorily designed with service loads, then checked by strength methods and found to have insufficient strength to resist the factored loads ($M_u = 1.2M_D + 1.6M_L$), nonprestressed reinforcement may be added to increase the factor of safety. The increase in T from these bars is assumed to equal $A_s f_y$ (ACI Code, Section 22.3.2.1). Section 9.6.2.1 of the ACI Code further states that the total amount of prestressed and non-prestressed reinforcement shall be sufficient to develop an ultimate moment equal to at least 1.2 times the cracking moment of the section. This cracking moment is calculated with the modulus of rupture of the concrete, except for flexural members with shear and flexural strength equal to at least twice that required to support the factored loads and for two-way, unbonded posttensioned slabs. This additional steel also will serve to reduce cracks. (The 1.2 times requirement may be waived for two-way unbonded posttensioned slabs and for flexural members with shear and flexural strength at least equal to twice that required by Sections 7.6.2.2 and 9.6.2.2 of the ACI Code.)

19.9 Deflections

The deflections of prestressed concrete beams must be calculated very carefully. Some members that are completely satisfactory in all other respects are not satisfactory for practical use because of the magnitudes of their deflections.

In previous chapters, one method used for limiting deflections was to specify minimum depths for various types of members (see Table 4.1 in Chapter 4). These minimum depths, however, are applicable only to non-prestressed sections. The actual deflection calculations are made as they are for members made of other materials, such as structural steel, reinforced concrete, and so on. However, the same problem exists for reinforced concrete members, and that is the difficulty of determining the modulus of elasticity to be used in the calculations. The modulus varies with age, with different stress levels, and with other factors. For Class U slabs and beams, the gross moments of inertia are used for deflection calculations. For prestressed Class T and Class C slabs and beams, deflection calculations are based on the effective moment of inertia, I_e, given in Section 6.6 of this text. In applying the equation for I_e, the term f_r in the calculation of M_{cr} is modified by adding the effective prestress to the modulus of rupture. The modified equation is $M_{cr} = (f_r + f_{pe})\frac{I_g}{y_t}$.

The deflection from the force in a set of straight tendons is considered first in this section, with reference being made to Figure 19.11(a). The prestress forces cause a negative moment equal to Pe and, thus, an upward deflection or camber of the beam. The centerline deflection can be calculated by taking moments at the point desired when the conjugate beam is loaded with the M/EI diagram. At the centerline the deflection equals

$$-\left(\frac{Pe\ell}{2EI}\right)\left(\frac{\ell}{2}-\frac{\ell}{4}\right) = -\frac{Pe\ell^2}{8EI}\uparrow$$

Should the cables not be straight, the deflection will be different because of the different negative moment diagram produced by the cable force. If the cables are bent down or curved, as shown in parts (b) and (c) of Figure 19.11, the conjugate beam can again be applied to compute the deflections. The resulting values are shown in the figure.

The deflections from the tendon stresses will change with time. First of all, the losses in stress in the prestress tendons will reduce the negative moments they produce and, thus, the upward deflections. The long-term compressive stresses in the bottom of the beam from the prestress negative moments will cause creep and, therefore, increase the upward deflections.

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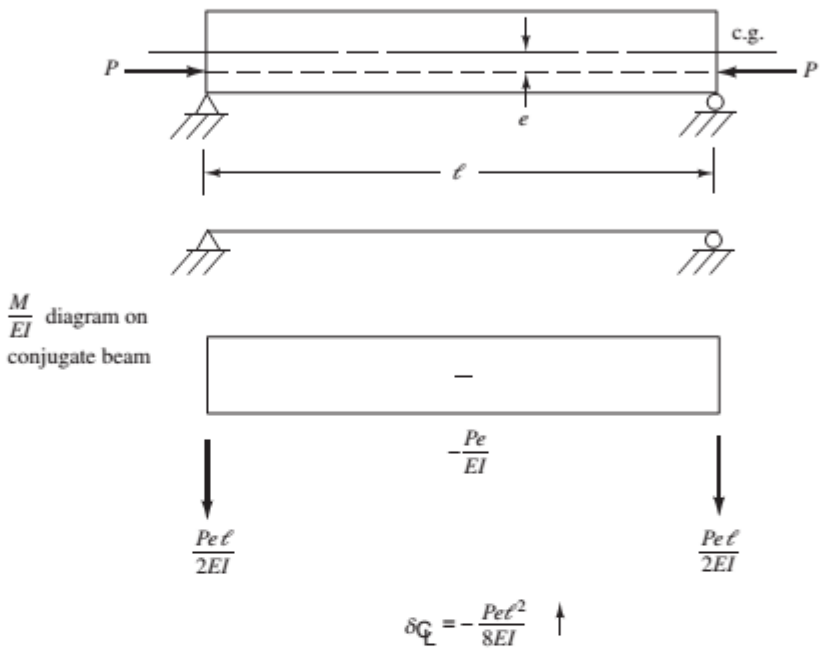
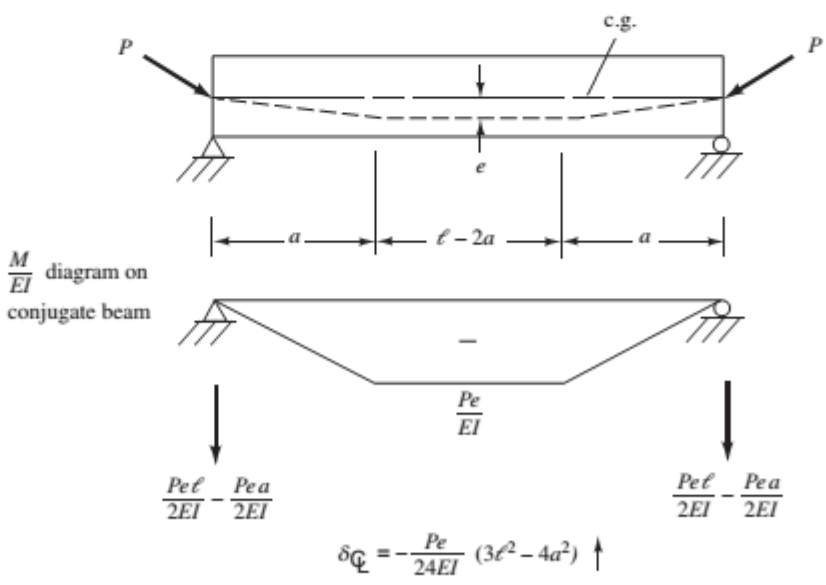
Segmental prestressed concrete bridge construction.

In addition to the deflections caused by the tendon stresses, there are deflections from the beam's own weight and from the additional dead and live loads subsequently applied to the beam. These deflections can be computed and superimposed on the ones caused by the tendons. Figure 19.11(d) shows the centerline deflection of a uniformly loaded simple beam obtained by taking moments at the centerline when the conjugate beam is loaded with the M/EI diagram.

Example 19.4 shows the initial and long-term deflection calculations for a rectangular pretensioned beam.



(a)



(b)

(*continues*)

FIGURE 19.11 Deflections in prestressed beams.

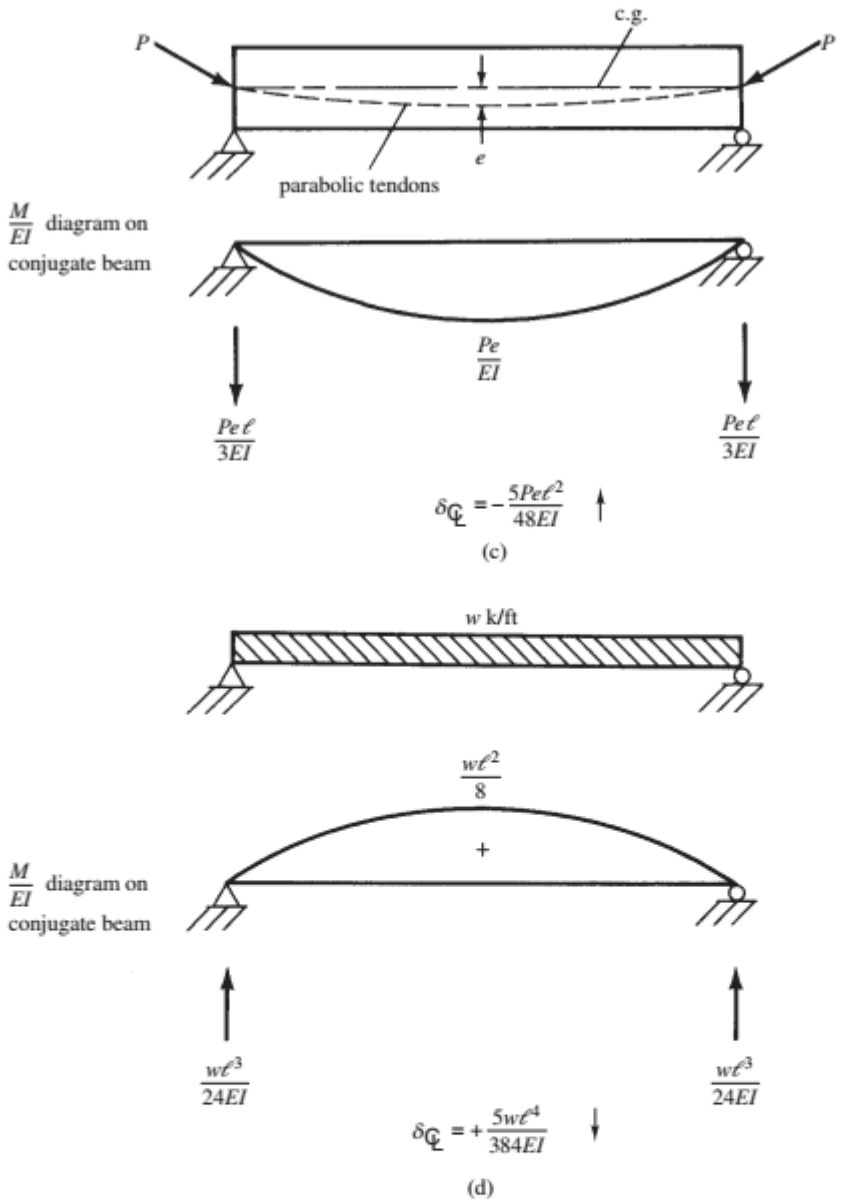


FIGURE 19.11 (*continued*)

Example 19.4

The pretensioned rectangular beam shown in Figure 19.12 has straight cables with initial stresses of 175 ksi and final stresses after losses of 140 ksi. Determine the deflection at the beam centerline immediately after the cables are cut. $E_c = 4 \times 10^6$ psi. Assume that the concrete is uncracked.

SOLUTION

$$I_g = \left(\frac{1}{12}\right)(12 \text{ in.})(20 \text{ in.})^3 = 8000 \text{ in.}^4$$

$$e = 6 \text{ in.}$$

$$\text{Beam weight} = \frac{(12 \text{ in.})(20 \text{ in.})}{144 \text{ in.}^2/\text{ft}^2}(150 \text{ pcf}) = 250 \text{ lb/ft}$$

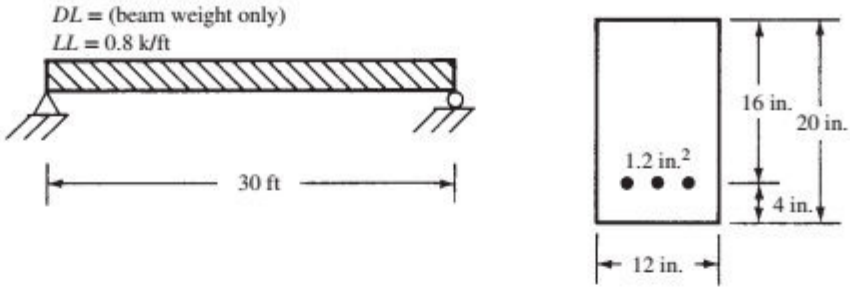


FIGURE 19.12 Information for Example 19.4.

Deflection Immediately after Cables Are Cut

$$\delta \text{ due to cable} = -\frac{Pe\ell^2}{8EI} = -\frac{(1.2 \text{ in.}^2 \times 175{,}000 \text{ psi})\,(6 \text{ in.})\,(12 \text{ in./ft} \times 30 \text{ ft})^2}{(8)\,(4 \times 10^6 \text{ psi})\,(8000 \text{ in.}^4)} = -0.638 \text{ in.} \uparrow$$

$$\delta \text{ due to beam weight} = +\frac{5w\ell^4}{384EI} = \frac{(5)\left(\frac{250 \text{ lb/ft}}{12 \text{ in./ft}}\right)(12 \text{ in./ft} \times 30 \text{ ft})^4}{(384)\,(4 \times 10^6 \text{ psi})\,(8000 \text{ in.}^4)} = \underline{+0.142 \text{ in.} \downarrow}$$

$$\text{Total deflection} = -0.496 \text{ in.} \uparrow$$

Additional Deflection Comments

Long-term deflections can be computed as previously described in Chapter 6. From the preceding example, it can be seen that, not counting external loads, the beam is initially cambered upward by 0.496 in.; as time goes by, this camber increases because of creep in the concrete. Such a camber is often advantageous in offsetting deflections caused by the superimposed loads. In some members, however, the camber can be quite large, particularly for long spans and where lightweight aggregates are used. If this camber is too large, the results can be quite detrimental to the structure (warping of floors, damage to roofing, cracking and warping of partitions, etc.).

To illustrate one problem that can occur, it is assumed that the roof of a school is being constructed by placing 50-ft double T's made with a lightweight aggregate side by side over a classroom. The resulting cambers may be rather large, and, worse, they may not be equal in the different sections. It then becomes necessary to force the different sections to the same deflection and tie them together in some fashion so that a smooth surface is provided for roofing. Once the surface is even, the members may be connected by welding together metal inserts, such as angles that were cast in the edges of the different sections for this purpose.

Both reinforced concrete members and prestressed members with overhanging or cantilevered ends often have rather large deflections. The total deflections at the free end of these members are the result of the sum of the normal deflections plus the effect of support rotations. This latter effect may frequently be the larger of the two, and, as a result, the sum of the two deflections may be so large as to affect the appearance of the structure detrimentally. For this reason, many designers try to avoid cantilevered members in prestressed construction.

19.10 Shear in Prestressed Sections

Web reinforcement for prestressed sections is handled in a manner similar to that used for a conventional reinforced concrete beam. In the expressions that follow, b_w is the web width or the diameter of a circular section and d_p is the distance from the extreme fiber in compression to the centroid of the

tensile reinforcement. Should the reaction introduce compression into the end region of a prestressed member, sections of the beam located at distances less than $h/2$ from the face of the support may be designed for the shear computed at $h/2$, where h is the overall thickness of the member.

$$v_u = \frac{V_u}{\phi b_w d_p}$$

Section 22.5.8.2 of the ACI Code provides two methods for estimating the shear strength that the concrete of a prestressed section can resist. There is an approximate method, which can be used only when the effective prestress force is equal to at least 40% of the tensile strength of the flexural reinforcement, f_{pu}, and a more detailed analysis, which can be used regardless of the magnitude of the effective prestress force. These methods are discussed in the paragraphs to follow.

Approximate Method

With this method, the nominal shear capacity of a prestressed section can be taken as

$$V_c = \left(0.6\lambda\sqrt{f'_c} + \frac{700 V_u d_p}{M_u}\right) b_w d \qquad \text{(ACI Table 22.5.8.2a)}$$

Section 22.5.8.2 of the ACI Code states that regardless of the value given by this equation, V_c need not be taken as less than $2\lambda\sqrt{f'_c}b_w d$, nor may it be larger than $5\lambda\sqrt{f'_c}b_w d$. In this expression, V_u is the maximum design shear at the section being considered, M_u is the design moment at the same section occurring simultaneously with V_u, and d is the distance from the extreme compression fiber to the centroid of the prestressed tendons. The value of $V_u d_p/M_u$ is limited to a maximum value of 1.0.

More Detailed Analysis

If a more detailed analysis is desired (it will have to be used if the effective prestressing force is less than 40% of the tensile strength of the flexural reinforcement), the nominal shear force carried by the concrete is considered to equal the smaller of V_{ci} or V_{cw}, to be defined here. The term V_{ci} represents the nominal shear strength of a member provided by the concrete when diagonal cracking results from combined shear and moment. The term V_{cw} represents the nominal shear strength of the member provided by the concrete when diagonal cracking results from excessive principal tensile stress in the concrete. In both expressions to follow, d is the distance from the extreme compression fiber to the centroid of the prestressed tendons or is $0.8h$, whichever is greater (ACI Code, Section 22.5.8.3.1).

The estimated shear capacity, V_{ci}, can be computed by the following expression, given by Section 22.5.8.3.1 of the ACI Code:

$$V_{ci} = 0.6\lambda\sqrt{f'_c}b_w d_p + V_d + \frac{V_i M_{cre}}{M_{\max}} \text{ but need not be taken as less than } 1.7\lambda\sqrt{f'_c}b_w d$$

(ACI Equation 22.5.8.3.1a)

In this expression, V_d is the shear at the section in question due to service dead load, $M_{\max}$ is the factored maximum bending moment at the section from externally applied design loads, V_i is the shear that occurs simultaneously with $M_{\max}$, and M_{cre} is the cracking moment, which is to be determined as follows:

$$M_{cre} = \left(\frac{I}{y_t}\right)(6\lambda\sqrt{f'_c} + f_{pe} - f_d) \qquad \text{(ACI Equation 22.5.8.3.1c)}$$

where

I = the moment of inertia of the section that resists the externally applied loads
y_t = the distance from the centroidal axis of the gross section (neglecting the reinforcement) to the extreme fiber in tension
f_{pe} = the compressive stress in the concrete from prestressing after all losses at the extreme fiber of the section where the applied loads cause tension
f_d = the stress from unfactored dead load at the extreme fiber where the applied loads cause tension

From a somewhat simplified principal tension theory, the shear capacity of a beam is equal to the value given by the following expression but need not be less than $1.7\lambda\sqrt{f'_c}b_w d$.

$$V_{cw} = \left(3.5\lambda\sqrt{f'_c} + 0.3f_{pc}\right)b_w d_p + V_p \qquad \text{(ACI Equation 22.5.8.3.2)}$$

In this expression, f_{pc} is the calculated compressive stress (in pounds per square inch) in the concrete at the centroid of the section resisting the applied loads due to the effective prestress after all losses have occurred. (Should the centroid be in the flange, f_{pc} is to be computed at the junction of the web and flange.) V_p is the vertical component of the effective prestress at the section under consideration. Alternately, Section 22.5.8.3.3 of the ACI Code states that V_{cw} may be taken as the shear force that corresponds to a multiple of dead load plus live load, which results in a calculated principal tensile stress equal to $4\lambda\sqrt{f'_c}$ at the centroid of the member or at the intersection of the flange and web if the centroid falls in the web.

A further comment should be made here about the computation of f_{pc} for pretensioned members, since it is affected by the transfer length. Section 22.5.9.1 of the ACI Code states that the transfer length can be taken as 50 diameters for strand tendons and 100 diameters for wire tendons. The prestress force may be assumed to vary linearly from zero at the end of the tendon to a maximum at the aforesaid transfer distance. If the value of $h/2$ is less than the transfer length, it is necessary to consider the reduced prestress when V_{cw} is calculated (ACI Code, Section 22.5.9.3).

19.11 Design of Shear Reinforcement

Should the computed value of V_u exceed ϕV_c, the area of vertical stirrups (the code does not permit inclined stirrups or bent-up bars in prestressed members) must not be less than A_v as determined by the following expression from Section 22.5.10.5.3 of the ACI Code:

$$V_s = \frac{A_v f_{yt} d}{s} \qquad \text{(ACI Equation 22.5.10.5.3)}$$

As in conventional reinforced concrete design, a minimum area of shear reinforcement is required at all points where V_u is greater than $\frac{1}{2}\phi V_c$. This minimum area is to be determined from the expression to follow if the effective prestress is less than 40% of the tensile strength of the flexural reinforcement (ACI Code, Section 9.6.3.3):

$$A_{v\,\min} = 0.75\sqrt{f'_c}\frac{b_w s}{f_{yt}} \text{ but shall not be less than } \frac{50 b_w s}{f_{yt}} \qquad \text{(ACI Table 9.6.3.3)}$$

where b_w and s are in inches.

If the effective prestress is equal to or greater than 40% of the tensile strength of the flexural reinforcement, the following expression, in which A_{ps} is the area of prestressed reinforcement in the tensile zone, is to be used to calculate A_v:

$$A_{v\,\min} = \left(\frac{A_{ps}}{80}\right)\left(\frac{f_{pu}}{f_{yt}}\right)\left(\frac{s}{d}\right)\sqrt{\left(\frac{d}{b_w}\right)} \qquad \text{(ACI Table 9.6.3.3e)}$$

CC&A Collection, Courtesy of The Concrete Society.

Prestressed concrete Runnymede Bridge over Thames River near Egham, Surrey, England.

Section 9.7.6.2.2 of the ACI Code states that in no case may the maximum spacing exceed $0.75h$ or 24 in. Examples 19.5 and 19.6 illustrate the calculations necessary for determining the shear strength and for selecting the stirrups for a prestressed beam.

Example 19.5

Calculate the shearing strength of the section shown in Figure 19.13 at 4 ft from the supports, using both the approximate method and the more detailed method allowed by the ACI Code. Assume that the area of the prestressing steel is 1.0 in.^{2}, the effective prestress force is 250 k, and $f'_c = 4000$ psi, normal weight.

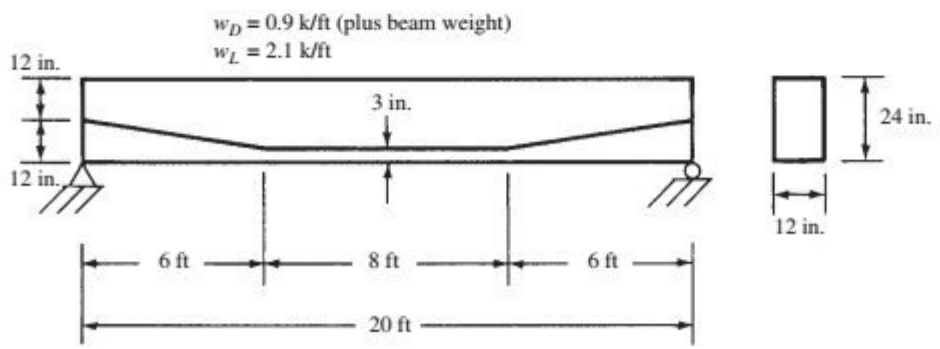


FIGURE 19.13 Information for Example 19.5.

SOLUTION

Approximate Method

$$\text{Beam weight} = \frac{(12\text{ in.})(24\text{ in.})}{144\text{ in.}^2/\text{ft}^2}(150\text{ lb/ft}^3) = 300\text{ plf}$$

$$w_u = (1.2)(0.9\text{ klf} + 0.3\text{ klf}) + (1.6)(2.1\text{ klf}) = 4.8\text{ k/ft}$$

$$V_u = (10\text{ ft})(4.8\text{ klf}) - (4\text{ ft})(4.8\text{ klf}) = 28.8\text{ k}$$

$$M_u = (10\text{ ft})(4.8\text{ klf})(4\text{ ft}) - (4\text{ ft})(4.8\text{ klf})(2\text{ ft}) = 153.6\text{ ft-k}$$

$$\frac{V_u d}{M_u} = \frac{(28.8\text{ k})(24\text{ in.} - 3\text{ in.} - 3\text{ in.})}{(12)(153.6\text{ ft-k})} = 0.281 < 1.0 \quad \underline{\text{OK}}$$

$$V_c = \left(0.6\lambda\sqrt{f'_c} + 700\frac{V_u d_p}{M_u}\right) b_w d \qquad \text{(ACI Table 22.5.8.2a)}$$

$$= \left[(0.6)\ (1.0)\ (\sqrt{4000}\text{ psi}) + (700)(0.281)\right]\ (12\text{ in.})(18\text{ in.}) = 50{,}684\text{ lb}$$

$$\text{Minimum } V_c = (2)(1.0)(\sqrt{4000}\text{ psi})(12\text{ in.})(18\text{ in.}) = 27{,}322\text{ lb} < 50{,}684\text{ lb}$$

$$\text{Maximum } V_c = (5)(1.0)(\sqrt{4000}\text{ psi})(12\text{ in.})(18\text{ in.}) = 68{,}305\text{ lb} > 50{,}684\text{ lb}$$

$$V_c = 50{,}684\text{ lb}$$

More Detailed Method

$$I = \left(\frac{1}{12}\right)(12\text{ in.})(24\text{ in.})^3 = 13{,}824\text{ in.}^4$$

$$y_t = 12\text{ in.}$$

f_{pe} = compressive stress in concrete due to prestress after all losses

$$= \frac{P}{A} + \frac{Pec}{I}$$

$$f_{pe} = \frac{250{,}000\text{ lb}}{(12\text{ in.})(24\text{ in.})} + \frac{(250{,}000\text{ lb})(6\text{ in.})(12\text{ in.})}{13{,}824\text{ in.}^4} = 2170\text{ psi}$$

M_d = dead load moment at 4-ft point = (10 ft)(1.2 klf)(4 ft) − (4 ft)(1.2 klf)(2 ft)

= 38.4 ft-k

$$f_d = \frac{M_d y_t}{I} = \text{stress due to the dead load moment} = \frac{(12\text{ in./ft})(38{,}400\text{ ft-lb})(12\text{ in.})}{13{,}824\text{ in.}^4}$$

= 400 psi

$$M_{cre} = \text{cracking moment} = \left(\frac{I}{y_t}\right)\left(6\lambda\sqrt{f'_c} + f_{pe} - f_d\right) \qquad \text{(ACI Equation 22.5.8.3.1c)}$$

$$= \left(\frac{13{,}824\text{ in.}^4}{12\text{ in.}}\right)\left[(6)(1.0)\left(\sqrt{4000}\text{ psi} + 2170\text{ psi} - 400\text{ psi}\right)\right] = 2{,}476{,}193\text{ in.-lb}$$

= 206,349 ft-lb

w_u not counting beam weight = (1.2)(1.2 klf − 0.3 klf) + (1.6)(2.1 klf) = 4.44 k/ft

M_{max} = (10 ft)(4.44 klf)(4 ft) − (4 ft)(4.44 klf)(2 ft) = 142.08 ft-k = 142,080 ft-lb

V_i because of w_u occurring same time as M_{max} = (10 ft) (4.44 klf) – (4 ft) (4.44 klf)

= 26.64 k = 26,640 lb

V_d = dead load shear = (10 ft) (1.2 klf) – (4 ft) (1.2 klf) = 7.2 k = 7200 lb

d = 24 in. – 3 in. – 3 in. = 18 in. or (0.8) (24) = 19.2 in.

$$V_{ci} = 0.6\lambda\sqrt{f'_c}b_w d_p + V_d + \frac{V_i M_{cre}}{M_{max}} \qquad \text{(ACI Equation 22.5.8.3.1a)}$$

$$= (0.6)\,(1.0)\,(\sqrt{4000}\text{ psi})\,(12\text{ in.})\,(19.2\text{ in.}) + 7200\text{ lb} + \frac{(26{,}640\text{ lb})\,(206{,}349\text{ ft-lb})}{142{,}080\text{ ft-lb}}$$

= 54,634 lb ←

but need not be less than (1.7) (1.0) ($\sqrt{4000}$ psi) (12 in.) (19.2 in.) = 24,772 lb

Computing V_{cw}

f_{pc} = calculated compressive stress in psi at the centroid of the concrete because of the effective prestress

$$= \frac{250{,}000\text{ lb}}{(12\text{ in.})\,(24\text{ in.})} = 868\text{ psi}$$

V_p = vertical component of effective prestress at section = $\dfrac{9\text{ in.}}{\sqrt{9\text{ in.}^2 + 72\text{ in.}^2}}(250{,}000\text{ lb})$

$$= \left(\frac{9\text{ in.}}{72.56\text{ in.}}\right)(250{,}000\text{ lb}) = 31{,}009\text{ lb}$$

$$V_{cw} = (3.5\lambda\sqrt{f'_c} + 0.3f_{pc})\,b_w d + V_p \qquad \text{(ACI Equation 22.5.8.3.2)}$$

$$= (3.5)\,(1.0)\,(\sqrt{4000}\text{ psi} + 0.3 \times 868\text{ psi})\,(12\text{ in.})\,(19.2\text{ in.}) + 31{,}009\text{ lb} = 142{,}006\text{ lb}$$

Using Lesser of V_{ci} or V_{cw}

$$V_c = 54{,}634\text{ lb}$$

Example 19.6

Determine the spacing of #3 ⊔ stirrups required for the beam of Example 19.5 at 4 ft from the end support if f_{pu} is 250 ksi for the prestressing steel and f_{yt} for the stirrups is 40 ksi. Use the value of V_c obtained by the approximate method, 54,634 lb.

SOLUTION

$$w_u = (1.2)\,(1.2\text{ klf}) + (1.6)\,(2.1\text{ klf}) = 4.8\text{ k/ft}$$

$$V_u = (10\text{ ft})\,(4.8\text{ klf}) - (4\text{ ft})\,(4.8\text{ klf}) = 28.8\text{ k}$$

$$\phi V_c = (0.75)\,(54{,}634\text{ lb}) = 40{,}976\text{ lb}$$

$$> V_u = 28{,}800\text{ lb}$$

$$V_u > \frac{\phi V_c}{2} = 20{,}488\text{ lb} < \phi V_c$$

A minimum amount of reinforcement is needed.

Since effective prestress is greater than 40% of tensile strength of reinforcement,

$$A_v = \left(\frac{A_{ps}}{80}\right)\left(\frac{f_{pu}}{f_{yt}}\right)\left(\frac{s}{d}\right)\sqrt{\left(\frac{d}{b_w}\right)} \qquad \text{(ACI Table 9.6.3.3e)}$$

$$(2)\,(0.11) = \left(\frac{1.0\text{ in.}^2}{80}\right)\left(\frac{250{,}000\text{ psi}}{40{,}000\text{ psi}}\right)\left(\frac{s}{18\text{ in.}}\right)\sqrt{\frac{18\text{ in.}}{12\text{ in.}}}$$

$$s = 41.38\text{ in., but maximum } s = \left(\frac{3}{4}\right)(24\text{ in.}) = 18\text{ in.} \qquad \underline{\text{Use 18 in.}}$$

19.12 Additional Topics

This chapter has presented a brief discussion of prestressed concrete. A number of other important topics have been omitted from this introductory material. Several of these items are briefly mentioned in the paragraphs that follow.

Stresses in End Blocks

The part of a prestressed member around the end anchorages of the steel tendons is called the *end block*. In this region, the prestress forces are transferred from very concentrated areas out into the whole beam cross section. It has been found that the length of transfer for posttensioned members is less than the height of the beam and in fact is probably much less.

For posttensioned members, there is direct-bearing compression at the end anchorage; therefore, solid end blocks are usually used there to spread out the concentrated prestress forces. To prevent bursting of the block, either wire mesh or a grid of vertical and horizontal reinforcing bars is placed near the end face of the beam. In addition, both vertical and horizontal reinforcing bars are placed throughout the block.

For pretensioned members where the prestress is transferred to the concrete by bond over a distance approximately equal to the beam depth, a solid end block is probably not necessary, but spaced stirrups are needed. A great deal of information on the subject of end block stresses for posttensioned and pretensioned members is available.[5]

Composite Construction

Precast prestressed sections are frequently used in buildings and bridges in combination with cast-in-place concrete. Should such members be properly designed for shear transfer so that the two parts will act together as a unit, they are called *composite sections*. Examples of such members were previously shown in parts (e) and (f) of Figure 19.8. In composite construction, the parts that are difficult to form and that contain most of the reinforcing bars are precast, whereas the slabs and perhaps the top of the beams, which are relatively easy to form, are cast in place.

The precast sections are normally designed to support their own weights plus the green cast-in-place concrete in the slabs plus any other loads applied during construction. The dead and live loads applied after the slab hardens are supported by the composite section. The combination of the two parts will yield a composite section that has a very large moment of inertia and thus a very large resisting moment. It is usually quite economical to use (a) a precast prestressed beam made with a high-strength concrete and (b) a slab made with an ordinary grade of concrete. If this practice is followed, it will be necessary to account for the different moduli of elasticity of the two materials in calculating the composite properties (thus it becomes a transformed area problem).

[5] Nawy, *Prestressed Concrete*, p. 173.

Continuous Members

Continuous prestressed sections may be cast in place completely with their tendons running continuously from one end to the other. It should be realized for such members that where the service loads tend to cause positive moments, the tendons should produce negative moments and vice versa. This means that the tendons should be below the member's center of gravity in normally positive moment regions and above the center of gravity in normally negative moment regions. To produce the desired stress distributions, it is possible to use curved tendons and members of constant cross section or straight tendons with members of variable cross section. In Figure 19.14, several continuous beams of these types are shown.

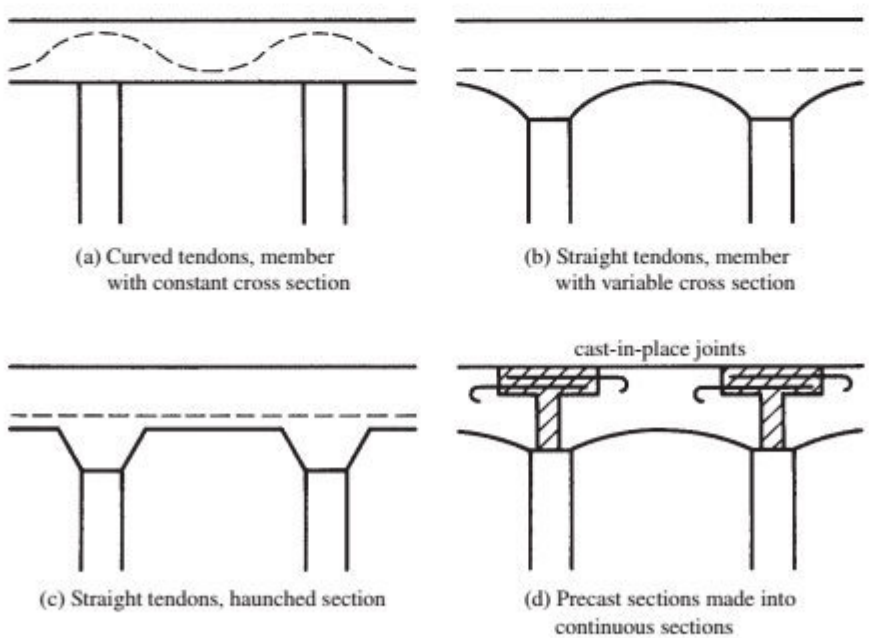
Another type of continuous section that has been used very successfully in the United States, particularly for bridge construction, involves the use of precast prestressed members made into continuous sections with cast-in-place concrete and regular reinforcing steel. Figure 19.14(d) shows such a case. For such construction, the precast section resists a portion of the dead load, whereas the continuous member resists the live load and the dead load that are applied after the cast-in-place concrete hardens.

Partial Prestressing

During the early days of prestressed concrete, the objective of the designer was to proportion members that could never be subject to tension when service loads were applied. Such members are said to be *fully prestressed*. Subsequent investigations of fully prestressed members have shown that they often have an appreciable amount of extra strength. As a result, many designers now believe that certain amounts of tensile stresses can be permitted under service loads. Members that are permitted to have some tensile stresses are said to be *partially prestressed*.

A major advantage of a partially prestressed beam is a decrease in camber. This is particularly important when the beam load or the dead load is quite low compared to the total design load.

To provide additional safety for partially prestressed beams, it is common practice to add some conventional reinforcement. This reinforcement will increase the ultimate flexural strength of the members and will help in carrying the tensile stresses in the beam.[6]



(a) Curved tendons, member with constant cross section

(b) Straight tendons, member with variable cross section

(c) Straight tendons, haunched section

(d) Precast sections made into continuous sections

FIGURE 19.14 Continuous beams.

[6] Lin, T. Y. and Burns, N. H., 1981, *Design of Prestressed Concrete Structures*, 3rd ed. (Hoboken, NJ: John Wiley & Sons), pp. 325–344.

19.13 Computer Example

Example 19.7

Use the Excel spreadsheet provided for Chapter 19 to solve Example 19.2.

SOLUTION

Open the Chapter 19 Excel spreadsheet, and open the worksheet Stress Calculations. Enter values only for cells highlighted in yellow (only in the Excel spreadsheets, not in the printed example). Results are shown below. This spreadsheet does many more calculations. The ones to be compared to Example 19.2 are circled.

Stresses in prestressed concrete beams

P_i =	294.118	kips				
R =	0.85					
$e_{midspan}$ =	9	in.				
$e_{support}$ =	9	in.				
tendon dia. =	0.5	in.				
S_1 =	1152	$in.^3$				
S_2 =	1152	$in.^3$				
A =	288	$in.^2$				
f'_{ci} =	3500	psi	f_{ti} =	177	psi	
f'_c =	5000	psi	f_{ts} =	849	psi	
γ_c =	145	pcf	β_1 =	0.8		
ℓ =	20	ft				
w_D =	1700	plf				
w_L =	1000	plf				
% of w_L sustained	50.00					
$e_{0.4l}$ =	9	in.				
$e_{50\ diam}$	9.00	in.				
w_o =	300.00	plf				
M_o =	180	in.-k				
M_s =	1500	in.-k				
M_T =	1800	in.-k				

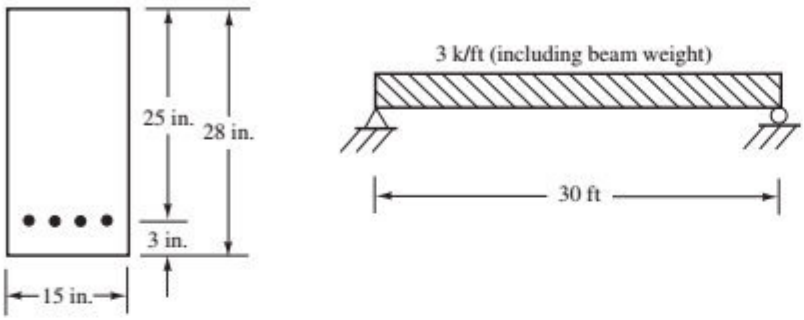
Stress calculation at release

				stress					
	$-P_i/A$	$+P_ie/S_1$	$-M_o/S_1$	at midspan	at x = 0.4	at 50 dia.	at support	allowable	
f_1 =	−1021	2298	−156.25	1120	1127	1277	1277	177	= f_{ti} midspan
								355	= f_{ti} ends
	$-P_i/A$	$-P_ie/S_2$	$+M_o/S_2$						
f_2 =	−1021	−2298	156	−3163	−3169	−3319	−3319	−2100	= f_{ci}
	$-P_e/A$	$+P_ee/S_1$	$-M_T/S_1$						
Loaded (M_t) f_1 =	−868	1953	−1562.5	−477	−415	1085	1085	−3000	= f_{cs} (for M_t)
	$-P_e/A$	$+P_ee/S_1$	$-M_s/S_1$						
Loaded (M_s) f_1 =	−868	1953	−1302.1	−217	−165	1085	1085	−2250	= f_{cs} (for M_s)
	$-P_e/A$	$-P_ee/S_2$	$+M_T/S_1$						
f_2 =	−868	−1953	1563	−1259	−1321	−2821	−2821	849	

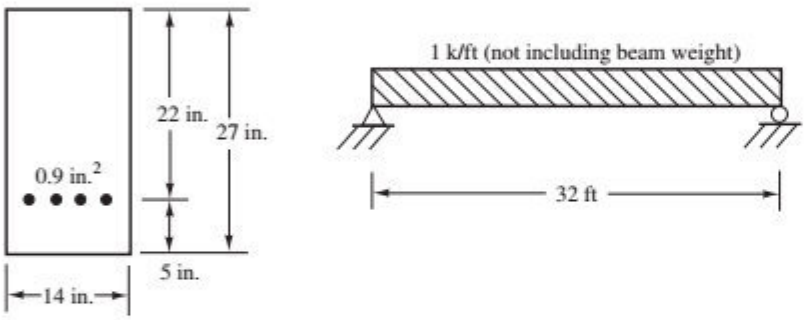
PROBLEMS

For all problems, assume concrete is normal weight unless otherwise stated.

Problem 19.1 The beam shown has an effective total prestress of 240 k. Calculate the fiber stresses at top and bottom of the beam shown at the ends and centerline. The tendons are assumed to be straight. (*Ans.* $f_{top} = -1.291$ ksi, $f_{bot} = +0.148$ ksi at the centerline)



Problem 19.2 Compute the stresses at top and bottom of the beam shown at the ends and centerline immediately after the cables are cut. Assume straight cable and 10% losses. Initial prestress is 170 ksi.

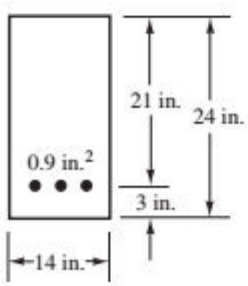


Problem 19.3 The beam shown has a 30-ft simple span, $f'_c = 5000$ psi, $f_{pu} = 250{,}000$ psi, and the initial prestress is 160,000 psi, $f_{py} = 0.85f_{pu}$, and 10% losses.

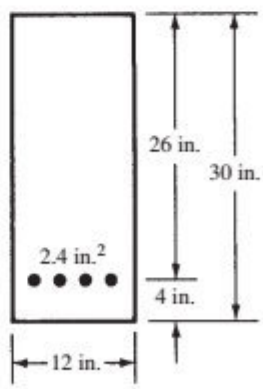
(a) Calculate the concrete stresses at top and bottom of the beam at midspan immediately after the tendons are cut. (*Ans.* $f_{top} = +0.183$ ksi, $f_{bot} = -1.041$ ksi)

(b) Recalculate the stresses at midspan after assumed prestress losses in the tendons of 20%. (*Ans.* $f_{top} = +0.076$ ksi, $f_{bot} = -0.762$ ksi)

(c) What maximum service live load can this beam support in addition to its own weight if allowable stresses of $0.6f'_c$ in compression and $12\sqrt{f'_c}$ in tension are permitted? (*Ans.* 1.605 k/ft)



Problem 19.4 Using the same allowable stresses permitted in Problem 19.3, what total uniform load can the beam shown support for a 50-ft simple span in addition to its own weight? Assume 20% prestress loss.



Problem 19.5 Compute the cracking moment and the permissible ultimate moment capacity of the beam of Problem 19.3 if $f'_c = 5000$ psi and $f_{py} = 200{,}000$ psi. (*Ans.* $M_{cr} = 144.74$ ft-k, $\phi M_n = 300$ ft-k)

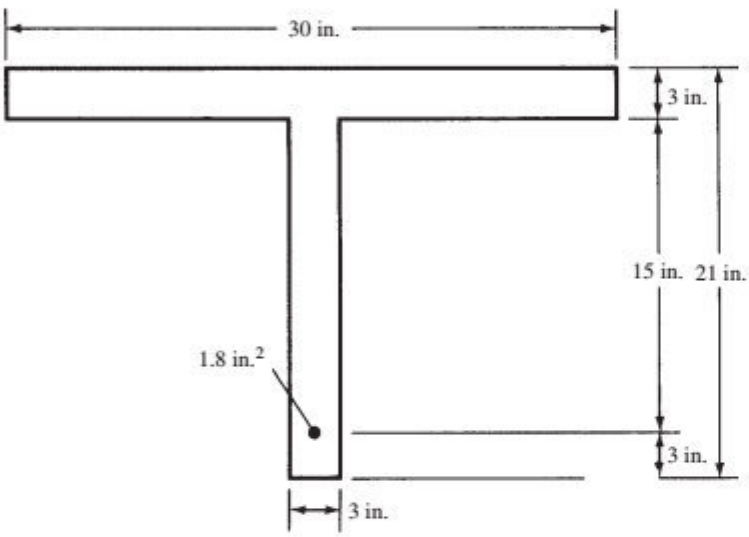
Problem 19.6 Compute the stresses at top and bottom of the beam of Problem 19.1 at the centerline if it is picked up at the centerline. Assume 100% impact. Concrete weighs 150 lb-ft^3.

Problem 19.7 Determine the stresses at the one-third points of the beam of Problem 19.6 if the beam is picked up at those points. (*Ans.* $f_{top} = +1.009$ ksi, $f_{bot} = -2.151$ ksi)

Problem 19.8 Calculate the design moment capacity of a 12-in. × 20-in. pretensioned beam that is prestressed with 1.2 in.^{2} of steel tendons stressed to an initial stress of 160 ksi. The center of gravity of the tendons is 3 in. above the bottom of the beam. $f'_c = 5000$ psi, $f_{py} = 200{,}000$ psi, and $f_{pu} = 250{,}000$ psi.

Problem 19.9 Compute the design moment capacity of the pretensioned beam of Problem 19.2 if $f'_c = 5000$ psi, $f_{py} = 200{,}000$ psi, and $f_{pu} = 225$ ksi. Assume $f_{py}/f_{pu} > 0.85$. (*Ans.* 293.1 ft-k)

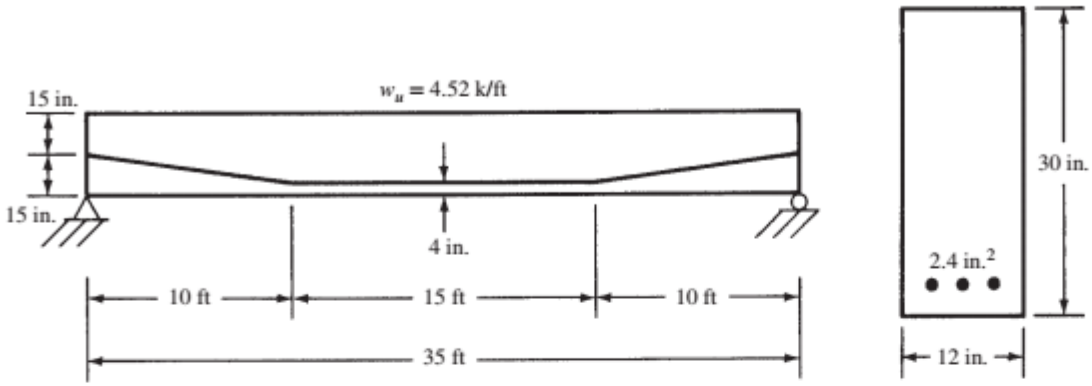
Problem 19.10 Compute the design moment capacity of the bonded T beam shown if $f'_c = 5000$ psi, $f_{pu} = 250{,}000$ psi, and the initial stress in the cables is 160,000 psi. Also, $f_{py} = 200{,}000$ psi.



Problem 19.11 Calculate the deflection at the centerline of the beam of Problem 19.3 immediately after the cables are cut. Assume cable stress = 160,000 psi. (*Ans.* −0.225 in. ↑)

Problem 19.12 Calculate the deflection at the centerline of the beam of Problem 19.1 immediately after the cables are cut. Assume P initial = 240 k and P after losses = 190 k. Repeat the calculation after losses if 20-k concentrated live loads are located at the one-third points of the beam and $f'_c = 5000$ psi. Assume that cables are straight and no loads are present other than the beam weight and the two 20-k loads. Use I_g for all calculations.

Problem 19.13 Determine the shearing strength of the beam shown 3 ft from the supports using the approximate method allowed by the ACI Code. Determine the required spacing of #3 ⊔ stirrups at the same section if $f'_c = 5000$ psi, $f_{pu} = 250{,}000$ psi, f_{yt} for stirrups = 50,000 psi, and $f_{se} = 200{,}000$ psi. (*Ans.* $s = 21.73$ in.)

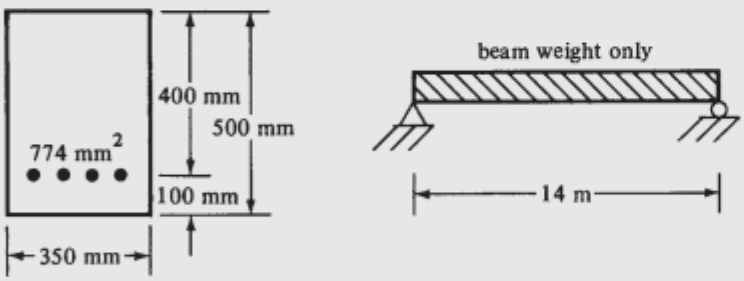


Problem 19.14 Repeat Problem 19.1 using Chapter 19 spreadsheet.

Problem 19.15 Repeat Problem 19.3 using Chapter 19 spreadsheet. (*Ans.* same as Problem 19.3)

Problems in SI Units

Problem 19.16 Immediately after cutting the cables in the beam shown, they have an effective prestress of 1.260 GPa. Determine the stresses at top and bottom of the beam at the ends and centerline. The concrete weighs 23.5 kN/m^3, cables are straight, $E = 27{,}800$ MPa, and $f_{py} = 0.8 f_{pu}$.

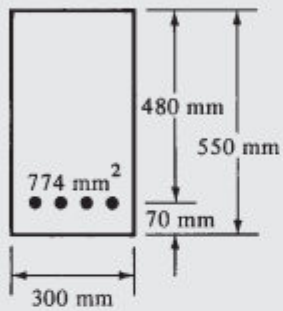


Problem 19.17 The beam shown has a 12-m simple span: $f'_c = 35$ MPa, $f_{pu} = 1.725$ GPa, and the initial prestress is 1.10 GPa.

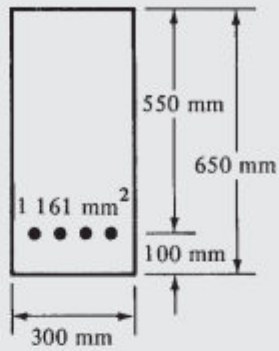
(a) Calculate the concrete stresses at top and bottom of the beam at midspan immediately after the tendons are cut. (*Ans.* $f_{top} = +1.766$ MPa, $f_{bot} = -12{,}086$ MPa)

(b) Recalculate the stresses after assumed losses in the tendons of 18%. (*Ans.* $f_{top} = +0.618$ MPa, $f_{bot} = -9.08$ MPa)

(c) What maximum service uniform live load can the beam support in addition to its own weight if allowable stresses of $0.45f'_c$ in compression and $0.5\sqrt{f'_c}$ in tension are permitted? (*Ans.* 10.095 kN/m)



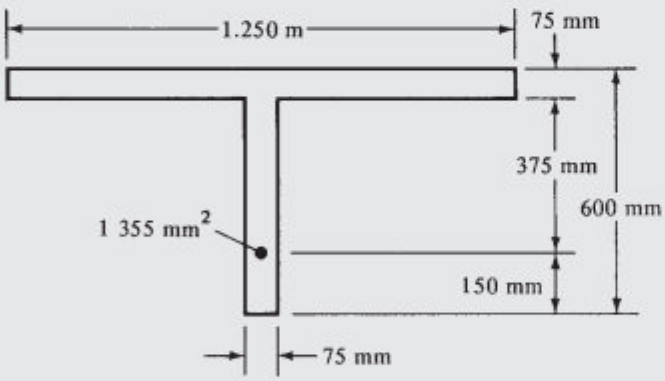
Problem 19.18 Using the same allowable stresses permitted and cable stresses as in Problem 19.17, what total uniform load, including beam weight, can the beam shown support for a 15-m simple span?



Problem 19.19 Compute the cracking moment and the design moment capacity of the bonded beam of Problem 19.17 if $f_{py} = 0.8f_{pu}$. (*Ans.* 181.68 kN·m, 438.07 kN·m)

Problem 19.20 Compute the stresses at top and bottom of the beam of Problem 19.16 if it is picked up at its one-third points. Assume an impact of 100%.

Problem 19.21 Compute the design moment capacity of the bonded T beam shown if $f'_c = 35$ MPa, $f_{pu} = 1.725$ GPa, $f_{py} = 0.8f_{pu}$, and the initial stress in the cables is 1.100 GPa. (*Ans.* 843.6 kN·m)



Problem 19.22 Calculate the deflection at the centerline of the beam of Problem 19.16 immediately after the cables are cut. Use I_g.

Tables and Graphs: U.S. Customary Units

APPENDIX A

TABLE A.1 Values of Modulus of Elasticity for Normal-Weight Concrete

U.S. Customary Units	
f'_c (psi)	E_c (psi)
3000	3,160,000
3500	3,410,000
4000	3,640,000
4500	3,870,000
5000	4,070,000

Source Notes:

Graphs 2 through 13 are reprinted from *Design Handbook, Volume 2, Columns* (SP-17A), 1978, with permission of American Concrete Institute.

Table A.3(a) and A.3(b) are reprinted from *Manual of Standard Practice*, 28th ed., 2009, with permission from Concrete Reinforcing Steel Institute, Schaumburg, IL.

Tables A.16 through A.20 are reprinted from Commentary on Building Code Requirements For Reinforced Concrete (ACI 318–77) with permission of the American Concrete Institute.

TABLE A.2 Designations, Areas, Perimeters, and Weights of Standard Bars

	U.S. Customary Units		
Bar No.	**Diameter (in.)**	**Cross-Sectional Area (in.^{2})**	**Unit Weight (lb/ft)**
3	0.375	0.11	0.376
4	0.500	0.20	0.668
5	0.625	0.31	1.043
6	0.750	0.44	1.502
7	0.875	0.60	2.044
8	1.000	0.79	2.670
9	1.128	1.00	3.400
10	1.270	1.27	4.303
11	1.410	1.56	5.313
14	1.693	2.25	7.650
18	2.257	4.00	13.600

TABLE A.3(a) Common Styles of Welded Wire Reinforcement Sheets

Style	Area ($in.^2/ft$)	Weight ($lb/100\ ft^2$)
4 × 4 – W1.4 × W1.4*	0.042	31
4 × 4 – W2.0 × W2.0*	0.060	44
4 × 4 – W2.9 × W2.9*	0.087	62
4 × 4 – W3.1 × W3.1	0.093	65
4 × 4 – W4.0 × W4.0*	0.120	88
6 × 6 – W1.4 × W1.4*	0.028	21
6 × 6 – W2.0 × W2.0*	0.040	30
6 × 6 – W2.9 × W2.9*	0.058	42
6 × 6 – W4.0 × W4.0*	0.080	58
6 × 6 – W4.2 × W4.2	0.084	60
6 × 6 – W4.4 × W4.4	0.088	63
6 × 6 – W4.7 × W4.7	0.094	68
6 × 6 – W7.5 × W7.5	0.150	108
6 × 6 – W8.1 × W8.1	0.162	116
6 × 6 – W8.3 × W8.3	0.166	119
12 × 12 – W8.3 × W8.3	0.083	63
12 × 12 – W8.8 × W8.8	0.088	66
12 × 12 – W9.1 × W9.1	0.091	69
12 × 12 – W9.4 × W9.4	0.094	71
12 × 12 – W15 × W15	0.150	113
12 × 12 – W16 × W16	0.160	120
12 × 12 – W16.6 × W16.6	0.166	125
12 × 12 – W17.1 × W17.1	0.171	128

*These styles may be obtained in roll form. It is recommended that rolls be straightened and cut to size before placement.

TABLE A.3(b) Cross-Sectional Area and Weight of Welded Wire Reinforcement (in.-lb)

Wire Size, W or D	Nom. Diam. (in.)	Nom. Wt (lb/ft)	A_s (in.^{2}/ft) per Wire Spacing (in.)						
			2	3	4	6	8	10	12
45	0.757	1.530	2.700	1.800	1.350	0.900	0.675	0.540	0.450
31	0.628	1.054	1.860	1.240	0.930	0.620	0.465	0.372	0.310
30	0.618	1.020	1.800	1.200	0.900	0.600	0.450	0.360	0.300
28	0.597	0.952	1.680	1.120	0.840	0.560	0.420	0.336	0.280
26	0.575	0.884	1.560	1.040	0.780	0.520	0.390	0.312	0.260
24	0.553	0.816	1.440	0.960	0.720	0.480	0.360	0.288	0.240
22	0.529	0.748	1.320	0.880	0.660	0.440	0.330	0.264	0.220
20	0.505	0.680	1.200	0.800	0.600	0.400	0.300	0.240	0.200
18	0.479	0.612	1.080	0.720	0.540	0.360	0.270	0.216	0.180
16	0.451	0.544	0.960	0.640	0.480	0.320	0.240	0.192	0.160
14	0.422	0.476	0.840	0.560	0.420	0.280	0.210	0.168	0.140
12	0.391	0.408	0.720	0.480	0.360	0.240	0.180	0.144	0.120
11	0.374	0.374	0.660	0.440	0.330	0.220	0.165	0.132	0.110
10.5	0.366	0.357	0.630	0.420	0.315	0.210	0.158	0.126	0.105
10	0.357	0.340	0.600	0.400	0.300	0.200	0.150	0.120	0.100
9.5	0.348	0.323	0.570	0.380	0.285	0.190	0.143	0.114	0.095
9	0.339	0.306	0.540	0.360	0.270	0.180	0.135	0.108	0.090
8.5	0.329	0.289	0.510	0.340	0.255	0.170	0.128	0.102	0.085
8	0.319	0.272	0.480	0.320	0.240	0.160	0.120	0.096	0.080
7.5	0.309	0.255	0.450	0.300	0.225	0.150	0.113	0.090	0.075
7	0.299	0.238	0.420	0.280	0.210	0.140	0.105	0.084	0.070
6.5	0.288	0.221	0.390	0.260	0.195	0.130	0.098	0.078	0.065
6	0.276	0.204	0.360	0.240	0.180	0.120	0.090	0.072	0.060
5.5	0.265	0.187	0.330	0.220	0.165	0.110	0.083	0.066	0.055
5	0.252	0.170	0.300	0.200	0.150	0.100	0.075	0.060	0.050
4.5	0.239	0.153	0.270	0.180	0.135	0.090	0.068	0.054	0.045
4	0.226	0.136	0.240	0.160	0.120	0.080	0.060	0.048	0.040
3.5	0.211	0.119	0.210	0.140	0.105	0.070	0.053	0.042	0.035
3	0.195	0.102	0.180	0.120	0.090	0.080	0.045	0.036	0.030
2.9	0.192	0.099	0.174	0.116	0.087	0.058	0.044	0.035	0.029
2.5	0.178	0.085	0.150	0.100	0.075	0.050	0.038	0.030	0.025
2	0.160	0.068	0.120	0.080	0.060	0.040	0.030	0.024	0.020
1.4	0.134	0.048	0.084	0.056	0.042	0.028	0.021	0.017	0.014

Notes:

1. The above listing of plain and deformed wire sizes represents wires normally selected to manufacture welded wire reinforcement to specific areas of reinforcement. Wire sizes other than those listed above, including larger sizes, may be available if the quantity required is sufficient to justify manufacture.
2. The nominal diameter of a deformed wire is equivalent to the diameter of a plain wire having the same weight per foor as the deformed wire.
3. The ACI Building Code requirements for tension development lengths and tension lap splice lengths of welded wire reinforcement are not included in this chapter. These design requirements are covered in *Reinforcing Bars: Anchorages and Splices* available from CRSI. For additional information, see *Manual of Standard Practice—Structural Welded Wire Reinforcement* and *Structural Detailing Manual,* both published by the Wire Reinforcement Institute.

TABLE A.4 Area of Groups of Bars, $in.^2$

Number of Bars	Bar Size Designation										
	#3	#4	#5	#6	#7	#8	#9	#10	#11	#14	#18
2	0.22	0.39	0.61	0.88	1.20	1.57	2.00	2.54	3.12	4.50	8.00
3	0.33	0.59	0.92	1.33	1.80	2.36	3.00	3.81	4.68	6.75	12.00
4	0.44	0.79	1.23	1.77	2.41	3.14	4.00	5.08	6.24	9.00	16.00
5	0.55	0.98	1.53	2.21	3.01	3.93	5.00	6.35	7.80	11.25	20.00
6	0.66	1.18	1.84	2.65	3.61	4.71	6.00	7.62	9.36	13.50	24.00
7	0.77	1.37	2.15	3.09	4.21	5.50	7.00	8.89	10.92	15.75	28.00
8	0.88	1.57	2.45	3.53	4.81	6.28	8.00	10.16	12.48	18.00	32.00
9	0.99	1.77	2.76	3.98	5.41	7.07	9.00	11.43	14.04	20.25	36.00
10	1.10	1.96	3.07	4.42	6.01	7.85	10.00	12.70	15.60	22.50	40.00
11	1.21	2.16	3.37	4.86	6.61	8.64	11.00	13.97	17.16	24.75	44.00
12	1.33	2.36	3.68	5.30	7.22	9.42	12.00	15.24	18.72	27.00	48.00
13	1.44	2.55	3.99	5.74	7.82	10.21	13.00	16.51	20.28	29.25	52.00
14	1.55	2.75	4.30	6.19	8.42	11.00	14.00	17.78	21.84	31.50	56.00
15	1.66	2.95	4.60	6.63	9.02	11.78	15.00	19.05	23.40	33.75	60.00
16	1.77	3.14	4.91	7.07	9.62	12.57	16.00	20.32	24.96	36.00	64.00
17	1.88	3.34	5.22	7.51	10.22	13.35	17.00	21.59	26.52	38.25	68.00
18	1.99	3.53	5.52	7.95	10.82	14.14	18.00	22.86	28.08	40.50	72.00
19	2.10	3.73	5.83	8.39	11.43	14.92	19.00	24.13	29.64	42.75	76.00
20	2.21	3.93	6.14	8.84	12.03	15.71	20.00	25.40	31.20	45.00	80.00
21	2.32	4.12	6.44	9.28	12.63	16.49	21.00	26.67	32.76	47.25	84.00
22	2.43	4.32	6.75	9.72	13.23	17.28	22.00	27.94	34.32	49.50	88.00

TABLE A.5 Minimum Width of Web for Multiple Bars, in., ACI 318-14[a,b,c]

Bar Size Designation	#3 "U" Shaped Stirrups Number of Bars in a Single Layer					#4 "U" Shaped Stirrups Number of Bars in a Single Layer					Add for Each Additional Bar (in.)
	2	3	4	5	6	2	3	4	5	6	
#4	6.75	8.25	9.75	11.25	12.75	7.50	9.00	10.50	12.00	13.50	1.50
#5	6.88	8.50	10.13	11.75	13.38	7.63	9.25	10.88	12.50	14.13	1.63
#6	7.00	8.75	10.50	12.25	14.00	7.75	9.50	11.25	13.00	14.75	1.75
#7	7.13	9.00	10.88	12.75	14.63	7.88	9.75	11.63	13.50	15.38	1.88
#8	7.25	9.25	11.25	13.25	15.25	8.00	10.00	12.00	14.00	16.00	2.00
#9	7.51	9.76	12.02	14.27	16.53	8.26	10.51	12.77	15.02	17.28	2.26
#10	7.79	10.33	12.87	15.41	17.95	8.54	11.08	13.62	16.16	18.70	2.54
#11	8.07	10.89	13.71	16.53	19.35	8.82	11.64	14.46	17.28	20.10	2.82
#14	8.83	12.22	15.60	18.99	22.37	9.39	12.77	16.16	19.54	22.93	3.39
#18	10.52	15.04	19.55	24.06	28.58	10.77	15.29	19.80	24.31	28.83	4.51

[a]Minimum spacing and bend radii based on ACI 318-2014, Sections 25.2.1 and 25.3.2 (3/4 in. max agg. size).
[b]Bars are not lap-spliced or bundled.
[c]Cover for stirrups assumed to be 1 1/2 in. (based on interior exposure).

TABLE A.6 Area of Bars per ft of Width for Slabs and Footings, $in.^2/ft$

Bar Spacing, in.	Bar Size Designation								
	3	4	5	6	7	8	9	10	11
3	0.44	0.79	1.23	1.77	2.41	3.14	4.00	5.08	6.24
3 1/2	0.38	0.67	1.05	1.51	2.06	2.69	3.43	4.35	5.35
4	0.33	0.59	0.92	1.33	1.80	2.36	3.00	3.81	4.68
4 1/2	0.29	0.52	0.82	1.18	1.60	2.09	2.67	3.39	4.16
5	0.27	0.47	0.74	1.06	1.44	1.88	2.40	3.05	3.74
5 1/2	0.24	0.43	0.67	0.96	1.31	1.71	2.18	2.77	3.40
6	0.22	0.39	0.61	0.88	1.20	1.57	2.00	2.54	3.12
6 1/2	0.20	0.36	0.57	0.82	1.11	1.45	1.85	2.34	2.88
7	0.19	0.34	0.53	0.76	1.03	1.35	1.71	2.18	2.67
7 1/2	0.18	0.31	0.49	0.71	0.96	1.26	1.60	2.03	2.50
8	0.17	0.29	0.46	0.66	0.90	1.18	1.50	1.91	2.34
9	0.15	0.26	0.41	0.59	0.80	1.05	1.33	1.69	2.08
10	0.13	0.24	0.37	0.53	0.72	0.94	1.20	1.52	1.87
11	0.12	0.21	0.33	0.48	0.66	0.86	1.09	1.39	1.70
12	0.11	0.20	0.31	0.44	0.60	0.79	1.00	1.27	1.56
13	0.10	0.18	0.28	0.41	0.56	0.72	0.92	1.17	1.44
14	0.09	0.17	0.26	0.38	0.52	0.67	0.86	1.09	1.34
15	0.09	0.16	0.25	0.35	0.48	0.63	0.80	1.02	1.25
16	0.08	0.15	0.23	0.33	0.45	0.59	0.75	0.95	1.17
17	0.08	0.14	0.22	0.31	0.42	0.55	0.71	0.90	1.10
18	0.07	0.13	0.20	0.29	0.40	0.52	0.67	0.85	1.04

TABLE A.7 Values of ρ Balanced, ρ to Achieve Various ϵ_t Values, and ρ Minimum for Flexure. All Values Are for Tensile Reinforced Rectangular Sections

f_y	f'_c	3000 psi $\beta_1 = 0.85$	4000 psi $\beta_1 = 0.85$	5000 psi $\beta_1 = 0.80$	6000 psi $\beta_1 = 0.75$
Grade 40 40,000 psi (275.8 MPa)	ρ balanced	0.0371	0.0495	0.0582	0.0655
	ρ when $\epsilon_t = 0.004$	0.0232	0.0310	0.0364	0.0410
	ρ when $\epsilon_t = 0.005$	0.0203	0.0271	0.0319	0.0359
	ρ when $\epsilon_t = 0.0075$	0.0155	0.0206	0.0243	0.0273
	ρ min for flexure	0.0050	0.0050	0.0053	0.0058
Grade 50 50,000 psi (344.8 MPa)	ρ balanced	0.0275	0.0367	0.0432	0.0486
	ρ when $\epsilon_t = 0.004$	0.0186	0.0248	0.0291	0.0328
	ρ when $\epsilon_t = 0.005$	0.0163	0.0217	0.0255	0.0287
	ρ when $\epsilon_t = 0.0075$	0.0124	0.0165	0.0194	0.0219
	ρ min for flexure	0.0040	0.0040	0.0042	0.0046
Grade 60 60,000 psi (413.7 MPa)	ρ balanced	0.0214	0.0285	0.0335	0.0377
	ρ when $\epsilon_t = 0.004$	0.0155	0.0206	0.0243	0.0273
	ρ when $\epsilon_t = 0.005$	0.0136	0.0181	0.0212	0.0239
	ρ when $\epsilon_t = 0.0075$	0.0103	0.0138	0.0162	0.0182
	ρ min for flexure	0.0033	0.0033	0.0035	0.0039
Grade 75 75,000 psi (517.1 MPa)	ρ balanced	0.0155	0.0207	0.0243	0.0274
	ρ when $\epsilon_t = 0.004$	0.0124	0.0165	0.0194	0.0219
	ρ when $\epsilon_t = 0.005$	0.0108	0.0144	0.0170	0.0191
	ρ when $\epsilon_t = 0.0075$	0.0083	0.0110	0.0130	0.0146
	ρ min for flexure	0.0027	0.0027	0.0028	0.0031

TABLE A.8 f_y = 40,000 psi; f'_c = 3000 psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$		ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for temp. and shrinkage	0.0020	78.74	ρ_{min} for flexure	0.0050	192.1	0.0080	299.9	0.0110	401.9
	0.0021	82.62		0.0051	195.8	0.0081	303.4	0.0111	405.2
	0.0022	86.48		0.0052	199.5	0.0082	306.8	0.0112	408.5
	0.0023	90.34		0.0053	203.2	0.0083	310.3	0.0113	411.8
	0.0024	94.19		0.0054	206.8	0.0084	313.8	0.0114	415.1
	0.0025	98.04		0.0055	210.5	0.0085	317.3	0.0115	418.4
	0.0026	101.9		0.0056	214.1	0.0086	320.7	0.0116	421.7
	0.0027	105.7		0.0057	217.8	0.0087	324.2	0.0117	424.9
	0.0028	109.5		0.0058	221.4	0.0088	327.6	0.0118	428.2
	0.0029	113.4		0.0059	225.0	0.0089	331.1	0.0119	431.4
	0.0030	117.2		0.0060	228.7	0.0090	334.5	0.0120	434.7
	0.0031	121.0		0.0061	232.3	0.0091	337.9	0.0121	437.9
	0.0032	124.8		0.0062	235.9	0.0092	341.4	0.0122	441.2
	0.0033	128.6		0.0063	239.5	0.0093	344.8	0.0123	444.4
	0.0034	132.4		0.0064	243.1	0.0094	348.2	0.0124	447.6
	0.0035	136.2		0.0065	246.7	0.0095	351.6	0.0125	450.8
	0.0036	139.9		0.0066	250.3	0.0096	355.0	0.0126	454.0
	0.0037	143.7		0.0067	253.9	0.0097	358.4	0.0127	457.2
	0.0038	147.5		0.0068	257.4	0.0098	361.8	0.0128	460.4
	0.0039	151.2		0.0069	261.0	0.0099	365.2	0.0129	463.6
	0.0040	155.0		0.0070	264.6	0.0100	368.5	0.0130	466.8
	0.0041	158.7		0.0071	268.1	0.0101	371.9	0.0131	470.0
	0.0042	162.5		0.0072	271.7	0.0102	375.3	0.0132	473.2
	0.0043	166.2		0.0073	275.2	0.0103	378.6	0.0133	476.3
	0.0044	169.9		0.0074	278.8	0.0104	382.0	0.0134	479.5
	0.0045	173.6		0.0075	282.3	0.0105	385.3	0.0135	482.6
	0.0046	177.4		0.0076	285.8	0.0106	388.6	0.0136	485.8
	0.0047	181.1		0.0077	289.3	0.0107	392.0	0.0137	488.9
	0.0048	184.8		0.0078	292.9	0.0108	395.3	0.0138	492.1
	0.0049	188.5		0.0079	296.4	0.0109	398.6	0.0139	495.2

(*continues*)

TABLE A.8 (*Continued*)

	ρ	$\frac{M_u}{\phi bd^2}$		ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
	0.0140	498.3		0.0156	547.4	0.0172	594.9	0.0188	640.8
	0.0141	501.4		0.0157	550.4	0.0173	597.8	0.0189	643.6
	0.0142	504.5		0.0158	553.4	0.0174	600.7	0.0190	646.4
	0.0143	507.6		0.0159	556.4	0.0175	603.6	0.0191	649.2
	0.0144	510.7		0.0160	559.4	0.0176	606.5	0.0192	652.0
	0.0145	513.8		0.0161	562.4	0.0177	609.4	0.0193	654.8
	0.0146	516.9		0.0162	565.4	0.0178	612.3	0.0194	657.6
	0.0147	520.0		0.0163	568.4	0.0179	615.2	0.0195	660.3
	0.0148	523.1		0.0164	571.4	0.0180	618.0	0.0196	663.1
	0.0149	526.1		0.0165	574.3	0.0181	620.9	0.0197	665.9
	0.0150	529.2		0.0166	577.3	0.0182	623.8	0.0198	668.6
	0.0151	532.2		0.0167	580.2	0.0183	626.6	0.0199	671.4
	0.0152	535.3		0.0168	583.2	0.0184	629.5	0.0200	674.1
	0.0153	538.3		0.0169	586.1	0.0185	632.3	0.0201	676.9
	0.0154	541.4		0.0170	589.1	0.0186	635.1	0.0202	679.6
	0.0155	544.4		0.0171	592.0	0.0187	638.0	0.0203	682.3

TABLE A.9 $f_y = 40{,}000$ psi; $f'_c = 4000$ psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$		ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for temp. and shrinkage	0.0020	79.06	ρ_{min} for flexure	0.0050	194.1	0.0080	304.9	0.0110	411.4
	0.0021	82.96		0.0051	197.9	0.0081	308.5	0.0111	414.9
	0.0022	86.86		0.0052	201.6	0.0082	312.1	0.0112	418.4
	0.0023	90.75		0.0053	205.4	0.0083	315.7	0.0113	421.9
	0.0024	94.64		0.0054	209.1	0.0084	319.3	0.0114	425.3
	0.0025	98.53		0.0055	212.9	0.0085	322.9	0.0115	428.8
	0.0026	102.4		0.0056	216.6	0.0086	326.5	0.0116	432.2
	0.0027	106.3		0.0057	220.3	0.0087	330.1	0.0117	435.7
	0.0028	110.2		0.0058	224.1	0.0088	333.7	0.0118	439.1
	0.0029	114.0		0.0059	227.8	0.0089	337.3	0.0119	442.6
	0.0030	117.9		0.0060	231.5	0.0090	340.9	0.0120	446.0
	0.0031	121.7		0.0061	235.2	0.0091	344.5	0.0121	449.4
	0.0032	125.6		0.0062	238.9	0.0092	348.0	0.0122	452.9
	0.0033	129.4		0.0063	242.6	0.0093	351.6	0.0123	456.3
	0.0034	133.3		0.0064	246.3	0.0094	355.1	0.0124	459.7
	0.0035	137.1		0.0065	250.0	0.0095	358.7	0.0125	463.1
	0.0036	141.0		0.0066	253.7	0.0096	362.2	0.0126	466.5
	0.0037	144.8		0.0067	257.4	0.0097	365.8	0.0127	469.9
	0.0038	148.6		0.0068	261.1	0.0098	369.3	0.0128	473.3
	0.0039	152.4		0.0069	264.8	0.0099	372.9	0.0129	476.7
	0.0040	156.2		0.0070	268.4	0.0100	376.4	0.0130	480.1
	0.0041	160.0		0.0071	272.1	0.0101	379.9	0.0131	483.5
	0.0042	163.8		0.0072	275.8	0.0102	383.4	0.0132	486.9
	0.0043	167.6		0.0073	279.4	0.0103	387.0	0.0133	490.2
	0.0044	171.4		0.0074	283.1	0.0104	390.5	0.0134	493.6
	0.0045	175.2		0.0075	286.7	0.0105	394.0	0.0135	497.0
	0.0046	179.0		0.0076	290.4	0.0106	397.5	0.0136	500.3
	0.0047	182.8		0.0077	294.0	0.0107	401.0	0.0137	503.7
	0.0048	186.6		0.0078	297.6	0.0108	404.5	0.0138	507.0
	0.0049	190.4		0.0079	301.3	0.0109	408.0	0.0139	510.4

(*continues*)

TABLE A.9 (*Continued*)

	ρ	$\frac{M_u}{\phi bd^2}$		ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
	0.0140	513.7		0.0173	621.4	0.0206	723.8	0.0239	821.2
	0.0141	517.1		0.0174	624.5	0.0207	726.9	0.0240	824.1
	0.0142	520.4		0.0175	627.7	0.0208	729.9	0.0241	826.9
	0.0143	523.7		0.0176	630.9	0.0209	732.9	0.0242	829.8
	0.0144	527.1		0.0177	634.1	0.0210	735.9	0.0243	832.6
	0.0145	530.4		0.0178	637.2	0.0211	738.9	0.0244	835.5
	0.0146	533.7		0.0179	640.4	0.0212	741.9	0.0245	838.3
	0.0147	537.0		0.0180	643.5	0.0213	744.9	0.0246	841.2
	0.0148	540.3		0.0181	646.7	0.0214	747.9	0.0247	844.0
	0.0149	543.6		0.0182	649.8	0.0215	750.9	0.0248	846.8
	0.0150	546.9		0.0183	653.0	0.0216	753.9	0.0249	849.7
	0.0151	550.2		0.0184	656.1	0.0217	756.9	0.0250	852.5
	0.0152	553.5		0.0185	659.2	0.0218	759.8	0.0251	855.3
	0.0153	556.7		0.0186	662.3	0.0219	762.8	0.0252	858.1
	0.0154	560.0		0.0187	665.5	0.0220	765.8	0.0253	860.9
	0.0155	563.3		0.0188	668.6	0.0221	768.7	0.0254	863.7
	0.0156	566.6		0.0189	671.7	0.0222	771.7	0.0255	866.5
	0.0157	569.8		0.0190	674.8	0.0223	774.6	0.0256	869.3
	0.0158	573.1		0.0191	677.9	0.0224	777.6	0.0257	872.1
	0.0159	576.3		0.0192	681.0	0.0225	780.5	0.0258	874.9
	0.0160	579.6		0.0193	684.1	0.0226	783.4	0.0259	877.7
	0.0161	582.8		0.0194	687.2	0.0227	786.4	0.0260	880.5
	0.0162	586.1		0.0195	690.3	0.0228	789.3	0.0261	883.2
	0.0163	589.3		0.0196	693.3	0.0229	792.2	0.0262	886.0
	0.0164	592.5		0.0197	696.4	0.0230	795.1	0.0263	888.7
	0.0165	595.7		0.0198	699.5	0.0231	798.1	0.0264	891.5
	0.0166	599.0		0.0199	702.5	0.0232	801.0	0.0265	894.3
	0.0167	602.2		0.0200	705.6	0.0233	803.9	0.0266	897.0
	0.0168	605.4		0.0201	708.6	0.0234	806.8	0.0267	899.7
	0.0169	608.6		0.0202	711.7	0.0235	809.7	0.0268	902.5
	0.0170	611.8		0.0203	714.7	0.0236	812.5	0.0269	905.2
	0.0171	615.0		0.0204	717.8	0.0237	815.4	0.0270	907.9
	0.0172	618.2		0.0205	720.8	0.0238	818.3	0.0271	910.7

TABLE A.10 $f_y = 50{,}000$ psi; $f'_c = 3000$ psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for temp. and shrinkage	0.0020	98.04	0.0056	264.6	0.0092	418.4	0.0128	559.4
	0.0021	102.8	0.0057	269.0	0.0093	422.5	0.0129	563.2
	0.0022	107.6	0.0058	273.5	0.0094	426.6	0.0130	566.9
	0.0023	112.4	0.0059	277.9	0.0095	430.6	0.0131	570.6
	0.0024	117.2	0.0060	282.3	0.0096	434.7	0.0132	574.3
	0.0025	121.9	0.0061	286.7	0.0097	438.7	0.0133	578.0
	0.0026	126.7	0.0062	291.1	0.0098	442.8	0.0134	581.7
	0.0027	131.4	0.0063	295.5	0.0099	446.8	0.0135	585.4
	0.0028	136.2	0.0064	299.9	0.0100	450.8	0.0136	589.1
	0.0029	140.9	0.0065	304.2	0.0101	454.8	0.0137	592.7
	0.0030	145.6	0.0066	308.6	0.0102	458.8	0.0138	596.4
	0.0031	150.3	0.0067	312.9	0.0103	462.8	0.0139	600.0
	0.0032	155.0	0.0068	317.3	0.0104	466.8	0.0140	603.6
	0.0033	159.7	0.0069	321.6	0.0105	470.8	0.0141	607.2
	0.0034	164.3	0.0070	325.9	0.0106	474.8	0.0142	610.9
	0.0035	169.0	0.0071	330.2	0.0107	478.7	0.0143	614.5
	0.0036	173.6	0.0072	334.5	0.0108	482.6	0.0144	618.0
	0.0037	178.3	0.0073	338.8	0.0109	486.6	0.0145	621.6
	0.0038	182.9	0.0074	343.1	0.0110	490.5	0.0146	625.2
ρ_{min} for flexure	0.0039	187.5	0.0075	347.3	0.0111	494.4	0.0147	628.7
	0.0040	192.1	0.0076	351.6	0.0112	498.3	0.0148	632.3
	0.0041	196.7	0.0077	355.8	0.0113	502.2	0.0149	635.8
	0.0042	201.3	0.0078	360.1	0.0114	506.1	0.0150	639.4
	0.0043	205.9	0.0079	364.3	0.0115	510.0	0.0151	642.9
	0.0044	210.5	0.0080	368.5	0.0116	513.8	0.0152	646.4
	0.0045	215.0	0.0081	372.7	0.0117	517.7	0.0153	649.9
	0.0046	219.6	0.0082	376.9	0.0118	521.5	0.0154	653.4
	0.0047	224.1	0.0083	381.1	0.0119	525.4	0.0155	656.9
	0.0048	228.7	0.0084	385.3	0.0120	529.2	0.0156	660.3
	0.0049	233.2	0.0085	389.5	0.0121	533.0	0.0157	663.8
	0.0050	237.7	0.0086	393.6	0.0122	536.8	0.0158	667.3
	0.0051	242.2	0.0087	397.8	0.0123	540.6	0.0159	670.7
	0.0052	246.7	0.0088	401.9	0.0124	544.4	0.0160	674.1
	0.0053	251.2	0.0089	406.1	0.0125	548.2	0.0161	677.5
	0.0054	255.7	0.0090	410.2	0.0126	551.9	0.0162	681.0
	0.0055	260.1	0.0091	414.3	0.0127	555.7	0.0163	684.4

TABLE A.11 $f_y = 50{,}000$ psi; $f'_c = 4000$ psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for	0.0020	98.53	0.0059	282.2	0.0098	454.6	0.0137	615.8
temp. and	0.0021	103.4	0.0060	286.7	0.0099	458.9	0.0138	619.8
shrinkage	0.0022	108.2	0.0061	291.3	0.0100	463.1	0.0139	623.7
	0.0023	113.0	0.0062	295.8	0.0101	467.4	0.0140	627.7
	0.0024	117.9	0.0063	300.4	0.0102	471.6	0.0141	631.7
	0.0025	122.7	0.0064	304.9	0.0103	475.9	0.0142	635.6
	0.0026	127.5	0.0065	309.4	0.0104	480.1	0.0143	639.6
	0.0027	132.3	0.0066	313.9	0.0105	484.3	0.0144	643.5
	0.0028	137.1	0.0067	318.4	0.0106	488.6	0.0145	647.5
	0.0029	141.9	0.0068	322.9	0.0107	492.8	0.0146	651.4
	0.0030	146.7	0.0069	327.4	0.0108	497.0	0.0147	655.3
	0.0031	151.5	0.0070	331.9	0.0109	501.2	0.0148	659.2
	0.0032	156.2	0.0071	336.4	0.0110	505.4	0.0149	663.1
	0.0033	161.0	0.0072	340.9	0.0111	509.6	0.0150	667.0
	0.0034	165.7	0.0073	345.3	0.0112	513.7	0.0151	670.9
	0.0035	170.5	0.0074	349.8	0.0113	517.9	0.0152	674.8
	0.0036	175.2	0.0075	354.3	0.0114	522.1	0.0153	678.7
	0.0037	180.0	0.0076	358.7	0.0115	526.2	0.0154	682.5
	0.0038	184.7	0.0077	363.1	0.0116	530.4	0.0155	686.4
ρ_{min} for	0.0039	189.4	0.0078	367.6	0.0117	534.5	0.0156	690.3
flexure	0.0040	194.1	0.0079	372.0	0.0118	538.6	0.0157	694.1
	0.0041	198.8	0.0080	376.4	0.0119	542.8	0.0158	697.9
	0.0042	203.5	0.0081	380.8	0.0120	546.9	0.0159	701.8
	0.0043	208.2	0.0082	385.2	0.0121	551.0	0.0160	705.6
	0.0044	212.9	0.0083	389.6	0.0122	555.1	0.0161	709.4
	0.0045	217.5	0.0084	394.0	0.0123	559.2	0.0162	713.2
	0.0046	222.2	0.0085	398.4	0.0124	563.3	0.0163	717.0
	0.0047	226.9	0.0086	402.7	0.0125	567.4	0.0164	720.8
	0.0048	231.5	0.0087	407.1	0.0126	571.4	0.0165	724.6
	0.0049	236.1	0.0088	411.4	0.0127	575.5	0.0166	728.4
	0.0050	240.8	0.0089	415.8	0.0128	579.6	0.0167	732.1
	0.0051	245.4	0.0090	420.1	0.0129	583.6	0.0168	735.9
	0.0052	250.0	0.0091	424.5	0.0130	587.7	0.0169	327.4
	0.0053	254.6	0.0092	428.8	0.0131	591.7	0.0170	743.4
	0.0054	259.2	0.0093	433.1	0.0132	595.7	0.0171	747.2
	0.0055	263.8	0.0094	437.4	0.0133	599.8	0.0172	750.9
	0.0056	268.4	0.0095	441.7	0.0134	603.8	0.0173	754.6
	0.0057	273.0	0.0096	446.0	0.0135	607.8	0.0174	758.3
	0.0058	277.6	0.0097	450.3	0.0136	611.8	0.0175	762.1

TABLE A.11 (*Continued*)

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
	0.0176	765.8	0.0187	806.0	0.0198	845.4	0.0208	880.5
	0.0177	769.5	0.0188	809.7	0.0199	849.0	0.0209	883.9
	0.0178	773.2	0.0189	813.3	0.0200	852.5	0.0210	887.4
	0.0179	776.8	0.0190	816.9	0.0201	856.0	0.0211	890.8
	0.0180	780.5	0.0191	820.5	0.0202	859.5	0.0212	894.3
	0.0181	784.2	0.0192	824.1	0.0203	863.0	0.0213	897.7
	0.0182	787.8	0.0193	827.6	0.0204	866.5	0.0214	901.1
	0.0183	791.5	0.0194	831.2	0.0205	870.0	0.0215	904.5
	0.0184	795.1	0.0195	834.8	0.0206	873.5	0.0216	907.9
	0.0185	798.8	0.0196	838.3	0.0207	877.0	0.0217	911.3
	0.0186	802.4	0.0197	841.9				

TABLE A.12 $f_y = 60{,}000$ psi; $f'_c = 3000$ psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for temp. and shrinkage	0.0018	105.7	0.0048	271.7	0.0078	424.9	0.0108	565.4
	0.0019	111.5	0.0049	277.0	0.0079	429.8	0.0109	569.9
	0.0020	117.2	0.0050	282.3	0.0080	434.7	0.0110	574.3
	0.0021	122.9	0.0051	287.6	0.0081	439.5	0.0111	578.8
	0.0022	128.6	0.0052	292.9	0.0082	444.4	0.0112	582.3
	0.0023	134.3	0.0053	298.1	0.0083	449.2	0.0113	587.6
	0.0024	139.9	0.0054	303.4	0.0084	454.0	0.0114	592.0
	0.0025	145.6	0.0055	308.6	0.0085	458.8	0.0115	596.4
	0.0026	151.2	0.0056	313.8	0.0086	463.6	0.0116	600.7
	0.0027	156.9	0.0057	319.0	0.0087	468.4	0.0117	605.1
	0.0028	162.5	0.0058	324.2	0.0088	473.2	0.0118	609.4
	0.0029	168.1	0.0059	329.4	0.0089	477.9	0.0119	613.7
	0.0030	173.7	0.0060	334.5	0.0090	482.6	0.0120	618.0
	0.0031	179.2	0.0061	339.7	0.0091	487.4	0.0121	622.3
	0.0032	184.8	0.0062	344.8	0.0092	492.1	0.0122	626.6
ρ_{min} for flexure	0.0033	190.3	0.0063	349.9	0.0093	496.8	0.0123	630.9
	0.0034	195.8	0.0064	355.0	0.0094	501.4	0.0124	635.1
	0.0035	201.3	0.0065	360.1	0.0095	506.1	0.0125	639.4
	0.0036	206.8	0.0066	365.2	0.0096	510.7	0.0126	643.6
	0.0037	212.3	0.0067	370.2	0.0097	515.4	0.0127	647.8
	0.0038	217.8	0.0068	375.3	0.0098	520.0	0.0128	652.0
	0.0039	223.2	0.0069	380.3	0.0099	524.6	0.0129	656.2
	0.0040	228.7	0.0070	385.3	0.0100	529.2	0.0130	660.9
	0.0041	234.1	0.0071	390.3	0.0101	533.8	0.0131	664.5
	0.0042	239.5	0.0072	395.3	0.0102	538.3	0.0132	668.6
	0.0043	244.9	0.0073	400.3	0.0103	542.9	0.0133	672.8
	0.0044	250.3	0.0074	405.2	0.0104	547.4	0.0134	676.9
	0.0045	255.7	0.0075	410.2	0.0105	551.9	0.0135	681.0
	0.0046	261.0	0.0076	415.1	0.0106	556.4	0.0136	685.0
	0.0047	266.4	0.0077	420.0	0.0107	560.9		

TABLE A.13 $f_y = 60{,}000$ psi; $f'_c = 4000$ psi—U.S. Customary Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for temp. and shrinkage	0.0018	106.3	0.0059	335.5	0.0100	546.9	0.0141	740.4
	0.0019	112.1	0.0060	340.9	0.0101	551.8	0.0142	744.9
	0.0020	117.1	0.0061	346.2	0.0102	556.7	0.0143	749.4
	0.0021	123.7	0.0062	351.6	0.0103	561.7	0.0144	753.9
	0.0022	129.4	0.0063	356.9	0.0104	566.6	0.0145	758.3
	0.0023	135.2	0.0064	362.2	0.0105	571.5	0.0146	762.8
	0.0024	141.0	0.0065	367.6	0.0106	576.3	0.0147	767.2
	0.0025	146.7	0.0066	372.9	0.0107	581.2	0.0148	771.7
	0.0026	152.4	0.0067	378.2	0.0108	586.1	0.0149	776.1
	0.0027	158.1	0.0068	383.4	0.0109	590.9	0.0150	780.5
	0.0028	163.8	0.0069	388.7	0.0110	595.7	0.0151	784.9
	0.0029	169.5	0.0070	394.0	0.0111	600.6	0.0152	789.3
	0.0030	175.2	0.0071	399.2	0.0112	605.4	0.0153	793.7
	0.0031	180.9	0.0072	404.5	0.0113	610.2	0.0154	798.1
	0.0032	186.6	0.0073	409.7	0.0114	615.0	0.0155	802.4
ρ_{min} for flexure	0.0033	192.2	0.0074	414.9	0.0115	619.8	0.0156	806.8
	0.0034	197.9	0.0075	420.1	0.0116	624.5	0.0157	811.1
	0.0035	203.5	0.0076	425.3	0.0117	629.3	0.0158	815.4
	0.0036	209.1	0.0077	430.5	0.0118	634.1	0.0159	819.7
	0.0037	214.7	0.0078	435.7	0.0119	638.8	0.0160	824.1
	0.0038	220.3	0.0079	440.9	0.0120	643.5	0.0161	828.3
	0.0039	225.9	0.0080	446.0	0.0121	648.2	0.0162	832.6
	0.0040	231.5	0.0081	451.2	0.0122	653.0	0.0163	836.9
	0.0041	237.1	0.0082	456.3	0.0123	657.7	0.0164	841.2
	0.0042	242.6	0.0083	461.4	0.0124	662.3	0.0165	845.4
	0.0043	248.2	0.0084	466.5	0.0125	667.0	0.0166	849.7
	0.0044	253.7	0.0085	471.6	0.0126	671.7	0.0167	853.9
	0.0045	259.2	0.0086	476.7	0.0127	676.3	0.0168	858.1
	0.0046	264.8	0.0087	481.8	0.0128	681.0	0.0169	862.3
	0.0047	270.3	0.0088	486.9	0.0129	685.6	0.0170	866.5
	0.0048	275.8	0.0089	491.9	0.0130	690.3	0.0171	870.7
	0.0049	281.2	0.0090	497.0	0.0131	694.9	0.0172	874.9
	0.0050	286.7	0.0091	502.0	0.0132	699.5	0.0173	879.1
	0.0051	292.2	0.0092	507.1	0.0133	704.1	0.0174	883.2
	0.0052	297.6	0.0093	512.1	0.0134	708.6	0.0175	887.4
	0.0053	303.1	0.0094	517.1	0.0135	713.2	0.0176	891.5
	0.0054	308.5	0.0095	522.1	0.0136	717.8	0.0177	895.6
	0.0055	313.9	0.0096	527.1	0.0137	722.3	0.0178	899.7
	0.0056	319.3	0.0097	532.0	0.0138	726.9	0.0179	903.9
	0.0057	324.7	0.0098	537.0	0.0139	731.4	0.0180	907.9
	0.0058	330.1	0.0099	542.0	0.0140	735.9	0.0181	912.0

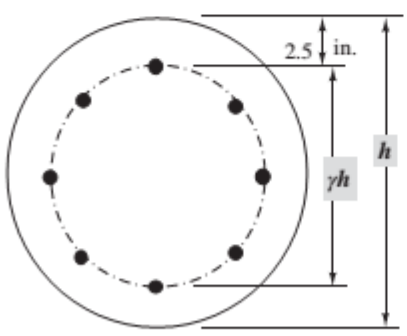
TABLE A.14 Maximum Spacing of 3/8-in. Spirals

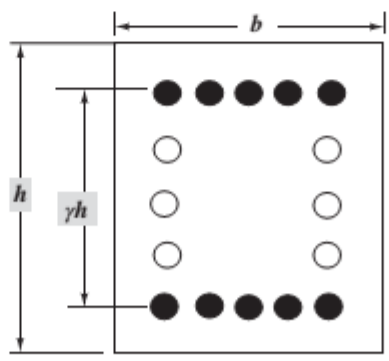
f'_c	ksi	4					5				
f_{yt}	ksi	60					60				
Spiral wire diameter d_b	in.	0.375					0.375				
a_s	in.^{2}	0.11					0.11				
Cover	in.	1.5					1.5				
Column outside diameter	in.	12	16	20	24	30	12	16	20	24	30
Core diameter, D_c	in.	9	13	17	21	27	9	13	17	21	27
A_g	in.^{2}	113.1	201.1	314.2	452.4	706.9	113.1	201.1	314.2	452.4	706.9
A_c	in.^{2}	63.6	132.7	227.0	346.4	572.6	63.6	132.7	227.0	346.4	572.6
min ρ_s		0.023	0.015	0.012	0.009	0.007	0.029	0.019	0.014	0.011	0.009
$s^* \rho_s$	in.	0.047	0.033	0.025	0.021	0.016	0.047	0.033	0.025	0.021	0.016
Maximum theoretical values	in.	2.02	2.14	2.21	2.25	2.29	1.61	1.71	1.76	1.80	1.83
Spacing rounded to 1/4 in.	in.	2	2	2	2.25	2.25	1.5	1.5	1.75	1.75	1.75

f'_c	ksi	4					5				
f_{yt}	ksi	75					75				
Spiral wire diameter d_b	in.	0.375					0.375				
a_s	in.^{2}	0.11					0.11				
Cover	in.	1.5					1.5				
Column outside diameter	in.	12	16	20	24	30	12	16	20	24	30
Core diameter, D_c	in.	9	13	17	21	27	9	13	17	21	27
A_g	in.^{2}	113.1	201.1	314.2	452.4	706.9	113.1	201.1	314.2	452.4	706.9
A_c	in.^{2}	63.6	132.7	227.0	346.4	572.6	63.6	132.7	227.0	346.4	572.6
min ρ_s		0.019	0.012	0.009	0.007	0.006	0.023	0.015	0.012	0.009	0.007
$s^* \rho_s$	in.	0.047	0.033	0.025	0.021	0.016	0.047	0.033	0.025	0.021	0.016
Maximum theoretical values	in.	2.52	2.67	2.76	2.81	2.87	2.02	2.14	2.21	2.25	2.29
Spacing rounded to 1/4 in.	in.	2.5	2.5	2.75	2.75	2.75	2	2	2	2.25	2.25

TABLE A.15 Areas, Weights, and Moments of Inertia for Square and Circular Columns, $(h - \gamma h) = 5$ in.

Circular Column						Square Column						
h	Area	Weight	I_c	A_s*	I_s*	$b = h$	Area	Weight	I_c	A_s*	I_{s1}, in.^{4*,1}	I_{s2}, in.^{4*,2}
in.	in.^{2}	lb/ft	in.^{4}	in.^{2}	in.^{4}	in.	in.^{2}	lb/ft	in.^{4}	in.^{2}	Steel Arranged in	
											2 Faces	4 Faces
12	113	118	1018	1.13	6.93	12	144	150	1728	1.44	17.64	11.76
13	133	138	1402	1.33	10.62	13	169	176	2380	1.69	27.04	18.03
14	154	160	1886	1.54	15.59	14	196	204	3201	1.96	39.69	26.46
15	177	184	2485	1.77	22.09	15	225	234	4219	2.25	56.25	37.50
16	201	209	3217	2.01	30.41	16	256	267	5461	2.56	77.44	51.63
17	227	236	4100	2.27	40.86	17	289	301	6960	2.89	104.04	69.36
18	254	265	5153	2.54	53.76	18	324	338	8748	3.24	136.89	91.26
19	284	295	6397	2.84	69.46	19	361	376	10860	3.61	176.89	117.93
20	314	327	7854	3.14	88.36	20	400	417	13333	4.00	225.00	150.00
21	346	361	9547	3.46	110.84	21	441	459	16207	4.41	282.24	188.16
22	380	396	11499	3.80	137.32	22	484	504	19521	4.84	349.69	233.13
23	415	433	13737	4.15	168.27	23	529	551	23320	5.29	428.49	285.66
24	452	471	16286	4.52	204.14	24	576	600	27648	5.76	519.84	346.56
25	491	511	19175	4.91	245.44	25	625	651	32552	6.25	625.00	416.67
26	531	553	22432	5.31	292.68	26	676	704	38081	6.76	745.29	496.86
27	573	596	26087	5.73	346.40	27	729	759	44287	7.29	882.09	588.06
28	616	641	30172	6.16	407.17	28	784	817	51221	7.84	1036.8	691.23
29	661	688	34719	6.61	475.58	29	841	876	58940	8.41	1211.0	807.36
30	707	736	39761	7.07	552.23	30	900	938	67500	9.00	1406.3	937.50
32	804	838	51472	8.04	732.87	32	1024	1067	87381	10.24	1866.2	1244.2
34	908	946	65597	9.08	954.45	34	1156	1204	111361	11.56	2430.5	1620.3
36	1018	1060	82448	10.18	1222.7	36	1296	1350	139968	12.96	3113.6	2075.8
38	1134	1181	102354	11.34	1543.8	38	1444	1504	173761	14.44	3931.3	2620.9
40	1257	1309	125664	12.57	1924.2	40	1600	1667	213333	16.00	4900.0	3266.7



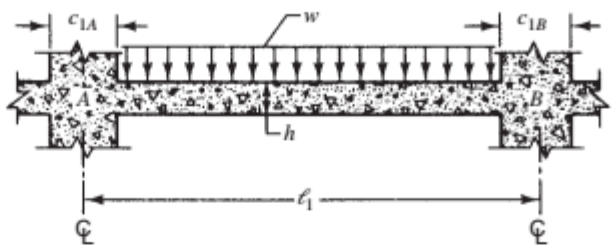


*Values shown for A_s, I_s, I_{s1} and I_{s2} are based on 1% reinforcing steel. For larger steel areas, multiply each by the ratio of the actual steel percentage to 1%.

[1] I_{s1} is for steel arranged as shown by only the solid circles in the drawing of the square column.

[2] I_{s2} is for steel arranged as shown by both the solid and hollow circles in the square column.

TABLE A.16 Moment Distribution Constants for Slabs without Drop Panels[a]



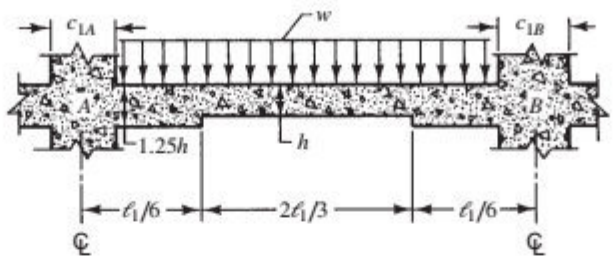
Column Dimension		Uniform Load FEM = Coef. ($w\ell_2\ell_1^2$)		Stiffness Factor[†]		Carryover Factor	
$\frac{c_{1A}}{\ell_1}$	$\frac{c_{1B}}{\ell_1}$	M_{AB}	M_{BA}	k_{AB}	k_{BA}	COF_{AB}	COF_{BA}
	0.00	0.083	0.083	4.00	4.00	0.500	0.500
	0.05	0.083	0.084	4.01	4.04	0.504	0.500
	0.10	0.082	0.086	4.03	4.15	0.513	0.499
	0.15	0.081	0.089	4.07	4.32	0.528	0.498
0.00	0.20	0.079	0.093	4.12	4.56	0.548	0.495
	0.25	0.077	0.097	4.18	4.88	0.573	0.491
	0.30	0.075	0.102	4.25	5.28	0.603	0.485
	0.35	0.073	0.107	4.33	5.78	0.638	0.478
	0.05	0.084	0.084	4.05	4.05	0.503	0.503
	0.10	0.083	0.086	4.07	4.15	0.513	0.503
	0.15	0.081	0.089	4.11	4.33	0.528	0.501
0.05	0.20	0.080	0.092	4.16	4.58	0.548	0.499
	0.25	0.078	0.096	4.22	4.89	0.573	0.494
	0.30	0.076	0.101	4.29	5.30	0.603	0.489
	0.35	0.074	0.107	4.37	5.80	0.638	0.481
	0.10	0.085	0.085	4.18	4.18	0.513	0.513
	0.15	0.083	0.088	4.22	4.36	0.528	0.511

TABLE A.16 (*Continued*)

Column Dimension		Uniform Load FEM = Coef. ($w\ell_2\ell_1^2$)		Stiffness Factor†		Carryover Factor	
$\frac{c_{1A}}{\ell_1}$	$\frac{c_{1B}}{\ell_1}$	M_{AB}	M_{BA}	k_{AB}	k_{BA}	COF_{AB}	COF_{BA}
0.10	0.20	0.082	0.091	4.27	4.61	0.548	0.508
	0.25	0.080	0.095	4.34	4.93	0.573	0.504
	0.30	0.078	0.100	4.41	5.34	0.602	0.498
	0.35	0.075	0.105	4.50	5.85	0.637	0.491
	0.15	0.086	0.086	4.40	4.40	0.526	0.526
	0.20	0.084	0.090	4.46	4.65	0.546	0.523
0.15	0.25	0.083	0.094	4.53	4.98	0.571	0.519
	0.30	0.080	0.099	4.61	5.40	0.601	0.513
	0.35	0.078	0.104	4.70	5.92	0.635	0.505
	0.20	0.088	0.088	4.72	4.72	0.543	0.543
0.20	0.25	0.086	0.092	4.79	5.05	0.568	0.539
	0.30	0.083	0.097	4.88	5.48	0.597	0.532
	0.35	0.081	0.102	4.99	6.01	0.632	0.524
	0.25	0.090	0.090	5.14	5.14	0.563	0.563
0.25	0.30	0.088	0.095	5.24	5.58	0.592	0.556
	0.35	0.085	0.100	5.36	6.12	0.626	0.548
0.30	0.30	0.092	0.092	5.69	5.69	0.585	0.585
	0.35	0.090	0.097	5.83	6.26	0.619	0.576
0.35	0.35	0.095	0.095	6.42	6.42	0.609	0.609

[a]Applicable when $c_1/\ell_1 = c_2/\ell_2$. For other relationships between these ratios, the constants will be slightly in error.
†Stiffness is $K_{AB} = k_{AB}E(\ell_2 h^3/12\ell_1)$ and $K_{BA} = k_{BA}E(\ell_2 h^3/12\ell_1)$

TABLE A.17 Moment Distribution Constants for Slabs with Drop Panels[a]



Column Dimension		Uniform Load FEM = Coef. ($w\ell_2\ell_1^2$)		Stiffness Factor†		Carryover Factor	
$\frac{c_{1A}}{\ell_1}$	$\frac{c_{1B}}{\ell_1}$	M_{AB}	M_{BA}	k_{AB}	k_{BA}	COF_{AB}	COF_{BA}
	0.00	0.088	0.088	4.78	4.78	0.541	0.541
	0.05	0.087	0.089	4.80	4.82	0.545	0.541
	0.10	0.087	0.090	4.83	4.94	0.553	0.541
0.00	0.15	0.085	0.093	4.87	5.12	0.567	0.540
	0.20	0.084	0.096	4.93	5.36	0.585	0.537
	0.25	0.082	0.100	5.00	5.68	0.606	0.534
	0.30	0.080	0.105	5.09	6.07	0.631	0.529
	0.05	0.088	0.088	4.84	4.84	0.545	0.545
	0.10	0.087	0.090	4.87	4.95	0.553	0.544
	0.15	0.085	0.093	4.91	5.13	0.567	0.543
0.05	0.20	0.084	0.096	4.97	5.38	0.584	0.541
	0.25	0.082	0.100	5.05	5.70	0.606	0.537
	0.30	0.080	0.104	5.13	6.09	0.632	0.532
	0.10	0.089	0.089	4.98	4.98	0.553	0.553
	0.15	0.088	0.092	5.03	5.16	0.566	0.551
0.10	0.20	0.086	0.094	5.09	5.42	0.584	0.549
	0.25	0.084	0.099	5.17	5.74	0.606	0.546
	0.30	0.082	0.103	5.26	6.13	0.631	0.541
	0.15	0.090	0.090	5.22	5.22	0.565	0.565
	0.20	0.089	0.094	5.28	5.47	0.583	0.563
0.15	0.25	0.087	0.097	5.37	5.80	0.604	0.559
	0.30	0.085	0.102	5.46	6.21	0.630	0.554
	0.20	0.092	0.092	5.55	5.55	0.580	0.580
0.20	0.25	0.090	0.096	5.64	5.88	0.602	0.577
	0.30	0.088	0.100	5.74	6.30	0.627	0.571
0.25	0.25	0.094	0.094	5.98	5.98	0.598	0.598
	0.30	0.091	0.098	6.10	6.41	0.622	0.593
0.30	0.30	0.095	0.095	6.54	6.54	0.617	0.617

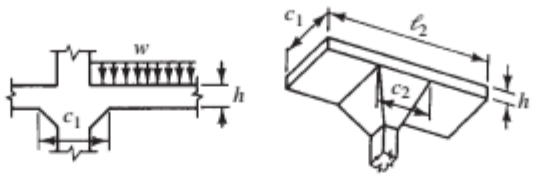
[a]Applicable when $c_1/\ell_1 = c_2/\ell_2$. For other relationships between these ratios, the constants will be slightly in error.
†Stiffness is $K_{AB} = k_{AB}E(\ell_2 h^3/12\ell_1)$ and $K_{BA} = k_{BA}E(\ell_2 h^3/12\ell_1)$

TABLE A.18 Moment Distribution Constants for Slab-Beam Members with Column Capitals

FEM (uniform load w) $= Mw\ell_2(\ell_1)^2$

K (stiffness) $= \dfrac{kE\ell_2 h^3}{12\ell_1}$

Carryover factor $= C$



$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	M	k	C	$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	M	k	C
	0.00	0.083	4.000	0.500	0.10	0.25	0.087	4.449	0.530
	0.05	0.083	4.000	0.500		0.30	0.087	4.535	0.535
	0.10	0.083	4.000	0.500		0.35	0.088	4.618	0.540
	0.15	0.083	4.000	0.500		0.40	0.088	4.698	0.545
	0.20	0.083	4.000	0.500		0.45	0.089	4.774	0.550
0.00	0.25	0.083	4.000	0.500		0.50	0.089	4.846	0.554
	0.30	0.083	4.000	0.500		0.00	0.083	4.000	0.500
	0.35	0.083	4.000	0.500		0.05	0.084	4.132	0.509
	0.40	0.083	4.000	0.500		0.10	0.085	4.267	0.517
	0.45	0.083	4.000	0.500		0.15	0.086	4.403	0.526
	0.50	0.083	4.000	0.500		0.20	0.087	4.541	0.534
	0.00	0.083	4.000	0.500	0.15	0.25	0.088	4.680	0.543
	0.05	0.084	4.047	0.503		0.30	0.089	4.818	0.550
	0.10	0.084	4.093	0.507		0.00	0.083	4.000	0.500
	0.15	0.084	4.138	0.510		0.05	0.085	4.170	0.511
	0.20	0.085	4.181	0.513		0.10	0.086	4.346	0.522
0.05	0.25	0.085	4.222	0.516		0.15	0.087	4.529	0.532
	0.30	0.085	4.261	0.518		0.20	0.088	4.717	0.543
	0.35	0.086	4.299	0.521	0.20	0.25	0.089	4.910	0.554
	0.40	0.086	4.334	0.523		0.30	0.090	5.108	0.564
	0.45	0.086	4.368	0.526		0.35	0.091	5.308	0.574
	0.50	0.086	4.398	0.528		0.40	0.092	5.509	0.584
	0.00	0.083	4.000	0.500		0.45	0.093	5.710	0.593
	0.05	0.084	4.091	0.506		0.50	0.094	5.908	0.602
	0.10	0.085	4.182	0.513		0.00	0.083	4.000	0.500
	0.15	0.085	4.272	0.519		0.05	0.085	4.204	0.512
	0.20	0.086	4.362	0.524		0.10	0.086	4.420	0.525

(*continues*)

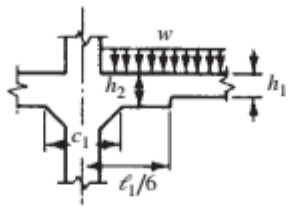
TABLE A.18 (*Continued*)

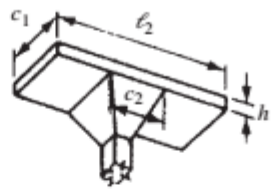
$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	M	k	C	$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	M	k	C
	0.15	0.087	4.648	0.538		0.20	0.090	5.348	0.563
	0.20	0.089	4.887	0.550	0.40	0.25	0.092	5.778	0.580
0.25	0.25	0.090	5.138	0.563		0.30	0.094	6.255	0.598
	0.30	0.091	5.401	0.576		0.35	0.095	6.782	0.617
	0.35	0.093	5.672	0.588		0.40	0.097	7.365	0.635
	0.40	0.094	5.952	0.600		0.45	0.099	8.007	0.654
	0.45	0.095	6.238	0.612		0.50	0.100	8.710	0.672
	0.50	0.096	6.527	0.623		0.00	0.083	4.000	0.500
	0.00	0.083	4.000	0.500		0.05	0.085	4.311	0.515
	0.05	0.085	4.235	0.514		0.10	0.087	4.658	0.530
	0.10	0.086	4.488	0.527		0.15	0.088	5.046	0.547
	0.15	0.088	4.760	0.542		0.20	0.090	5.480	0.564
	0.20	0.089	5.050	0.556	0.45	0.25	0.092	5.967	0.583
0.30	0.25	0.091	5.361	0.571		0.35	0.095	6.416	0.609
	0.30	0.092	5.692	0.585		0.40	0.096	6.888	0.626
	0.35	0.094	6.044	0.600		0.45	0.098	7.395	0.642
	0.40	0.095	6.414	0.614		0.50	0.099	7.935	0.658
	0.45	0.096	6.802	0.628		0.30	0.094	6.517	0.602
	0.50	0.098	7.205	0.642		0.35	0.096	7.136	0.621
	0.00	0.083	4.000	0.500		0.40	0.098	7.836	0.642
	0.05	0.085	4.264	0.514		0.45	0.100	8.625	0.662
	0.10	0.087	4.551	0.529		0.50	0.101	9.514	0.683
	0.15	0.088	4.864	0.545		0.00	0.083	4.000	0.500
	0.20	0.090	5.204	0.560		0.05	0.085	4.331	0.515
0.35	0.25	0.091	5.575	0.576		0.10	0.087	4.703	0.530
	0.30	0.093	5.979	0.593		0.15	0.088	5.123	0.547
	0.35	0.090	4.955	0.558		0.20	0.090	5.599	0.564
	0.40	0.090	5.090	0.565	0.50	0.25	0.092	6.141	0.583
	0.45	0.091	5.222	0.572		0.30	0.094	6.760	0.603
	0.50	0.092	5.349	0.579		0.35	0.096	7.470	0.624
	0.00	0.083	4.000	0.500		0.40	0.098	8.289	0.645
	0.05	0.085	4.289	0.515		0.45	0.100	9.234	0.667
	0.10	0.087	4.607	0.530		0.50	0.102	10.329	0.690
	0.15	0.088	4.959	0.546					

TABLE A.19 Moment Distribution Constants for Slab-Beam Members with Column Capitals and Drop Panels

$$\text{FEM (uniform load } w) = Mw\ell(\ell_1^2)$$

$$K \text{ (stiffness)} = \frac{kE\ell_2 h^3}{12\ell_1}$$





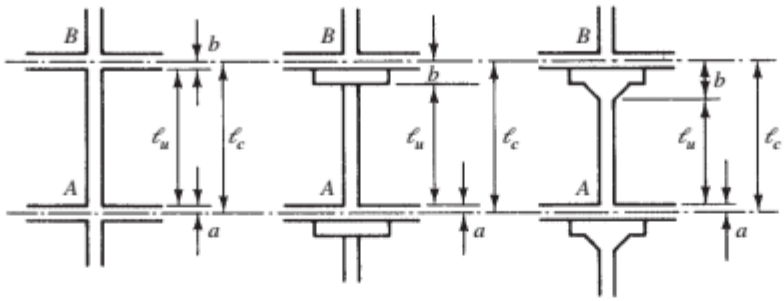
$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	Constants for $h_2 = 1.25h_1$			Constants for $h_2 = 1.5h_2$		
		M	**k**	**C**	**M**	**k**	**C**
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.088	4.795	0.542	0.093	5.837	0.589
	0.10	0.088	4.795	0.542	0.093	5.837	0.589
0.00	0.15	0.088	4.795	0.542	0.093	5.837	0.589
	0.20	0.088	4.795	0.542	0.093	5.837	0.589
	0.25	0.088	4.795	0.542	0.093	5.837	0.589
	0.30	0.088	4.797	0.542	0.093	5.837	0.589
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.088	4.846	0.545	0.093	5.890	0.591
	0.10	0.089	4.896	0.548	0.093	5.942	0.594
0.05	0.15	0.089	4.944	0.551	0.093	5.993	0.596
	0.20	0.089	4.990	0.553	0.094	6.041	0.598
	0.25	0.089	5.035	0.556	0.094	6.087	0.600
	0.30	0.090	5.077	0.558	0.094	6.131	0.602
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.088	4.894	0.548	0.093	5.940	0.593
	0.10	0.089	4.992	0.553	0.094	6.042	0.598
0.10	0.15	0.090	5.039	0.559	0.094	6.142	0.602
	0.20	0.090	5.184	0.564	0.094	6.240	0.607
	0.25	0.091	5.278	0.569	0.095	6.335	0.611
	0.30	0.091	5.368	0.573	0.095	6.427	0.615
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.089	4.938	0.550	0.093	5.986	0.595
	0.10	0.090	5.082	0.558	0.094	6.135	0.602

(*continues*)

TABLE A.19 (*Continued*)

$\frac{c_1}{\ell_1}$	$\frac{c_1}{\ell_2}$	Constants for $h_2 = 1.25h_1$			Constants for $h_2 = 1.5h_2$		
		M	*k*	*C*	*M*	*k*	*C*
0.15	0.15	0.090	5.228	0.565	0.095	6.284	0.608
	0.20	0.091	5.374	0.573	0.095	6.432	0.614
	0.25	0.092	5.520	0.580	0.096	6.579	0.620
	0.30	0.092	5.665	0.587	0.096	6.723	0.626
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.089	4.978	0.552	0.093	6.027	0.597
	0.10	0.090	5.167	0.562	0.094	6.221	0.605
0.20	0.15	0.091	5.361	0.571	0.095	6.418	0.613
	0.20	0.092	5.558	0.581	0.096	6.616	0.621
	0.25	0.093	5.760	0.590	0.096	6.816	0.628
	0.30	0.094	5.962	0.590	0.097	7.015	0.635
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.089	5.015	0.553	0.094	6.065	0.598
	0.10	0.090	5.245	0.565	0.094	6.300	0.608
0.25	0.15	0.091	5.485	0.576	0.095	6.543	0.617
	0.20	0.092	5.735	0.587	0.096	6.790	0.626
	0.25	0.094	5.994	0.598	0.097	7.043	0.635
	0.30	0.095	6.261	0.600	0.098	7.298	0.644
	0.00	0.088	4.795	0.542	0.093	5.837	0.589
	0.05	0.089	5.048	0.554	0.094	6.099	0.599
	0.10	0.090	5.317	0.567	0.095	6.372	0.610
0.30	0.15	0.092	5.601	0.580	0.096	6.657	0.620
	0.20	0.093	5.902	0.593	0.097	6.953	0.631
	0.25	0.094	6.219	0.605	0.098	7.258	0.641
	0.30	0.095	6.550	0.618	0.099	7.571	0.651

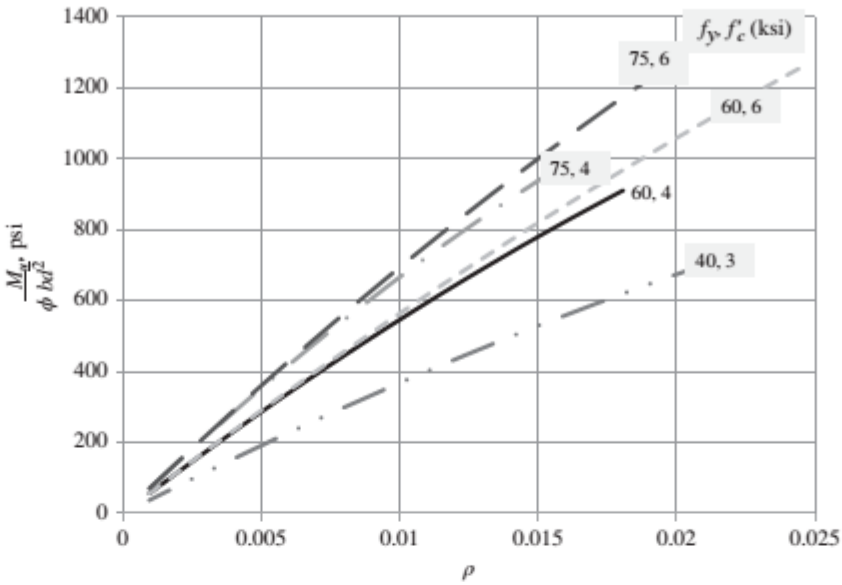
TABLE A.20 Stiffness Factors and Carryover Factors for Columns



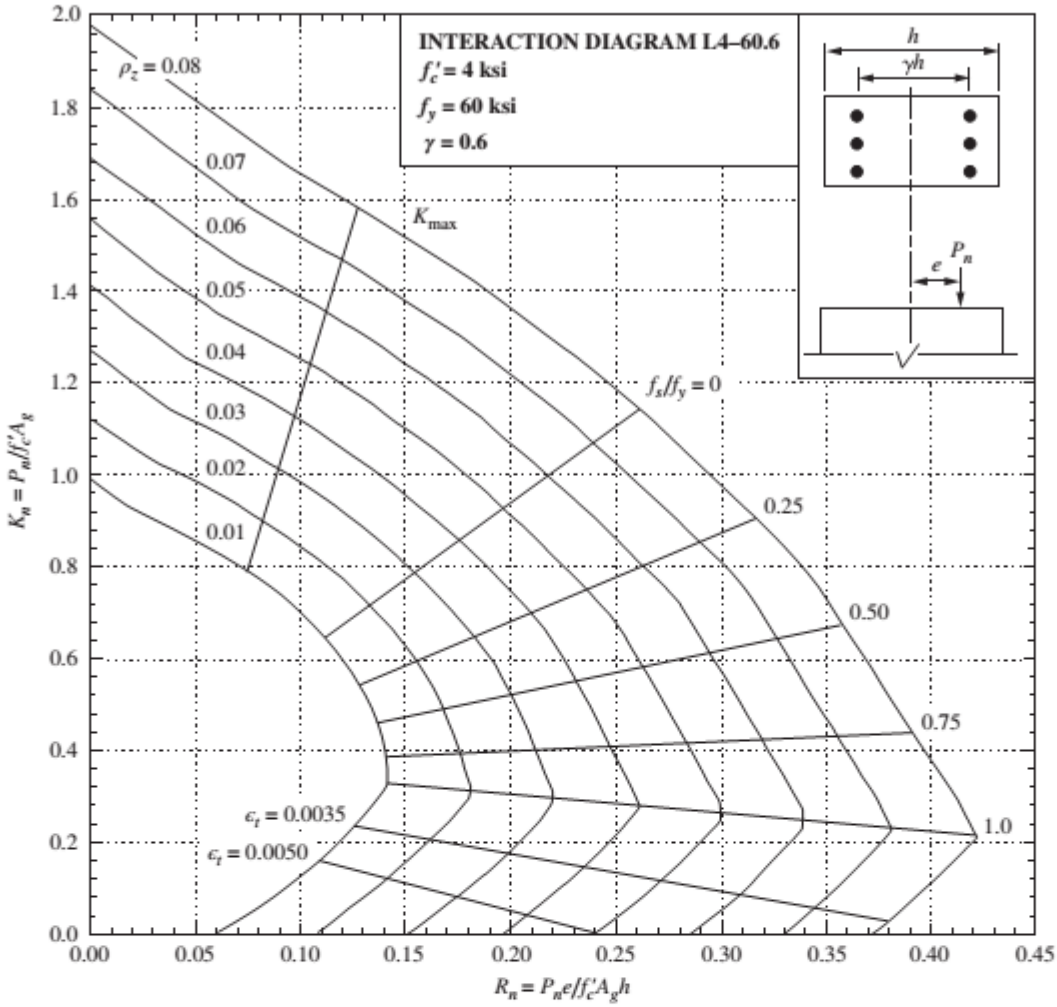
$\frac{a}{b}$	$\frac{\ell_u}{\ell_n}$	0.95	0.90	0.85	0.80	0.75
0.20	k_{AB}	4.32	4.70	5.33	5.65	6.27
	C_{AB}	0.57	0.64	0.71	0.80	0.89
0.40	k_{AB}	4.40	4.89	5.45	6.15	7.00
	C_{AB}	0.56	0.61	0.68	0.74	0.81
0.60	k_{AB}	4.46	5.02	5.70	6.54	7.58
	C_{AB}	0.55	0.60	0.65	0.70	0.76
0.80	k_{AB}	4.51	5.14	5.90	6.85	8.05
	C_{AB}	0.54	0.58	0.63	0.67	0.72
1.00	k_{AB}	4.55	5.23	6.06	7.11	8.44
	C_{AB}	0.54	0.57	0.61	0.65	0.68
1.20	k_{AB}	4.58	5.30	6.20	7.32	8.77
	C_{AB}	0.53	0.57	0.60	0.63	0.66
1.40	k_{AB}	4.61	5.36	6.31	7.51	9.05
	C_{AB}	0.53	0.56	0.59	0.61	0.64
1.60	k_{AB}	4.63	5.42	6.41	7.66	9.29
	C_{AB}	0.53	0.55	0.58	0.60	0.62
1.80	k_{AB}	4.65	5.46	6.49	7.80	9.50
	C_{AB}	0.53	0.55	0.57	0.59	0.60
2.00	k_{AB}	4.67	5.51	6.56	7.92	9.68
	C_{AB}	0.52	0.54	0.56	0.58	0.59

Notes:

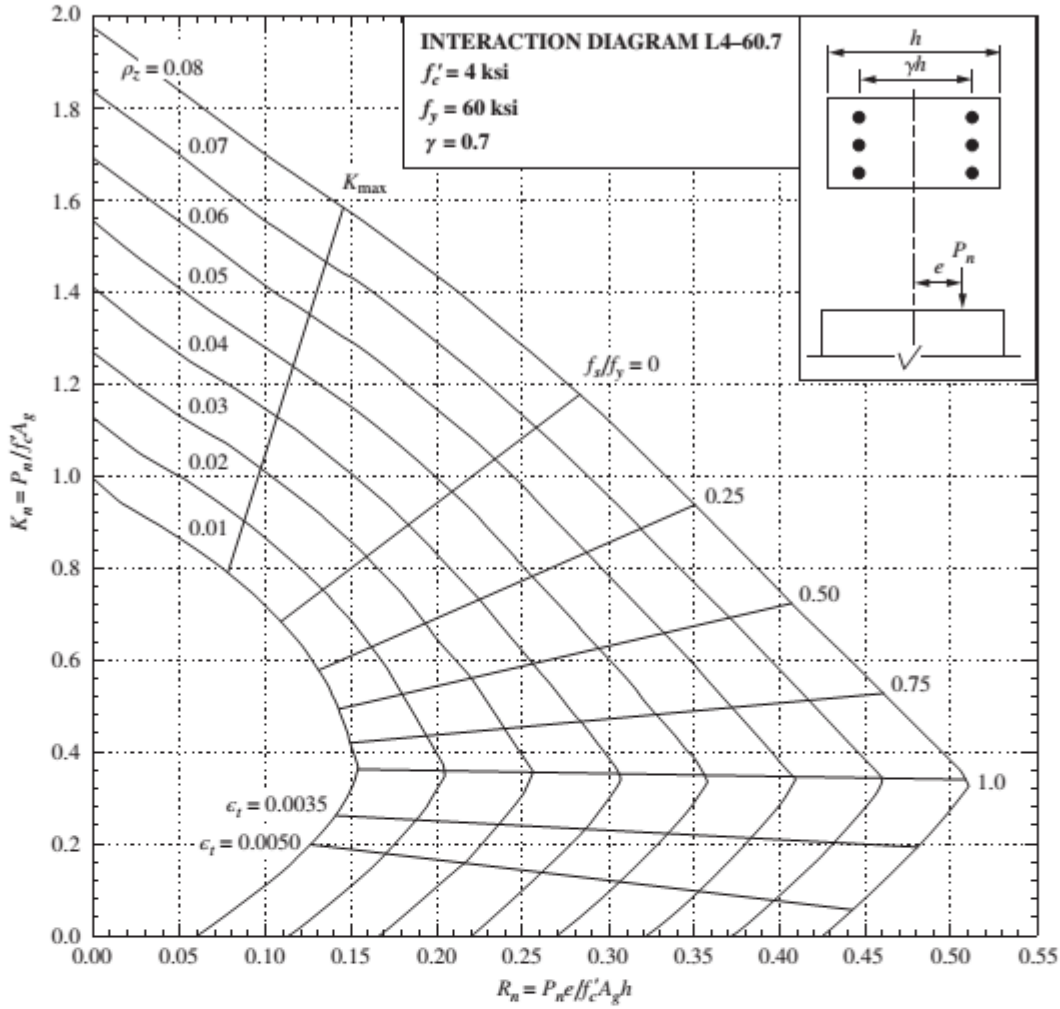
1. Values computed by column analogy method.
2. $k_c = k_{AB}$ from table(EI_0/ℓ_n).



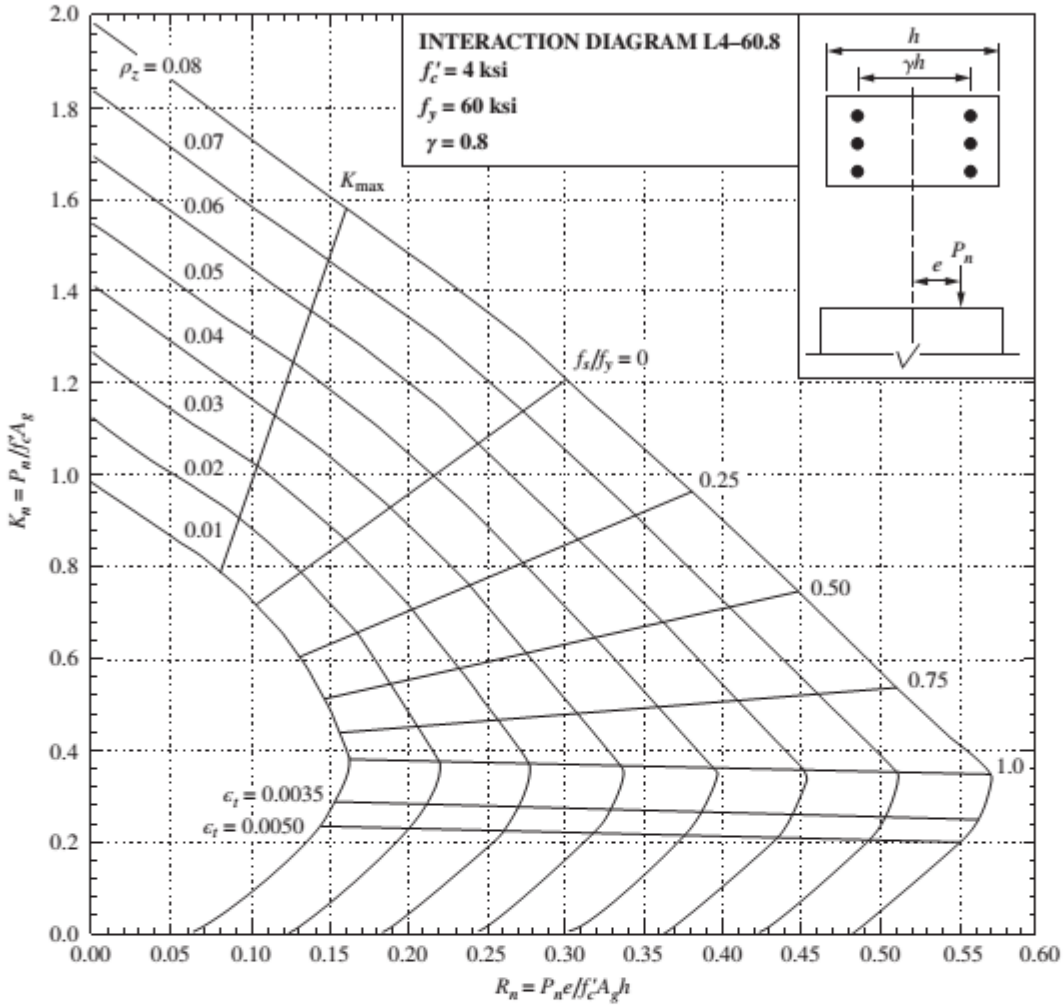
GRAPH 1 Normalized moment capacity vs. reinforcement ratio (ρ) for various values of f_y and f'_c.



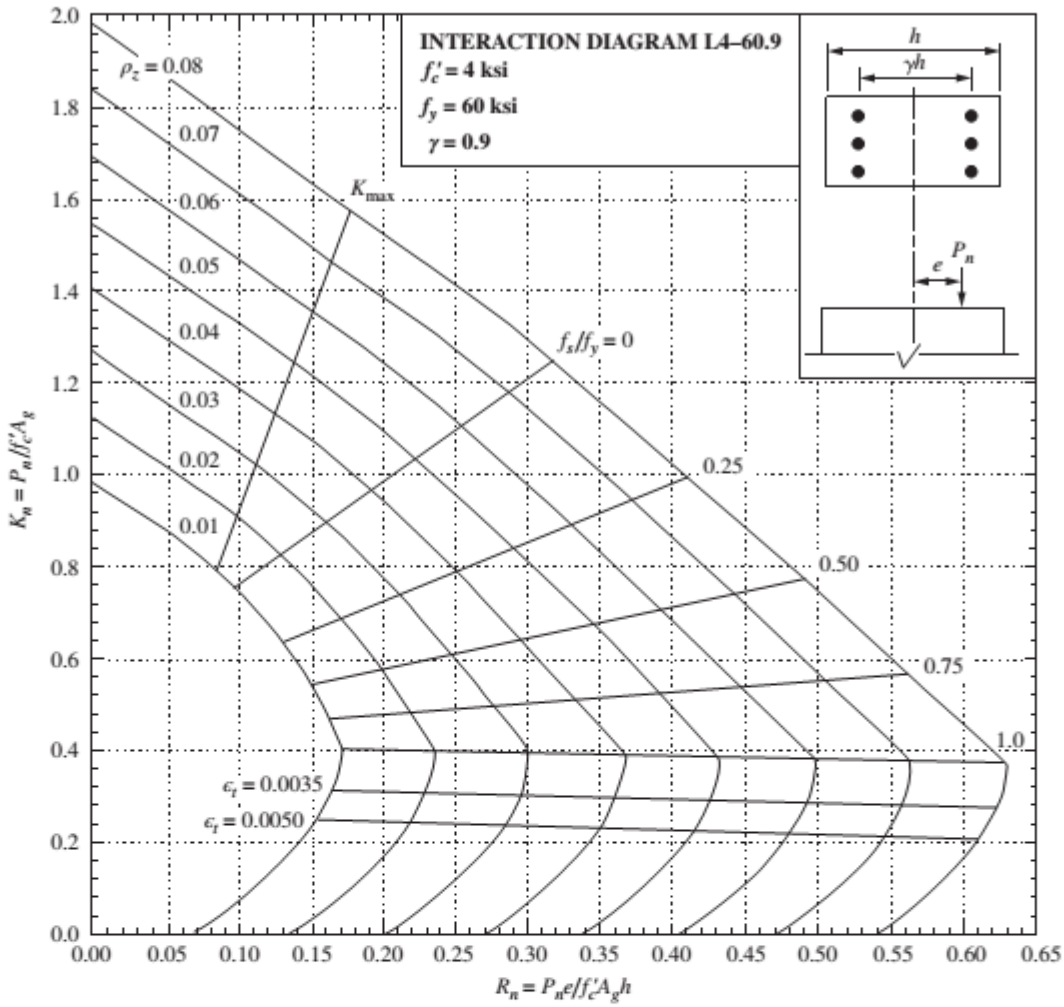
GRAPH 2 Column interaction diagrams for rectangular tied columns with bars on end faces only. (Published with the permission of the American Concrete Institute.)



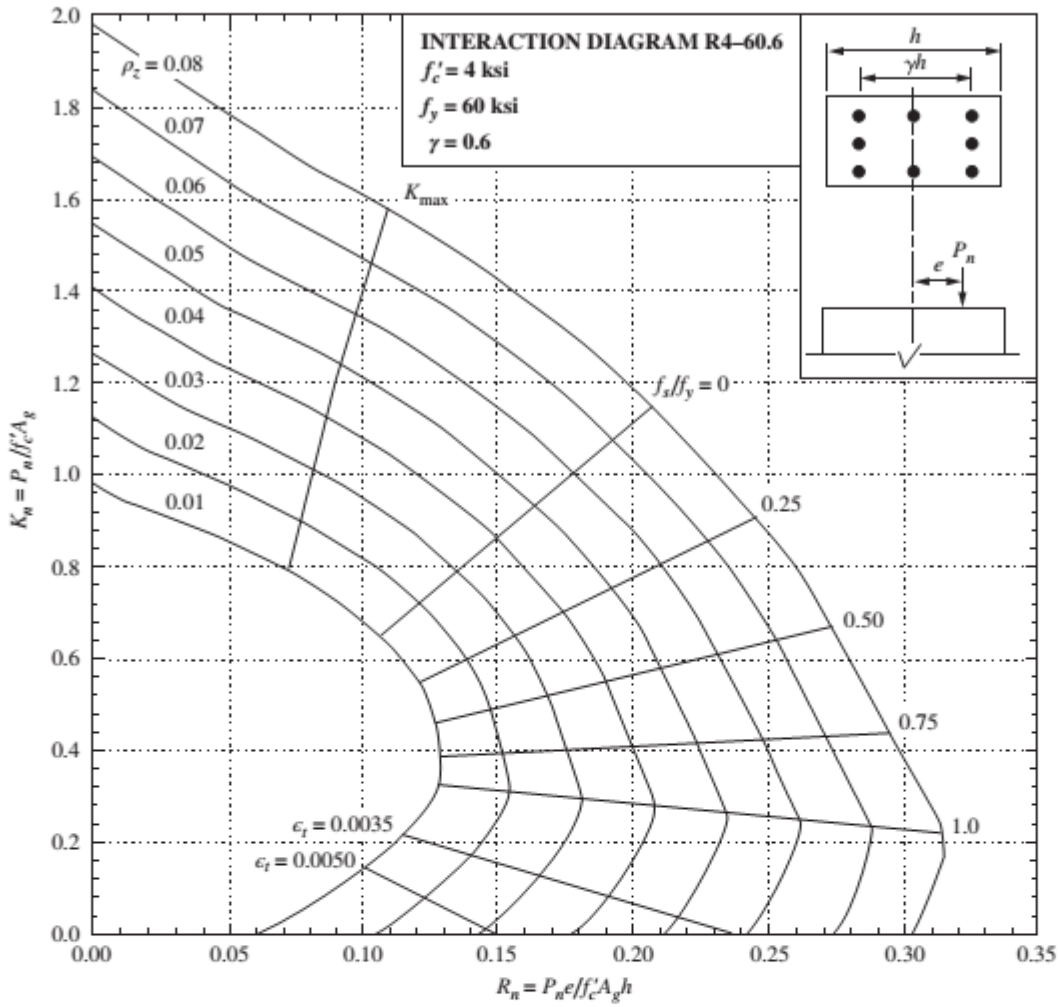
GRAPH 3 Column interaction diagrams for rectangular tied columns with bars on end faces only. (Published with the permission of the American Concrete Institute.)



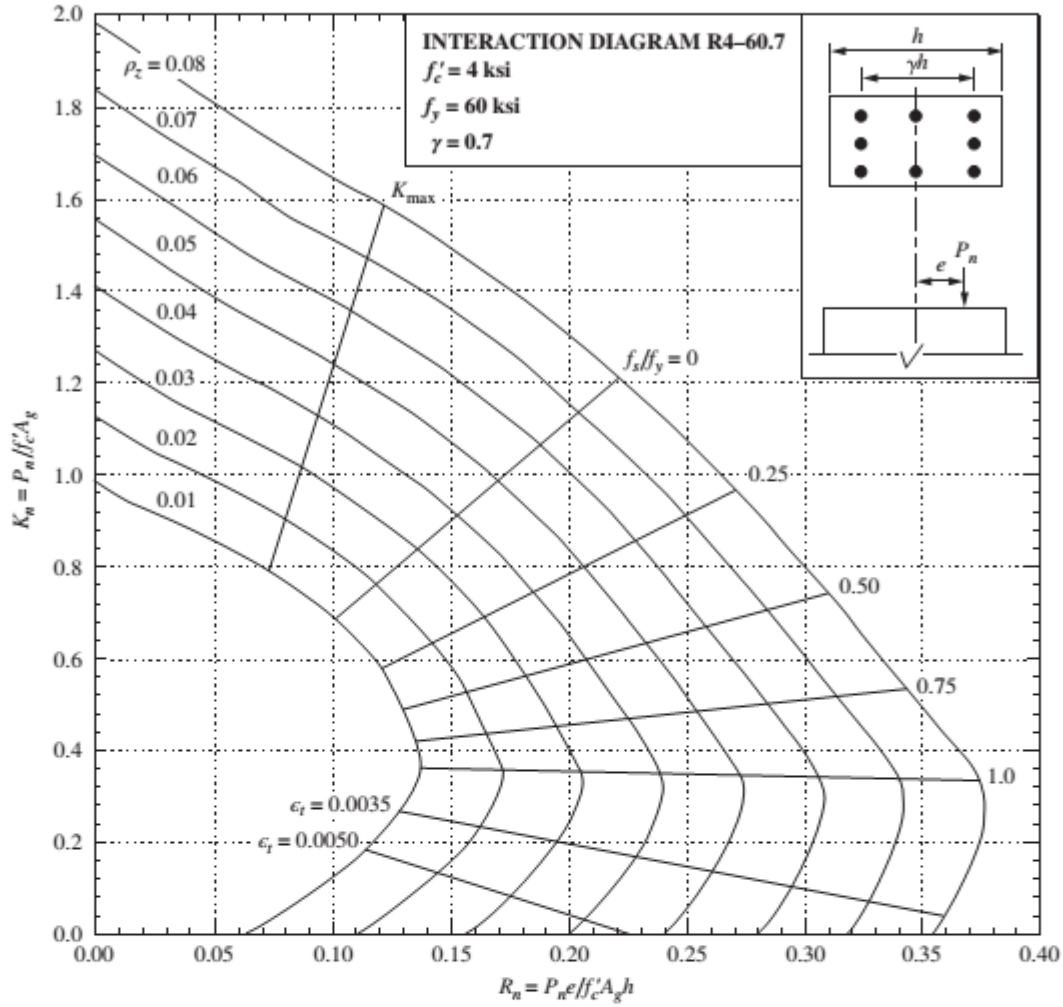
GRAPH 4 Column interaction diagrams for rectangular tied columns with bars on end faces only. (Published with the permission of the American Concrete Institute.)



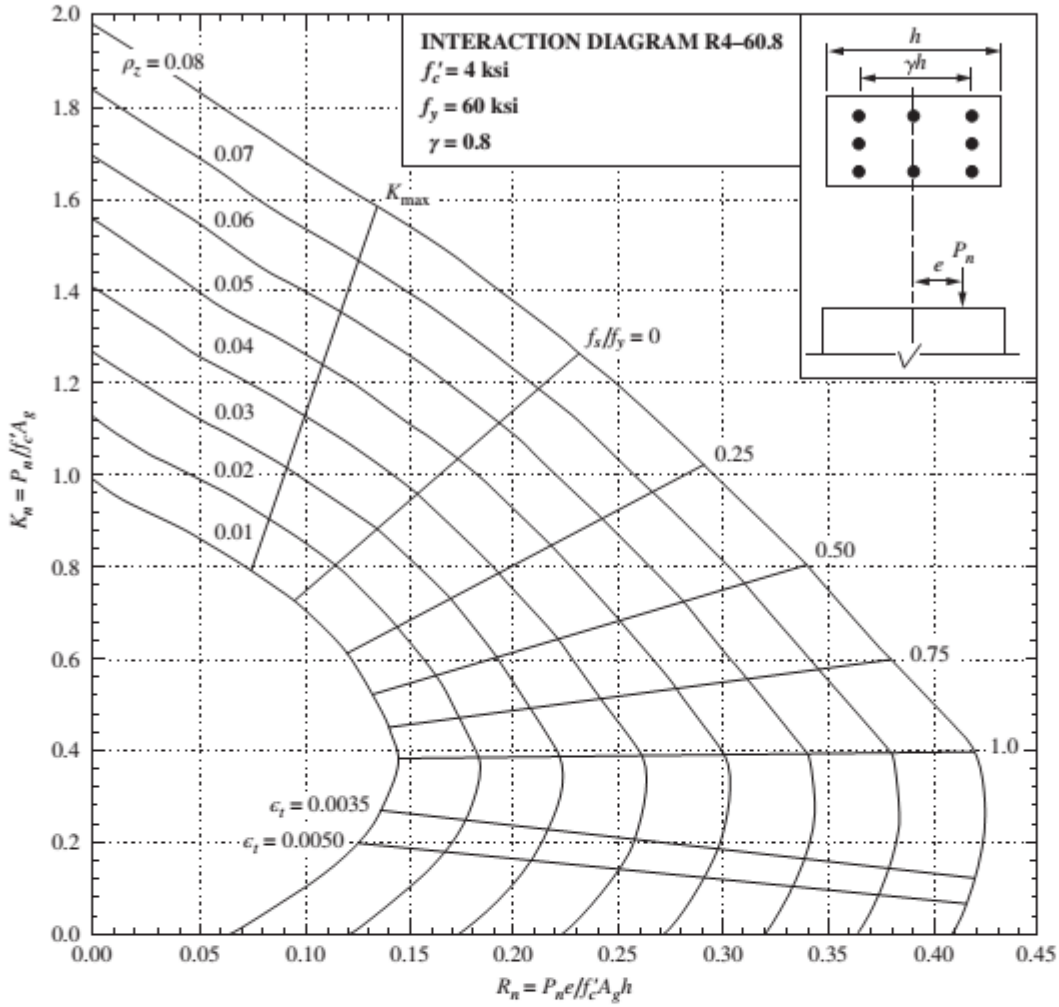
GRAPH 5 Column interaction diagrams for rectangular tied columns with bars on end faces only. (Published with the permission of the American Concrete Institute.)



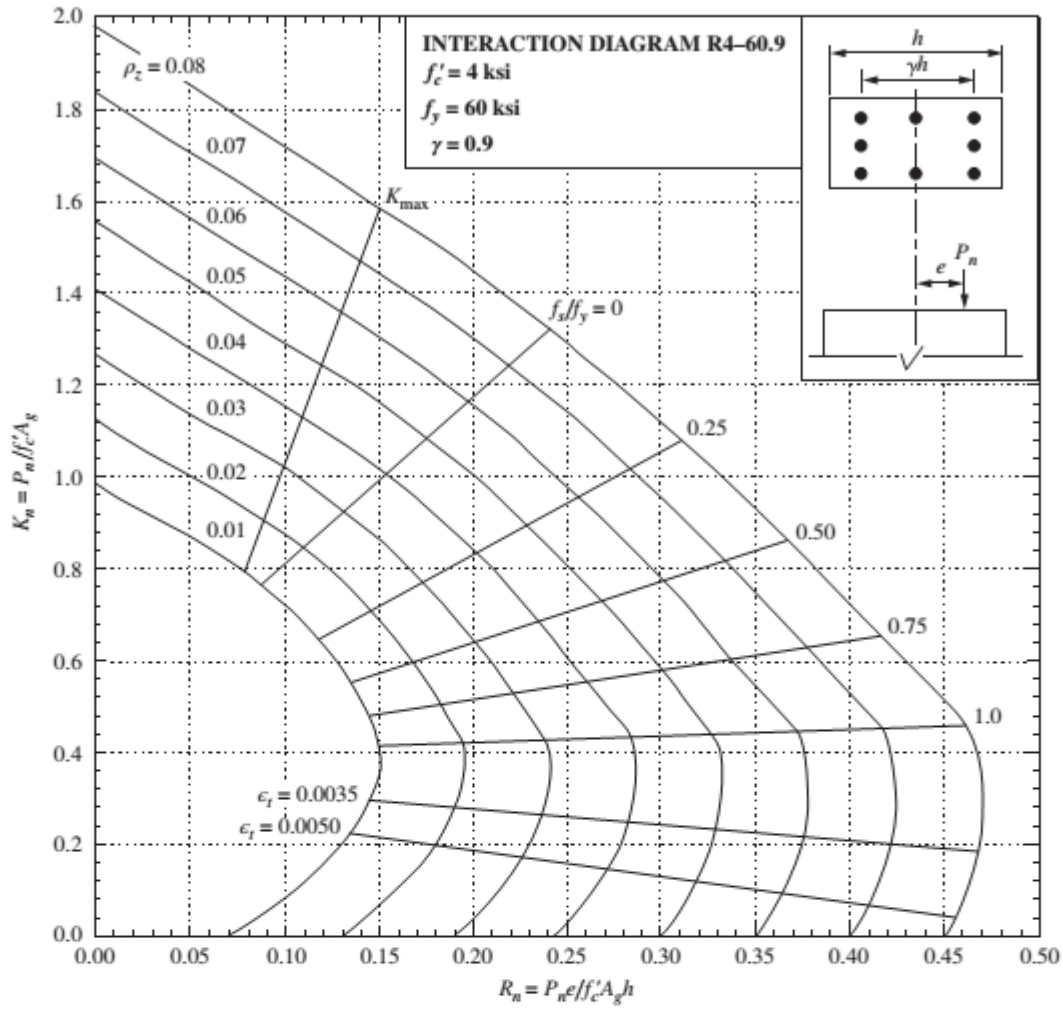
GRAPH 6 Column interaction diagrams for rectangular tied columns with bars on all four faces. (Published with the permission of the American Concrete Institute.)



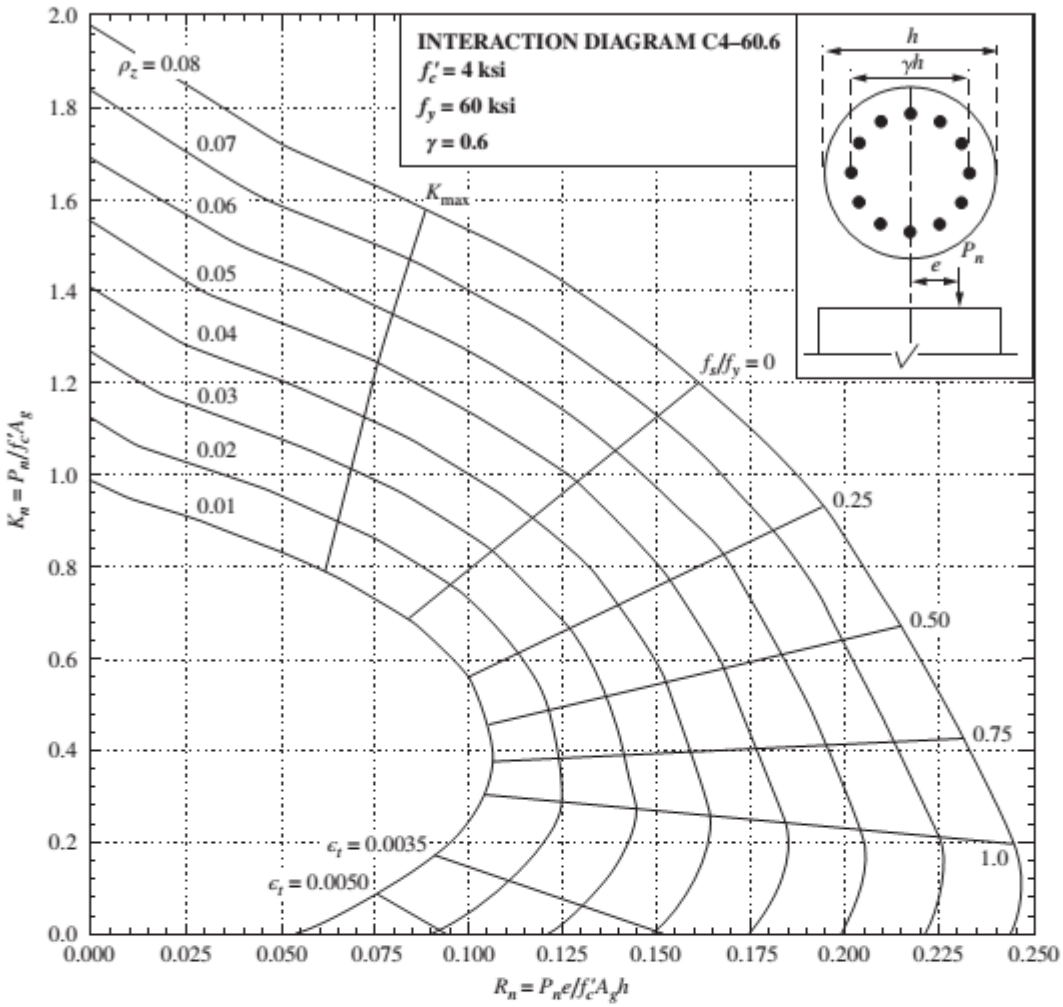
GRAPH 7 Column interaction diagrams for rectangular tied columns with bars on all four faces. (Published with the permission of the American Concrete Institute.)



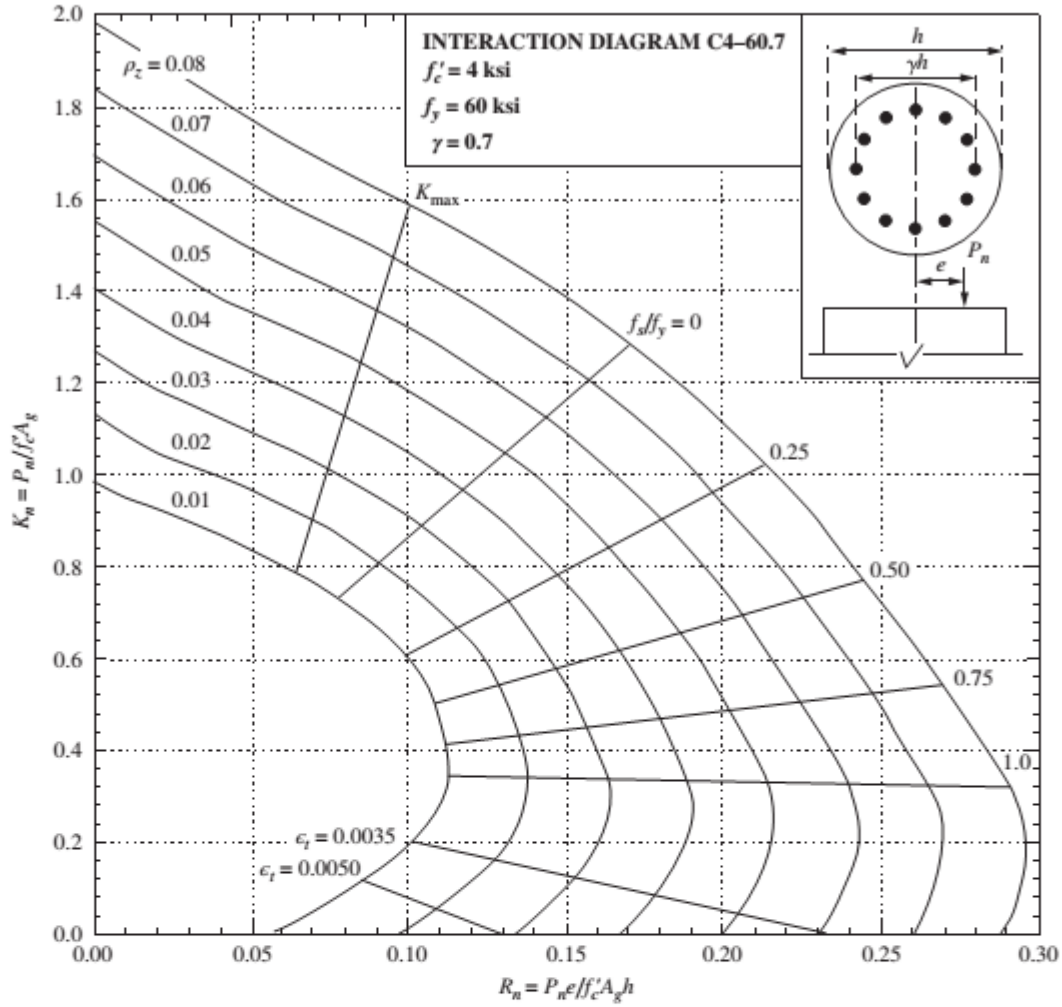
GRAPH 8 Column interaction diagrams for rectangular tied columns with bars on all four faces. (Published with the permission of the American Concrete Institute.)



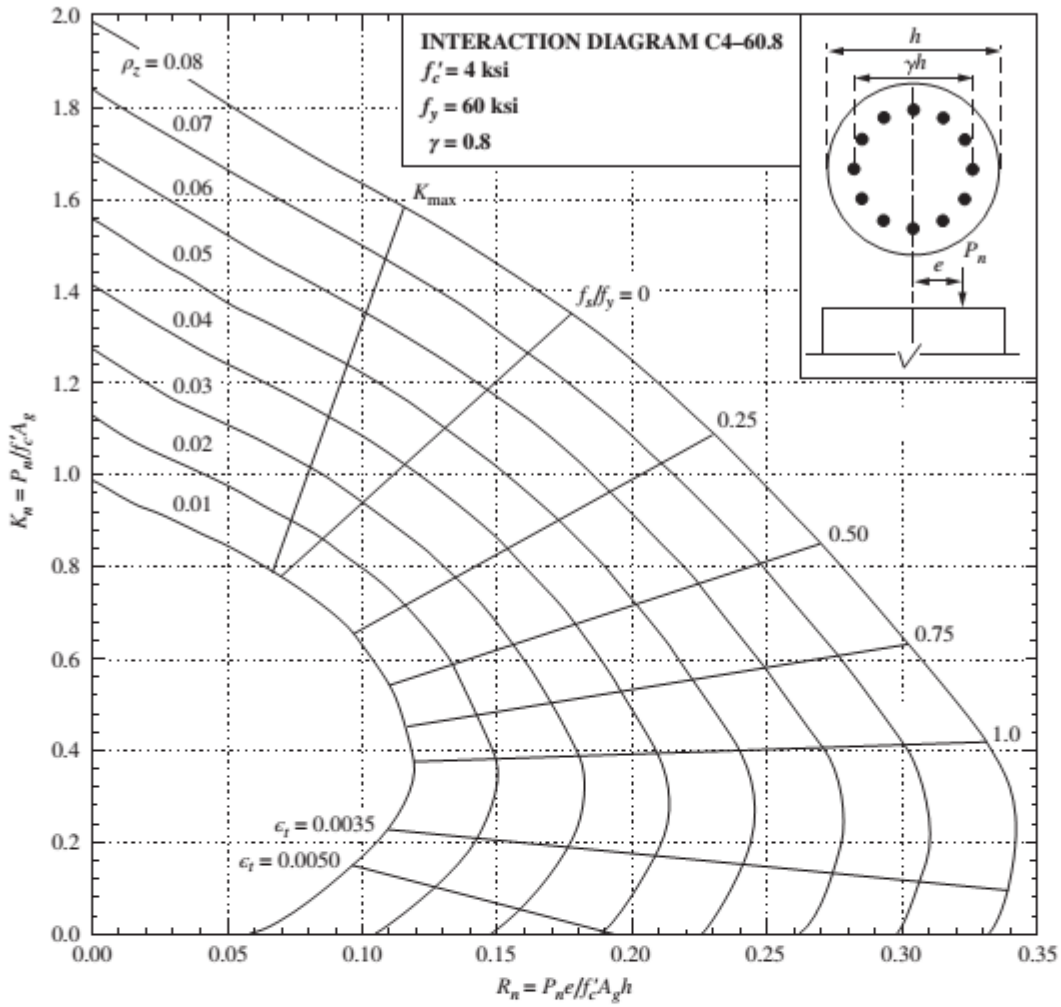
GRAPH 9 Column interaction diagrams for rectangular tied columns with bars on all four faces. (Published with the permission of the American Concrete Institute.)



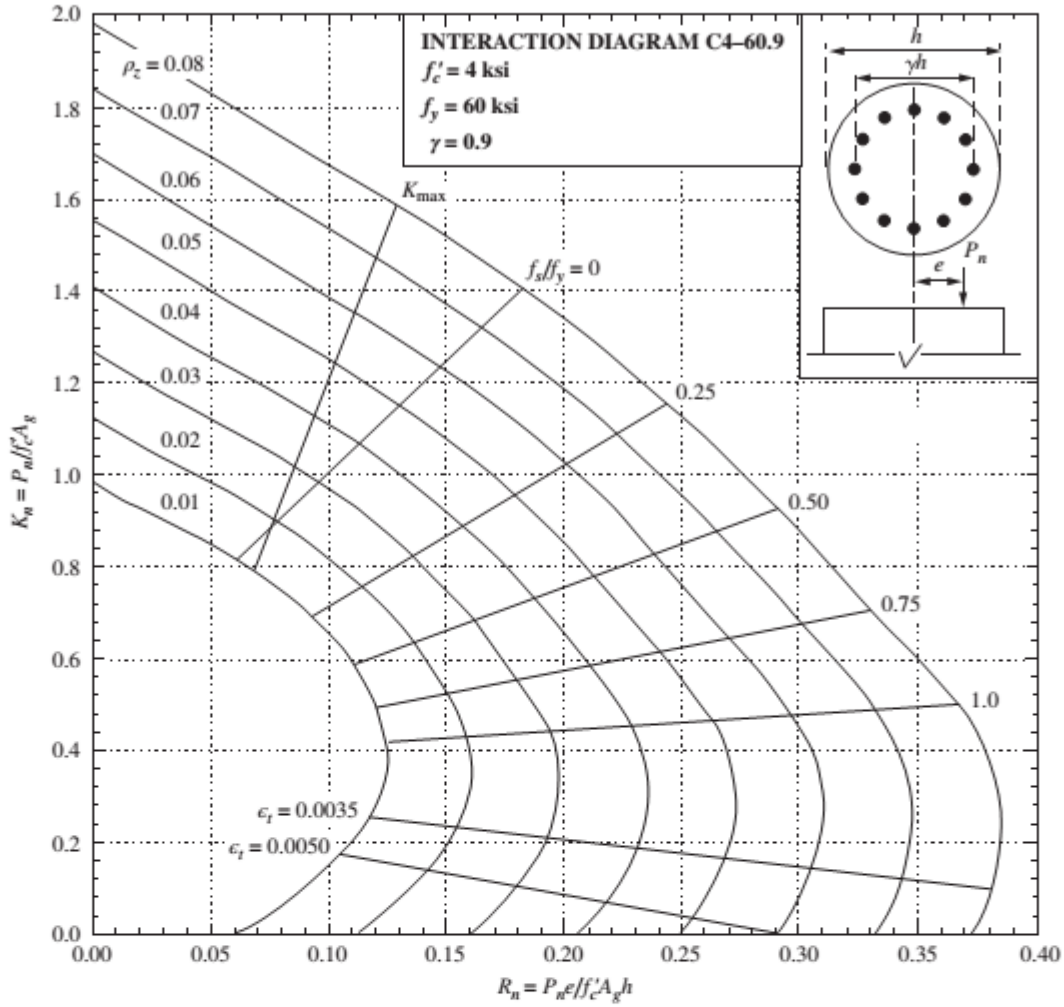
GRAPH 10 Column interaction diagrams for circular spiral columns.
(Published with the permission of the American Concrete Institute.)



GRAPH 11 Column interaction diagrams for circular spiral columns.
(Published with the permission of the American Concrete Institute.)



GRAPH 12 Column interaction diagrams for circular spiral columns.
(Published with the permission of the American Concrete Institute.)



GRAPH 13 Column interaction diagrams for circular spiral columns.
(Published with the permission of the American Concrete Institute.)

Tables in SI Units

APPENDIX B

TABLE B.1 Values of Modulus of Elasticity for Normal-Weight Concrete

f'_c (MPa)	E_c (MPa)
17	17,450
21	21,500
24	23,000
28	24,900
35	27,800
42	30,450

TABLE B.2 Designations, Diameters, Areas, Perimeters, and Masses of Metric Bars

	Nominal Dimensions		
Bar No.	Diameter (mm)	Area (mm^2)	Mass (kg/m)
10	9.5	71	0.560
13	12.7	129	0.994
16	15.9	199	1.552
19	19.1	284	2.235
22	22.2	387	3.042
25	25.4	510	3.973
29	28.7	645	5.060
32	32.3	819	6.404
36	35.8	1006	7.907
43	43.0	1452	11.38
57	57.3	2581	20.24

TABLE B.3 Grades of Reinforcing Bars and Metric Bar Sizes Available for Each

ASTM No.	Steel Grade (MPa)	Bar Sizes
A615M Billet	300	#10–#19
	420	#10–#57
	520	#19–#57
A616M Rail	350	#10–#36
	420	#10–#36
A617M Axle	300	#10–#36
	420	#10–#36
A706M Low-Alloy	420	#10–#57

TABLE B.4 Areas of Groups of Standard Metric Bars (mm^2)

	Number of Bars								
Bar Designation	2	3	4	5	6	7	8	9	10
#10	142	213	284	355	426	497	568	639	710
#13	258	387	516	645	774	903	1032	1161	1290
#16	398	597	796	995	1194	1393	1592	1791	1990
#19	568	852	1136	1420	1704	1988	2272	2556	2840
#22	774	1161	1548	1935	2322	2709	3096	3483	3870
#25	1020	1530	2040	2550	3060	3570	4080	4590	5100
#29	1290	1935	2580	3225	3870	4515	5160	5805	6450
#32	1638	2457	3276	4095	4914	5733	6552	7371	8190
#36	2012	3018	4024	5030	6036	7042	8048	9054	10,060
#43	2904	4356	5808	7260	8712	10,162	11,616	13,068	14,520
#57	5162	7743	10,324	12,905	15,486	18,067	20,648	23,229	25,810

	Number of Bars									
Bar Designation	11	12	13	14	15	16	17	18	19	20
#10	781	852	923	994	1065	1136	1207	1278	1349	1420
#13	1419	1548	1677	1806	1935	2064	2193	2322	2451	2580
#16	2189	2388	2587	2786	2985	3184	3383	3582	3781	3980
#19	3124	3408	3692	3976	4260	4544	4828	5112	5396	5680
#22	4257	4644	5031	5418	5805	6192	6579	6966	7353	7740
#25	5610	6120	6630	7140	7650	8160	8670	9180	9690	10,200
#29	7095	7740	8385	9030	9675	10,320	10,965	11,610	12,255	12,900
#32	9009	9828	10,647	11,466	12,285	13,104	13,913	14,742	15,561	16,380
#36	11,066	12,072	13,078	14,084	15,090	16,096	17,102	18,108	19,114	20,120
#43	15,972	17,424	18,876	20,328	21,780	23,232	24,684	26,136	27,588	29,040
#57	28,391	30,972	33,553	36,134	38,715	41,296	43,877	46,458	49,039	51,620

TABLE B.5 Minimum Beam Width (mm) for Beams with Inside Exposure (2014 ACI Metric Code)[a,b,c]

Size of Bars	Number of Bars in Single Layer of Reinforcement							Add for Each Additional Bar
	2	3	4	5	6	7	8	
#13	175	213	251	288	326	364	401	37.7
#16	178	219	260	301	342	383	424	40.9
#19	182	226	270	314	358	402	446	44.1
#22	185	232	279	326	373	421	468	47.2
#25	188	239	290	341	391	442	493	50.8
#29	195	252	310	367	424	482	539	57.4
#32	202	267	331	396	460	525	590	64.6
#36	209	281	353	424	496	567	639	71.6
#43	228	314	400	486	572	658	744	86.0
#57	271	386	501	615	730	844	959	114.6

[a]Minimum beam widths for beams were calculated using #10 stirrups.
[b]Maximum aggregate sizes were assumed not to exceed three-fourths of the clear spacing between the bars (ACI Code, Section 26.4.2.1).
[c]The horizontal distance from the center of the outside longitudinal bars to the inside of the stirrups was assumed to be equal to the larger of two times the stirrup diameter (ACI Code, Section 25.3.2) or half the longitudinal bar diameter.

TABLE B.6 Areas of Bars in Slabs (mm^2/m)

Spacing (mm)	Bar Number								
	10	13	16	19	22	25	29	32	36
75	947	1720	2653	3787	5160	6800	8600	10 920	13 413
90	789	1433	2211	3156	4300	5667	7167	9100	11 178
100	710	1290	1990	2840	3870	5100	6450	8190	10 060
115	617	1122	1730	2470	3365	4435	5609	7122	8748
130	546	992	1531	2185	2977	3923	4962	6300	7738
140	507	921	1421	2029	2764	3643	4607	5850	7186
150	473	860	1327	1893	2580	3400	4300	5460	6707
165	430	782	1206	1721	2345	3091	3909	4964	6097
180	394	717	1106	1578	2150	2833	3583	4550	5589
190	374	679	1047	1495	2037	2684	3395	4311	5295
200	355	645	995	1420	1935	2550	3225	4095	5030
225	316	573	884	1262	1720	2267	2867	3640	4471
250	284	516	796	1136	1548	2040	2580	3276	4024
300	237	430	663	947	1290	1700	2150	2730	3353

TABLE B.7 Values of ρ Balanced, ρ to Achieve Various ϵ_t Values, and ρ Minimum for Flexure. All Values are for Tensile Reinforced Rectangular Sections

f_y (MPa)	f'_c (MPa)	21 $\beta_1 = 0.85$	28 $\beta_1 = 0.85$	35 $\beta_1 = 0.814$	42 $\beta_1 = 0.764$
300	ρ balanced	0.0337	0.0450	0.0538	0.0606
	ρ when $\epsilon_t = 0.004$	0.0217	0.0289	0.0346	0.0390
	ρ when $\epsilon_t = 0.005$	0.0190	0.0253	0.0303	0.0341
	ρ when $\epsilon_t = 0.075$	0.0144	0.0193	0.0231	0.0260
	ρ min for flexure	0.0047	0.0047	0.0049	0.0054
350	ρ balanced	0.0274	0.0365	0.0437	0.0492
	ρ when $\epsilon_t = 0.004$	0.0186	0.0248	0.0297	0.0334
	ρ when $\epsilon_t = 0.005$	0.0163	0.0217	0.0259	0.0292
	ρ when $\epsilon_t = 0.0075$	0.0124	0.0165	0.0198	0.0223
	ρ min for flexure	0.0040	0.0040	0.0042	0.0046
420	ρ balanced	0.0212	0.0283	0.0339	0.0382
	ρ when $\epsilon_t = 0.004$	0.0155	0.0206	0.0247	0.0278
	ρ when $\epsilon_t = 0.005$	0.0135	0.0181	0.0216	0.0244
	ρ when $\epsilon_t = 0.0075$	0.0103	0.0138	0.0165	0.0186
	ρ min for flexure	0.0033	0.0033	0.0035	0.0039
520	ρ balanced	0.0156	0.0208	0.0249	0.0281
	ρ when $\epsilon_t = 0.004$	0.0125	0.0167	0.0200	0.0225
	ρ when $\epsilon_t = 0.005$	0.0109	0.0146	0.0175	0.0197
	ρ when $\epsilon_t = 0.0075$	0.0083	0.0111	0.0133	0.0150
	ρ min for flexure	0.0027	0.0027	0.0028	0.0031

TABLE B.8 $f_y = 420$ MPa; $f'_c = 21$ MPa—SI Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for	0.0018	0.740	0.0048	1.902	0.0078	2.975	0.0107	3.928
temp. and	0.0019	0.780	0.0049	1.939	0.0079	3.010	0.0108	3.960
shrinkage	0.0020	0.820	0.0050	1.976	0.0080	3.044	0.0109	3.991
	0.0021	0.860	0.0051	2.013	0.0081	3.078	0.0110	4.022
	0.0022	0.900	0.0052	2.050	0.0082	3.112	0.0111	4.053
	0.0023	0.940	0.0053	2.087	0.0083	3.146	0.0112	4.084
	0.0024	0.980	0.0054	2.124	0.0084	3.179	0.0113	4.115
	0.0025	1.019	0.0055	2.161	0.0085	3.213	0.0114	4.146
	0.0026	1.059	0.0056	2.197	0.0086	3.247	0.0115	4.177
	0.0027	1.098	0.0057	2.233	0.0087	3.280	0.0116	4.207
	0.0028	1.137	0.0058	2.270	0.0088	3.313	0.0117	4.238
	0.0029	1.176	0.0059	2.306	0.0089	3.347	0.0118	4.268
	0.0030	1.216	0.0060	2.342	0.0090	3.380	0.0119	4.298
	0.0031	1.255	0.0061	2.378	0.0091	3.413	0.0120	4.328
	0.0032	1.293	0.0062	2.414	0.0092	3.446	0.0121	4.359
ρ_{min} for	0.0033	1.332	0.0063	2.450	0.0093	3.479	0.0122	4.389
flexure	0.0034	1.371	0.0064	2.486	0.0094	3.511	0.0123	4.418
	0.0035	1.409	0.0065	2.521	0.0095	3.544	0.0124	4.448
	0.0036	1.448	0.0066	2.557	0.0096	3.577	0.0125	4.478
	0.0037	1.486	0.0067	2.592	0.0097	3.609	0.0126	4.508
	0.0038	1.525	0.0068	2.628	0.0098	3.641	0.0127	4.537
	0.0039	1.563	0.0069	2.663	0.0099	3.674	0.0128	4.566
	0.0040	1.601	0.0070	2.698	0.0100	3.706	0.0129	4.596
	0.0041	1.639	0.0071	2.733	0.0101	3.738	0.0130	4.625
	0.0042	1.677	0.0072	2.768	0.0102	3.770	0.0131	4.654
	0.0043	1.715	0.0073	2.803	0.0103	3.802	0.0132	4.683
	0.0044	1.752	0.0074	2.837	0.0104	3.834	0.0133	4.712
	0.0045	1.790	0.0075	2.872	0.0105	3.865	0.0134	4.741
	0.0046	1.827	0.0076	2.907	0.0106	3.897	0.0135	4.769
	0.0047	1.865	0.0077	2.941				

TABLE B.9 f_y = 420 MPa; f'_c = 28 MPa—SI Units

	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$	ρ	$\frac{M_u}{\phi bd^2}$
ρ_{min} for	0.0018	0.744	0.0059	2.349	0.0100	3.829	0.0141	5.185
temp. and	0.0019	0.785	0.0060	2.387	0.0101	3.864	0.0142	5.217
shrinkage	0.0020	0.825	0.0061	2.424	0.0102	3.898	0.0143	5.248
	0.0021	0.866	0.0062	2.462	0.0103	3.933	0.0144	5.280
	0.0022	0.906	0.0063	2.499	0.0104	3.967	0.0145	5.311
	0.0023	0.946	0.0064	2.536	0.0105	4.001	0.0146	5.342
	0.0024	0.987	0.0065	2.573	0.0106	4.036	0.0147	5.373
	0.0025	1.027	0.0066	2.611	0.0107	4.070	0.0148	5.404
	0.0026	1.067	0.0067	2.648	0.0108	4.104	0.0149	5.435
	0.0027	1.107	0.0068	2.685	0.0109	4.138	0.0150	5.466
	0.0028	1.147	0.0069	2.722	0.0110	4.172	0.0151	5.497
	0.0029	1.187	0.0070	2.758	0.0111	4.205	0.0152	5.528
	0.0030	1.227	0.0071	2.795	0.0112	4.239	0.0153	5.558
	0.0031	1.266	0.0072	2.832	0.0113	4.273	0.0154	5.589
	0.0032	1.306	0.0073	2.869	0.0114	4.306	0.0155	5.620
ρ_{min} for	0.0033	1.346	0.0074	2.905	0.0115	4.340	0.0156	5.650
flexure	0.0034	1.385	0.0075	2.942	0.0116	4.373	0.0157	5.681
	0.0035	1.424	0.0076	2.978	0.0117	4.407	0.0158	5.711
	0.0036	1.464	0.0077	3.014	0.0118	4.440	0.0159	5.741
	0.0037	1.503	0.0078	3.051	0.0119	4.473	0.0160	5.771
	0.0038	1.542	0.0079	3.087	0.0120	4.506	0.0161	5.801
	0.0039	1.582	0.0080	3.123	0.0121	4.539	0.0162	5.831
	0.0040	1.621	0.0081	3.159	0.0122	4.572	0.0163	5.861
	0.0041	1.660	0.0082	3.195	0.0123	4.605	0.0164	5.891
	0.0042	1.699	0.0083	3.231	0.0124	4.638	0.0165	5.921
	0.0043	1.737	0.0084	3.267	0.0125	4.671	0.0166	5.951
	0.0044	1.776	0.0085	3.302	0.0126	4.704	0.0167	5.980
	0.0045	1.815	0.0086	3.338	0.0127	4.736	0.0168	6.010
	0.0046	1.854	0.0087	3.374	0.0128	4.769	0.0169	6.040
	0.0047	1.892	0.0088	3.409	0.0129	4.801	0.0170	6.069
	0.0048	1.931	0.0089	3.444	0.0130	4.834	0.0171	6.098
	0.0049	1.969	0.0090	3.480	0.0131	4.866	0.0172	6.128
	0.0050	2.007	0.0091	3.515	0.0132	4.898	0.0173	6.157
	0.0051	2.046	0.0092	3.550	0.0133	4.930	0.0174	6.186
	0.0052	2.084	0.0093	3.585	0.0134	4.963	0.0175	6.215
	0.0053	2.122	0.0094	3.621	0.0135	4.995	0.0176	6.244
	0.0054	2.160	0.0095	3.656	0.0136	5.027	0.0177	6.273
	0.0055	2.198	0.0096	3.690	0.0137	5.058	0.0178	6.302
	0.0056	2.236	0.0097	3.725	0.0138	5.090	0.0179	6.331
	0.0057	2.274	0.0098	3.760	0.0139	5.122	0.0180	6.359
	0.0058	2.311	0.0099	3.795	0.0140	5.154	0.0181	6.388

Glossary*

Aggregate interlock Shear or friction resistance by the concrete on opposite sides of a crack in a reinforced concrete member (obviously larger with narrower cracks).

Balanced failure condition The simultaneous occurrence of the crushing of the compression concrete on one side of a member and the yielding of the tensile steel on the other side.

Bond stresses The shear-type stresses produced on the surfaces of reinforcing bars as the concrete tries to slip on those bars.

Camber The construction of a member bent or arched in one direction so that it won't look so bad when the service loads bend it in the opposite direction.

Capacity reduction factors Factors that take into account the uncertainties of material strengths, approximations in analysis, and variations in dimensions and workmanship. They are multiplied by the nominal or theoretical strengths of members to obtain their permissible strengths.

Cast-in-place concrete Concrete cast at the building site in its final position.

Column capital A flaring or enlarging of a column underneath a reinforced concrete slab.

Composite column A concrete column that is reinforced longitudinally with structural steel shapes.

Compression reinforcement Reinforcement added to the compression side of beams to increase moment capacity, increase ductility, decrease long-term deflections, or provide hangars for shear reinforcement.

Concrete A mixture of sand, gravel, crushed rock, or other aggregates held together in a rock-like mass with a paste of cement and water.

Concrete masonry unit (CMU) A masonry unit made of portland cement, water, and mineral aggregates formed into a rectangular prism.

Cover A protective layer of concrete over reinforcing bars to protect them from fire and corrosion.

Cracking moment The bending moment in a member when the concrete tensile stress equals the modulus of rupture and cracks begin to occur.

Creep or plastic flow When a concrete member is subjected to sustained compression loads, it will continue to shorten for years. The shortening that occurs after the initial or instantaneous shortening is called *creep* or *plastic flow* and is caused by the squeezing of water from the pores of the concrete.

Dead load Loads of constant magnitude that remain in one position. Examples: weights of walls, floors, roofs, plumbing, fixtures, structural frames, and so on.

Development length The length of a reinforcing bar needed to anchor or develop its stress at a critical section.

Doubly reinforced beam Concrete beams that have both tensile and compression reinforcing.

Drop panels A thickening of a reinforced concrete slab around a column.

Effective depth The distance from the compression face of a flexural member to the center of gravity of the tensile reinforcing.

Factored load A load that has been multiplied by a load factor, thus providing a safety factor.

Finite Element Analysis A numerical modeling technique in which a structure is divided into a number of discrete elements for analysis.

Flat plate Solid concrete floor or roof slabs of uniform depths that transfer loads directly to supporting columns without the aid of beams or capitals or drop panels.

Flat slab Reinforced concrete slab with capitals and/or drop panels.

Formwork The mold in which semiliquid concrete is placed.

*ACI 318-14 (Section 2.3) contains a more complete list of terminology.

Grade 40 (60) reinforcement Reinforcement with a minimum yield stress of 40,000 psi (60,000 psi).

Grout A mixture of portland cement or other cement, sand, water, and often coarse aggregate and sometimes admixtures used to fill the hollow cells in masonry units and bond reinforcing to the masonry.

Headed deformed bars Deformed bars with heads attached at one or both ends.

Honeycomb Areas of concrete where there is segregation of the coarse aggregate or rock pockets where the aggregate is not surrounded with mortar. It is caused by the improper handling and placing of the concrete.

Inflection point A point in a flexural member where the bending moment is zero and where the moment is changing from one sign to the other.

Influence line A diagram whose ordinates show the magnitude and character of some function of a structure (shear, moment, etc.) as a load of unity moves across the structure.

Interaction curve A diagram showing the interaction or relationship between two functions of a member, usually axial column load and bending.

Joint reinforcement Wire reinforcing embedded in mortar joints used in masonry walls to reduce shrinkage cracking.

L beam A T beam at the edge of a reinforced concrete slab that has a flange on only one side.

Lightweight concrete Concrete where lightweight aggregate (such as zonolite, expanded shales, sawdust) is used to replace the coarse and/or fine aggregate.

Limit state A condition at which a structure or some part of that structure ceases to perform its intended function.

Lintel A beam made of structural steel, precast concrete, or masonry embedded in a masonry wall spanning over an opening such as a window or door.

Live loads Loads that change position and magnitude. They move or are moved. Examples: trucks, people, wind, rain, earthquakes, temperature changes, and so on.

Load factor A factor generally larger than one that is multiplied by a service or working load to provide a factor of safety.

Long columns *See* Slender columns.

Maximum considered earthquake (MCE) An extreme earthquake, considered to occur only once every 2500 years.

Microcrack A crack too fine to be seen with the naked eye.

Modulus of elasticity The ratio of stress to strain in elastic materials. The higher its value, the smaller the deformations in a member.

Modulus of rupture The flexural tensile strength of concrete.

Monolithic concrete Concrete cast in one piece or in different operations but with proper construction joints.

Mortar A cementitious material used to construct masonry by being placed between the individual masonry units to bond them together to form a masonry element such as a wall.

Nominal strength The theoretical ultimate strength of a member, such as M_n (nominal moment), V_n (nominal shear), and so on.

One-way slab A slab designed to bend in one direction.

Overreinforced members Members for which the tensile steel will not yield (nor will cracks and deflections appreciably change) before failure, which will be sudden and without warning due to crushing of the compression concrete.

***P-δ* moments** *See* Secondary moments.

Plain concrete Concrete with no reinforcing whatsoever.

Plastic centroid of column The location of the resultant force produced by the steel and the concrete.

Plastic deformation Permanent deformation occurring in a member after its yield stress is reached.

Plastic flow *See* Creep or plastic flow.

Poisson's ratio The lateral expansion or contraction of a member divided by its longitudinal shortening or lengthening when the member is subjected to tension or compression forces. (Average value for concrete is about 0.16.)

Posttensioned concrete Prestressed concrete for which the steel is tensioned after the concrete has hardened.

Precast concrete Concrete cast at a location away from its final position. It may be cast at the building site near its final position, but usually this is done at a concrete yard.

Prestressing The imposition of internal stresses into a structure that are of an opposite character to those that will be caused by the service or working loads.

Pretensioned Prestressed concrete for which the steel is tensioned before the concrete is placed.

Primary moments Computed moments in a structure that do not account for structure deformations.

Ready-mixed concrete Concrete that is mixed at a concrete plant and then is transported to the construction site.

Reinforced concrete A combination of concrete and steel reinforcing wherein the steel provides the tensile strength lacking in the concrete. (The steel reinforcing can also be used to resist compressive forces.)

Secondary moments Moments caused in a structure by its deformations under load. As a column bends laterally, a moment is caused equal to the axial load times the lateral deformation. It is called a *secondary* or *P-δ moment*.

Seismic Design Category A classification given to a structure based on its Occupancy Category and the severity of the design earthquake ground motion at the site.

Serviceability Pertains to the performance of structures under normal service loads and is concerned with such items as deflections, vibrations, cracking, and slipping.

Service loads The actual loads that are assumed to be applied to a structure when it is in service (also called *working loads*).

Shearheads Cross-shaped elements, such as steel channels and I beams, placed in reinforced concrete slabs above columns to increase their shear strength.

Shearwall A wall usually made of reinforced concrete or reinforced masonry loaded with lateral loads in its own plane.

Shores The temporary members (probably wood or metal) that are used for vertical support for formwork into which fresh concrete is placed.

Short columns Columns with such small slenderness ratios that secondary moments are negligible.

Strut-and-tie method A method of analysis for shear in reinforced concrete members whereby a truss model of a member, consisting of struts and ties connected at nodes, replaces the member, for purposes of analysis.

Slender columns (or long columns) Columns with sufficiently large slenderness ratios that secondary moments appreciably weaken them (to the ACI, an appreciable reduction in strength in columns is more than 5%).

Spalling The breaking off or flaking off of a concrete surface.

Spiral column A column that has a helical spiral made from bars or heavy wire wrapped continuously around its longitudinal reinforcing bars.

Spirals Closely spaced wires or bars wrapped in a continuous spiral around the longitudinal bars of a member to hold them in position.

Split-cylinder test A test used to estimate the tensile strength of concrete.

Stirrups Vertical reinforcement added to reinforced concrete beams to increase their shear capacity.

Strength design A method of design whereby the estimated dead and live loads are multiplied by certain load or safety factors. The resulting so-called *factored loads* are used to proportion the members.

T beam A reinforced concrete beam that incorporates a portion of the slab that it supports.

Tendons Wires, strands, cables, or bars used to prestress concrete.

Tied column A column with a series of closed steel ties wrapped around its longitudinal bars to hold them in place.

Ties Individual pieces of wires or bars wrapped at intervals around the longitudinal bars of a member to hold them in position.

Top bars Horizontal reinforcing bars that have at least 12 in. of fresh concrete placed beneath them.

Transformed area The cross-sectional area of one material theoretically changed into an equivalent area of another material by multiplying it by the ratio of the moduli of elasticity of the two materials. For illustration, an area of steel is changed to an equivalent area of concrete, expressed as nA_s.

Two-way slabs Floor or roof slabs supported by columns or walls arranged so that the slab bends in two directions.

Underreinforced member A member that is designed so that the tensile steel will begin to yield (resulting in appreciable deflections and large visible cracks) while the compression concrete has not yet reached its limiting compressive strain. Thus, a warning is provided before failure occurs.

Web reinforcement Shear reinforcement in flexural members.

Working loads The actual loads that are assumed to be applied to a structure when it is in service (also called *service loads*).

Working-stress design A method of design where the members of a structure are so proportioned that the design forces based on Allowable Stress Load Combinations do not cause elastically computed stresses to exceed certain specified values. Method is also referred to as *allowable stress design, elastic design, alternate design method* or *service load design*.

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Frequently Used Notation*

a = depth of equivalent compression rectangular stress block for flexural members
A_b = cross-sectional area of a reinforcing bar
A_c = area of core of a spirally reinforcing column measured out to out of spiral
A_g = gross cross-sectional area of a concrete member
A_l = total area of longitudinal reinforcing to resist torsion
A_s = area of nonprestressed tensile reinforcing
A'_s = area of compression reinforcement
A_{sk} = area of skin reinforcement for a deep beam per unit of height on one side face of beam
A_{st} = total area of nonprestressed longitudinal reinforcing (bars or steel shapes)
A_t = area of one leg of a closed stirrup resisting torsion within a distance s
A_v = cross-sectional area of shear reinforcing in a distance s in a flexural member
A_1 = loaded area
A_2 = maximum area of a supporting surface that is geometrically similar and concentric with the loaded area A_1
b = width of the compression face of a flexural member
b = effective width of the flange of a T or L beam
b_o = perimeter of critical section for shear for slabs or footings
b_w = web width or diameter of a circular section
c = distance from extreme compression fiber to neutral axis
c_b = smaller of: (a) the distance from center of a bar or wire to nearest concrete surface, and (b) one-half the center-to-center spacing of bars or wires being developed
C_m = a factor relating an equivalent uniform moment diagram to the actual diagram
d = effective depth of a section measured from extreme compression fiber to centroid of tensile reinforcement
d' = distance from extreme compression fiber to centroid of compression steel
d_b = bar diameter
d_c = concrete cover thickness measured from extreme tensile fiber to closed reinforcing bar or wire
D_c = diameter of core of spiral column measured out to out of column
E_c = modulus of elasticity of concrete
E_s = modulus of elasticity of steel
f_c = computed compression flexural fiber stress at service loads
f'_c = specified compression strength of concrete
f'_m = specified compressive strength of masonry
f_{ct} = average splitting tensile strength of lightweight aggregate concrete
f_r = modulus of rupture of concrete
f_s = computed flexural stress in tensile steel at service loads
f'_s = computed flexural stress in compression steel
f_t = computed tensile flexural stress in concrete
f_y = specified yield strength of nonprestressed reinforcing
f_{yt} = specified yield strength of transverse reinforcement, psi (MPa)
h = total thickness of member
h_f = thickness of compression flange of a T, L, or I section
I_{cr} = transformed moment of inertia of cracked concrete section

*ACI 318-14 (Section 2.2) contains a more complete list of notation.

I_e = effective moment of inertia of a section used for deflection calculations
I_g = gross moment of inertia of a section about its centroidal axis
k = effective length factor for a compression member
l_d = development length of a straight bar embedded in confined concrete
l_{db} = basic development length
l_{dh} = development length of a bar with a standard hook
l_{dt} = development length in tension of headed deformed bar
l_{hb} = basic development length of a standard hook in tension
l_n = clear span measured face to face of supports
l_u = unsupported length of a compression member
M_{cr} = cracking moment of concrete
M_1 = smaller end factored moment in a compression member, negative if double curvature
M_2 = larger end factored moment in a compression member
M_{1ns} = smaller factored end moment in a compression member due to loads that result in no appreciable sidesway
M_{2ns} = larger factored end moment in a compression member due to loads that result in no appreciable sidesway
M_a = maximum moment in member due to service loads at stage deflection is computed
M_o = total factored static moment
M_u = factored moment at section
n = modular ratio (ratio of modulus of elasticity of steel to that of concrete)
P_c = Euler buckling load of column
P_{no} = pure axial load capacity of column
P_0 = nominal axial load strength of a member with no eccentricity
q_a = allowable soil pressure
q_e = effective soil pressure
R_n = a term used in required percentage of steel expression for flexural members ($M_u/\phi bd^2$)
s = spacing of shear or torsional reinforcing parallel to longitudinal reinforcing
V_{nm} = nominal shear strength provided by masonry (see Chapter 20)
V_{ns} = nominal shear strength provided by shear reinforcement (see Chapter 20)
V_{tu} = torsional stress
w = crack width
w_c = unit weight of concrete
y_t = distance from centroidal axis of gross section to extreme fiber in tension
z = a term used to estimate crack sizes and specify distribution of reinforcing
β = ratio of long to short dimensions: clear spans for two-way slabs; sides of column, concentrated load or reaction area; or sides of a footing
β_{dns} = ratio used to account for reduction of stiffness of columns due to sustained axial loads
β_{ds} = ratio used to account for reduction of stiffness of columns due to sustained lateral loads
β_1 = a factor to be multiplied by the depth d of a member to obtain the depth of the equivalent rectangular stress block
δ = moment magnification factor to reflect effects of member curvature between ends of compression member
δ_s = moment magnification factor for slender columns in frames not braced against sidesway
ϵ_c = strain in compression concrete
ϵ_s = strain in tension reinforcement
ϵ'_s = strain in compression reinforcement
λ = modification factor reflecting the reduced mechanical properties of lightweight concrete; all relative to normal weight concrete of the same compressive strength
λ_Δ = a multiplier used in computing long-term deflections
μ = coefficient of friction
ξ = a time-dependent factor for sustained loads used in computing long-term deflections
ρ = ratio of nonprestressed reinforcement in a section
ρ' = ratio of compression reinforcing in a section
ρ_b = ratio of tensile reinforcing producing balanced strain condition
ϕ = capacity reduction factor

Typical SI Quantities and Units

Quantity	Unit	Symbol
Length	meter	m
Area	square meter	m^2
Volume	cubic meter	m^3
Force	newton	N
Weight	newton per cubic meter	N/m^3

Quantity	Unit	Symbol
Stress	pascal (N/m^2)	Pa
Moment	newton meter	N•m
Work	newton meter	Nm
Density	kilogram per cubic meter	kg/m^3
Mass	kilogram	kg

SI Prefixes

Prefix	Symbol	Multiplication Factor
tera	T	$10^{12} = 1\,000\,000\,000\,000$
giga	G	$10^{9} = 1\,000\,000\,000$
mega	M	$10^{6} = 1\,000\,000$
kilo	k	$10^{3} = 1\,000$
hecto	h	$10^{2} = 100$
deca	da	$10^{1} = 10$
deci	d	$10^{-1} = 0.100$
centi	c	$10^{-2} = 0.010$
milli	m	$10^{-3} = 0.001$
micro	μ	$10^{-6} = 0.000\,001$
nano	n	$10^{-9} = 0.000\,000\,001$
pico	p	$10^{-12} = 0.000\,000\,000\,001$

Conversion of U.S. Customary Units to SI Units

U.S. Customary Units	SI Units
1 in.	25.400 mm = 0.025 400 m
1 $in.^2$	645.16 $mm^2 = 6.451\,600\ m^2 \times 10^{-4}$
1 ft	304.800 mm = 0.304 800 m
1 lb	4.448 222 N
1 kip	4 448 222 N = 4.448 222 kN
1 psi	$6.894\,757\ kN/m^2 = 0.006\,895\ MN/m^2 = 0.006\,895\ N/mm^2$
1 psf	$47.880\ N/m^2 = 0.047\,800\ kN/m^2$
1 ksi	$6.894\,757\ MN/m^2 = 6.894\,757$ MPa
1 in-lb	0.112 985 N•m
1 ft-lb	1.355 818 N•m
1 in-k	112.985 N•m
1 ft-k	1 355.82 N•m = 1.355 82 kN•m

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